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SHORT-WAVE RADIATION PHENOMENA

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SHORT-WAVE RADIATION PHENOMENA

by

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SHORT-WAVE RADIATION PHENOMENA

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PREFACE

The purpose of this text is to present short-wave radiation phenomena with many useful applications which play a part in modern radio communication and in the use of electromagnetic waves for guiding objects. Safe aviation often depends upon reliable communication as well as upon radio beams along a desired path, frequently influenced by atmospheric conditions, the behavior of the earth crust, and other wave obstacles. There is also a new field of telemetering through distances.

The book gives the student in the field of radio many practical and numerical applications, together with classical as well as with more recent wave-propagation concepts. This is indeed a large order. Such information should be equally useful to the many radio amateurs who have done so much to provoke scientific interest in electromagnetic wave propagation, particularly with reference to electromagnetic skip distances due to ionospheric wave reflections. Electromagnetic waves have always been a most fascinating field for the experimenter as well as for the theoretical physicist. Both types of contributors are needed for the further enrichment of electromagnetic-field achievements, as they were in the days of J. C. Maxwell, Heinrich Hertz, Sir Oliver Lodge, and many others. We cannot separate the experimenter from the worker at the desk.

Inasmuch as a text of this type deals with radio experimenting and radio engineering as well as with radio science, many details are involved. Without such details no further progress could be encouraged by bringing out such a book. For this reason the various sections are all *self-contained*, with subjects indicated by titles. Therefore under the corresponding section title it is possible to find sufficient information dealing with the desired subject. It is admitted that even with such a subdivision in the text we may read, for instance, on the general wave medium (dissipative medium) in several places, but the cross references given will help the reader to find additional information in other sections.

Today we know from actual experience that commercial radiators and field interceptors as well as electromagnetic wave propagation are more involved than they appeared to be in the earlier days of radio. We find, for instance, that the physics of the atmosphere can explain certain propagation phenomena and their anomalies. Many of the anomalies may occur often and for quite long intervals, so that we cannot classify

them as mere exceptions to the rule. Some anomalies can actually be used for certain work.

We can no longer accept unobstructed free-space propagation in many applications, even when an airplane at a high altitude tries to communicate with another aircraft or to make radio contact with some ground station. We must take into consideration the actual electromagnetic behavior of a particular portion of the earth crust as well as of the wave medium between the radiator and the wave pickup. As a matter of fact the *electromagnetic properties* of all wave obstacles, radiators, and wave interceptors must be understood.

This is the reason why "Short-wave Radiation Phenomena" begins not with customary definitions and introductions but with numerical electromagnetic properties and their applications. The customary definitions and explanations are not overlooked, however, but are given in places where they belong. An attempt is made to start with applications showing the usefulness of certain modern electromagnetic concepts, and later on, after having introduced the subject, to bring in fundamental details and formulas which are useful.

Thus on reaching page 11, the reader has probably developed a curiosity as to why such properties are used. We can then quote without boundary conditions the fundamental Maxwell equations, leaving the details to more advanced sections given later. It is then possible to continue on page 12 with a cube of the general wave medium, which may be, for example, a portion of the earth crust or a portion of a brick wall. By means of such practical propagation examples we can bring out what is meant by conduction and displacement currents generally, explain field energy concepts and circulations of electric and magnetic fields, mention line integrals, and lead up to the concept of the *equivalent* dielectric constant of the wave medium with dissipation. This is the trend of the entire book.

The reader will finally acquire the knowledge of many field concepts, and even of formulas, by means of applications. He will probably find that he gradually understands not only the "know-why" of facts but also the "know-how" of many electromagnetic phenomena. Therefore a text of this type, even though not written in customary order, attempts to explain mathematical procedures together with actual computations. The numerical evaluations in this book have only pocket-slide-rule accuracy, which is sufficient for most practical applications.

By the term *short waves* in this text we mean that we deal with operating carrier frequencies, for which we can use, for instance, customary half-wave dipoles without prohibitive dimensions. For short waves, even when referring only to space waves corresponding to frequencies as low

as 50 megacycles per second, an obstacle need not be big to exercise its interference on wave propagation. It is the dimension in terms of the operating wavelength that counts, and the electromagnetic properties of the wave obstacle as well as of antenna systems.

Many radio workers of recent years are pursuing their activities along lines having a bearing on wave propagation. It is also for them that this text is written. For this reason many numerical applications and illustrations with descriptions are given, so that *self-study* is possible, and many can gradually acquire basic established facts when these are presented together with material which substantiates them.

Self-study is, no doubt, a good way of improving one's ability in a certain vocation. It seems far better than spoon-fed instruction, even if it takes more time. Experienced workers in the radio profession should admit that we must dig in by ourselves in order to master the subject. We cannot learn everything in a classroom, since the art of radio changes rapidly. But a classroom course can teach us certain fundamentals and show us how to think and how to reason, so that we can study by ourselves and do something with basic information. The sooner we acquire the habit of self-study, the sooner we shall gain the confidence to contribute to our profession. No matter what educational background a worker in the radio field has had, he must always keep abreast with the progress of the art. The writer of this text is no better off.

Today we are fortunate to have such good texts, in mks units, as "Electromagnetic Theory" by J. A. Stratton and "Electromagnetic Waves" by S. A. Schelkunoff. These texts are difficult to replace. We have also "Electromagnetic Engineering," Vol. I, by R. W. P. King. Stratton applies general and more or less rigorous treatments. Schelkunoff treats his subjects without unduly devoting himself to exacting details, but with respect to the engineering point of view. The texts of King and "Electromagnetic Field" (classical units) by M. Mason and W. Weaver deal likewise with rigorous mathematical treatments. Such references, as well as other recent texts given in the List of References, are good collateral reading with "Short-wave Radiation Phenomena" when used for senior and graduate students.

In current literature there seems to be much misunderstanding with respect to the significance of auxiliary vectors $\underset{\cdot}{D}$ and $\underset{\cdot}{H}$ related to $\underset{\cdot}{\mathcal{E}}$ and $\underset{\cdot}{B}$, the dot indicating that we deal with a vector quantity. The fundamental analogy of $\underset{\cdot}{\mathcal{E}}$ and $\underset{\cdot}{B}$ seems to be the least confusing one.

In "Short-wave Radiation Phenomena" it will be found that the dimensional units are often given after the formulas. Units are required when using formulas; they also give a means for checking the correctness of a formula. The mks system of units is used throughout this text, as in

all modern books. A list of symbols used, as well as many references on wave propagation, is given in the Appendix. Pertinent references appear in footnotes to the text.

The nomenclature used is in many cases that which is commonly employed in radio texts, with the following exceptions. The Greek lower-case kappa (κ) is used in place of the lower-case epsilon (ϵ) for the dielectric constant. This is done in order not to confuse the dielectric constant with the base of the natural logarithm. The electric field is expressed by the script $\mathcal{E}$, to distinguish it from the customary symbol E for the voltage. The symbol σ is used for the conductivity of a dissipative wave medium. The vector potential is expressed by the Greek capital lambda (Λ). Since at different places in this text we deal with *magnetic* and *electric* vector potentials, the suffix m to Λ specifies the *magnetic* vector potential and the suffix e specifies the *electric* vector potential. When only the magnetic vector potential occurs, the symbol without a suffix is used (consult the List of Symbols). The Hertz electric polarization and vector is expressed as customarily by $\prod$, a large form of the Greek capital pi. If the magnetic polarization or magnetization vector appears, it is printed as $\prod_m$. When *magnetic* and *electric* polarization potentials appear in the same formula, the Hertz vector is written either as $\prod$ or as $\prod_e$. There can be no confusion, since such potentials are always described when used. The symbol $\prod$ is also used for the customary *Gauss* function, which, of course, has nothing to do with the polarization vector and belongs to a different section. The Greek capital gamma (Γ) is used as the customary symbol for the Gamma function. Whenever a new quantity is introduced, its symbol is given with the dimension in the system, which is explained in Sec. 2. As a matter of fact, the dimensions are given throughout the text as well as in tables and illustrations. A few pages of reading of the first section or any other section will show this.

Much stress has been placed on presenting the subjects in a *comparative* way. We can speak of an *electric* displacement current, for instance, as we can of a *magnetic* displacement current, the former being in amperes and the latter in volts. We have also *electric* as well as *magnetic* fields associated in an electromagnetic field. This can be seen from Eq. (1) on page 11. There are *electric* and *magnetic* potentials as well as such vector potentials. We can speak of a Hertz vector which is *electric* in nature as well as of a corresponding vector which is *magnetic* in nature. This is brought out in many expressions and tables. Similarly we can distinguish between *electric* and *magnetic dipoles* and such effects. This comparison is illustrated in Fig. 30 on page 161. We have also electric and magnetic current sheets.

"Short-wave Radiation Phenomena" consists of nine chapters and an appendix. Chapter III and the Appendix can be used for reference when dealing with theoretical procedures.

Chapter I is basic to all of the following chapters, as it deals with fundamental concepts and relations between currents and fields. Therefore it should be read first, whether the text is used for classroom work or for self-study.

Chapter II is a comparison of elementary electric and magnetic dipoles. Formulas given in this chapter can be used for deriving expressions for dipoles as well as for linear radiators which have larger dimensions. They can also be used for deriving expressions for larger magnetic dipoles (current loops). Actions of fields and currents are brought out in connection with Fig. 1, while the actions of a "curl" and of a "divergence" are explained in connection with Fig. 2.

Chapter IV deals with propagation characteristics, which play such a large part in wave propagation in feeders.

Chapter V explains lines and radiation, based on field concepts, whether guided or in open space. It deals also with traveling and standing waves on lines.

Chapter VI discusses unobstructed radiation in space, which is also basic to radiation effects in the presence of electromagnetic wave obstructions.

Chapter VII explains radiation effects in the presence of wave obstructions by means of the ray theory.

Chapter VIII describes wave obstructions and electromagnetic diffraction effects, where we may distinguish between *Fresnel* and *Fraunhofer* diffraction. It gives details of wave-front radiators based on electromagnetic Huygens wavelets and field effects in the obstructed as well as in the unobstructed region. It deals with diffraction radiators which are driven by radio-frequency voltages across slits, as is also brought out in the Appendix, by circular-ring slit radiators and narrow slits in the axial direction of metal cylinders.

Electric and magnetic displacement currents, described in Chap. IX for TE and TM waves, are the reason why a wave guide with only one conducting surrounding wall can support such complementary wave-guide actions.

The descriptions of electromagnetic diffraction should open up a wide field of application. It may be that in the near future we can use space itself as a radiator, and use the field conditions in special tubes with their accelerations and decelerations directly as radiation sources, thus avoiding a conversion into current and again a conversion into radiation fields. We may imagine devices which do this without tubes.

Also it seems a probability that wedge-shaped wave obstructions can be used for diverting a TV wave into a location below and behind a wave obstruction.

With respect to space radiators, we have already used diffraction effects from the mouth of an open wave guide, the end of an electromagnetic horn, and the electromagnetic parabolic projector, as can be seen in various publications by Schelkunoff and others. We can then explain such radiations by imaginary dipole distributions in the aperture or by Huygens wavelets of a primary tangential wave front occupying the aperture in a screen. It can also be done by electric and magnetic current sheets, which are based on the equivalence principle (Schelkunoff). Even though many of such explanations and treatments are not exactly rigorous, measurements in some cases seem to show that diffraction radiators open up a new field in short-wave radiation.

The author welcomes any corrections and suggestions for improvement.

AUGUST HUND

SANTA MONICA, CALIF.
November, 1951

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SHORT-WAVE RADIATION PHENOMENA

When dealing with short-wave radiation phenomena, we require an understanding of experimental facts as well as of theoretical expediencies. It was the Maxwellian theory that led to the discovery of short electromagnetic waves and the first successful experimental results. However, when we are carrying on theoretical speculations, it seems just as important to know how to apply simple mathematical expansions as well as to be able to apply classical and more recent electromagnetic relations. We require formulas that adapt themselves readily to experimentation.

CHAPTER I

FUNDAMENTAL CONCEPTS AND RELATIONS OF CURRENTS AND ELECTROMAGNETIC FIELDS

Magnetic and electric fields are associated with a current flow caused by electrons moving in a conductor.

In the case of a single current-carrying conductor, the magnetic lines of force are circles about the conductor. For a perfect or nearly perfect conductor, the electric lines of force are normal to the surface of the conductor. If the conductivity is only finite, electric lines of force leave or enter the conductor under an angle that is less than 90 deg, and the electric field component along the conductor surface accounts for the joulean heat loss in the conductor.

The component of the electric field normal to the conductor surface and the magnetic field around the conductor constitute the electric and magnetic fields about the conductor with energy that rests solely in the field region about the conductor of VHF and UHF currents. There is practically no field penetration into the conductor at such high frequencies.

If the region of the field represents a pure dielectric, the field energy is primarily wattless but may, according to Maxwell's theory and the Hertzian experiments, radiate into the distant space a portion that is quite distinct from the induction field, which is *always* interlinked with the conductor. We have then a free and a bound field. The distant free field imparts, as it were, an equivalent radiation conductivity, even in the case of a pure dielectric medium. This does *not* mean a local loss of field energy that turns into heat. It means an escape of field energy into the far distant region. It is the radiated field energy of customary antennas, for instance.

When the medium about a current-carrying conductor is not a perfect dielectric but also exhibits a finite conductivity, we have, in addition to the wattless interlinked induction field and the propagated radiation field, joulean heat losses in the medium of the electromagnetic field. Inasmuch as a current flow causes such fields, fields of this type must, in turn, produce a corresponding current flow. According to Maxwell, this may also be a displacement current but would occur in an ideal insulating medium as well as in a complex wave medium. Moreover, when electrons are accelerated, they require field energy; and when they

are decelerated, they may give up field energy. The latter is made use of in the klystron, for instance.

1. Electromagnetic Properties of Any Medium. Whether we deal with a medium through which electromagnetic waves are being propagated or a medium that represents a radiator or a receiving antenna, it is important to know what the electromagnetic properties of the particular medium are in order to secure the most efficient transfer of energy.

The electromagnetic properties of a medium are completely defined by the dielectric constant κ, the permeability μ, and the conductivity σ. Even for a vacuum, we still have two of these properties effective, namely, the dielectric constant κ_0 and the permeability μ_0 of the vacuum. The conductivity vanishes, and only a displacement current is possible. This current may be electric or magnetic in nature. Also a convection current is possible if a pure electron discharge occurs, as in high-vacuum tubes, or electrons are present and in motion, as may be the case in the ionized layers in the upper atmosphere. Without the existence of μ and κ, we could not account for electromagnetic wave propagation, since $1/\sqrt{\mu\kappa}$ meters per second denotes the velocity of wave propagation in a perfect dielectric medium.

In the Giorgi meter-kilogram-second(mks) system of units as used throughout this text, we have for a vacuum a permeability of $\mu_0 = 1.257 \times 10^{-6}$ henry/m, a dielectric constant $\kappa_0 = 8.854 \times 10^{-12}$ farad/m, and a conductivity σ of zero mhos/m. This means zero conductivity across opposite parallel sides, each of which is one meter square, of a cubic meter of vacuum.

Table 1 gives the orders of magnitude for the various electromagnetic properties of mediums that occur in nature. The ever-existing reference values for empty space μ_0 and κ_0 account for the maximum possible speed, $c = 1/\sqrt{\mu_0\kappa_0} = 2.998 \times 10^8$ m/sec, of electromagnetic wave propagation. This is also essentially the speed in the lower atmosphere. It is also owing to these reference values that empty as well as practically empty space, *i.e.*, customary atmosphere, offers to plane electromagnetic waves a characteristic wave impedance of $Z_0 = \sqrt{\mu_0/\kappa_0} = 376.7$ ohms. This means that for customary atmosphere and strictly so for vacuum, at any space point in the distant radiation field, the magnetic field intensity H in amperes per meter corresponds to $\mathcal{E}/376.7$, where $\mathcal{E}$ denotes the associated electric field intensity in volts per meter. It also means that the radiation energy flux[1] for each square meter of the plane wave front is $376.7H^2$ watts, which is the same as $\mathcal{E}^2/376.7$ watts. Hence, for

[1] We have similar conditions as for Ohm's law $E/I = R$ and the power relations, EI watts, RI^2 watts, E^2/R watts, since $\mathcal{E}/H = 376\ 7$ ohms corresponds to $E/I = R$ ohms. This means, that we deal with effective values for $\mathcal{E}$, H, E, and I.

TABLE 1. TYPICAL ELECTROMAGNETIC PROPERTIES AT 20°C

Medium	Dielectric constant, κ, farads/m	Conductivity σ, mhos/m
Vacuum (empty space)	$\kappa = \kappa_0 = 8.854 \times 10^{-12}$	$\sigma = \sigma_0 = 0$
Air (above ground)	$\cong \kappa_0 = 8.854 \times 10^{-12}$	0
	for most practical applications. For other details consult pages 29–31, 40–56, and 118–138.	
Dry soil of earth crust	$4\kappa_0$ to $10\kappa_0$	10^{-4} to 10^{-5}
Wet soil of earth crust	$9\kappa_0$ to $30\kappa_0$	10^{-2} to 10^{-3}
Dry sand	$10\kappa_0$	2×10^{-3}
Ocean water	About $80\kappa_0$	4 to 6
Average lake and river water	About $80\kappa_0$	10^{-2} to 10^{-3}
Fresh water	About $80\kappa_0$	10^{-3}
Distilled water	$81.1\kappa_0$	2×10^{-4} to 5×10^{-4} much affected by the least impurity
Slate	$6.8\kappa_0$ to $7.2\kappa_0$	10^{-6}
Marble	About $8.2\kappa_0$	10^{-7} to 10^{-9}
Bakelite	$5.4\kappa_0$	About 10^{-9}
Dry wood	$2.6\kappa_0$ to $7\kappa_0$	About 10^{-9}
	depending on the kind of wood and the directions of fibers	
Paraffined wood	About $5.5\kappa_0$	10^{-8} to 10^{-11}
Mica	$5\kappa_0$ to $9\kappa_0$	10^{-11} to 10^{-14}
Glass	$3.5\kappa_0$ to $4.5\kappa_0$	About 10^{-12}
	depending on the kind of glass	
Fused quartz	About $5\kappa_0$	10^{-12} to 10^{-17}
	Frequency range plays a part, and for crystal quartz orientation of possible piezo polarization plays a part	
Annealed copper	κ_0	5.8005×10^7
Hard-drawn copper	κ_0	5.65×10^7
Silver	κ_0	6.14×10^7
Aluminum	κ_0	3.54×10^7
Pure iron	κ_0	10^7
Steel	κ_0	5×10^6 to 10^7

Remarks. For metals, σ denotes the conductivity in mhos/m at 20°C (room temperature); for dielectrics σ represents the approximate value at this temperature. The relative dielectric constant κ_r referred to vacuum is given by the numeric ratio $\kappa_r = \kappa/\kappa_0$, where κ is the dielectric constant for the dielectric and κ_0 the dielectric constant for vacuum, both being expressed in farads/m. The relative numeric permeability μ_r is correspondingly $\mu_r = \mu/\mu_0$ where μ holds for the medium and $\mu_0 = 1.257 \times 10^{-6}$ henry/m for vacuum. For most nonferromagnetic mediums we have $\mu = \mu_0$; for good conductors, we may assume $\kappa \cong \kappa_0$ farads/m. For mediums with medium dissipation ($\sigma \neq 0$), we deal also with an *equivalent* dielectric constant

$$\kappa_e = \kappa - j\sigma/\omega \text{ farads/m}$$

where $\omega/6.28$ is the operating frequency in cycles/sec. The equivalent *relative* dielectric constant κ_e' then becomes

$$\kappa_e' = \frac{\kappa_e}{\kappa_0} = \kappa_r - j18{,}000\frac{\sigma}{f}$$

for f in Mc/sec. The figure of merit (Q value) of a wave medium is given by $Q = \omega\kappa/\sigma$ which is numeric.

Numerical values of this type are made use of throughout this text, and particularly so on pages 43–46, 52–56, 123, 126–128, and 140–143.

a measured field intensity of $\mathcal{E} = 50$ mv/m every second across each square meter, we have a power flow of $(50 \times 10^{-3})^2/376.7 = 6.61$ μw. From Table 1, we note that for average ground, we have a relative dielectric constant $\kappa_r \cong 13$, *i.e.*, an average value $\kappa = 13\kappa_0 = 1.15 \times 10^{-10}$ farad/m. For ocean water, we have $\kappa = 80\kappa_0 = 7.08 \times 10^{-10}$ farad/m for the dielectric constant for VHF and UHF field excitations. For frequencies that are extremely high, the relative value of κ_r may be as low as 20 for ocean water. In a medium in which efficient wave propagation is desired, the Q value of the medium must be very high at the operating frequency. Normally this requires a very small value for the conductivity σ in comparison with the value of $\omega\kappa$ mhos per meter. This means that the frequency also plays an important part. Humidity and temperature changes in the lower portion of the troposphere may cause the relative dielectric constant κ_r to vary within a few parts in a thousand. However, this will not have an appreciable effect on the velocity of wave propagation. But according to Secs. 175 178, it will change a linear path of wave propagation to a curved path because of a corresponding gradient of the atmospheric index of refraction to the electromagnetic waves. In many cases, the path is curved downward, so that wave propagation is possible into the geometrical shadow region below the horizon owing to such refraction. This must not be confused with extended propagation due to diffraction. Atmospheric refraction may also cause a linear path to curve upward, so that the actual service area of a particular radiator may be smaller in effect. We may also have to deal with a condition of superrefraction, as is brought out in Sec. 178 and in Sec. 239 of the Appendix.

In the lower troposphere close to the earth,[1] we may meet strata, or layers, of such dielectric properties that may cause atmospheric wave guide action between definite surfaces of separation. Such anomalous action may give rise to pronounced extended propagation ranges.

For a medium that is to serve as a transmitting or a receiving antenna, the very opposite is true with respect to the conductivity. A very high conductivity is necessary. As far as the dielectric constant is concerned, in most good conductors it is probably the same as that which holds for a vacuum. It would be very difficult to measure the value of κ, since the conduction current greatly outweighs the effect of any displacement current; and besides, at very high and ultrahigh frequencies, an electromagnetic field does not appreciably penetrate a good conductor.

To understand numerical applications, an apparently simple system

[1] A distinction is made between earth and ground surface. By "earth" is meant soil, rock, and water surface. By "ground" is meant any surface that is not water (such as ocean, lake, river).

of units for expressing the various quantities connecting electro-magnetic fields with current and power relations is briefly described in the following paragraphs.

2. The Giorgi MKS System of Units. For such units, electromagnetic relations are greatly simplified and the units correspond closely to quantities met in customary alternating-current practice, since we deal, for instance, with units such as volts per meter, amperes per meter, and watts per square meter. According to Giorgi, the practical system of units can be changed to an absolute system by taking as the unit length the international meter, as the unit of mass the kilogram, and as the unit of time the customary second and by employing another unit such as the coulomb of the practical system. We then deal strictly with a meter-kilogram-second-coulomb system. In field relations, the product $\mu_0\kappa_0$ and the ratio μ_0/κ_0 are essential and neither μ_0 nor κ_0 by itself is of consequence. This is true, since $1/\sqrt{\mu_0\kappa_0}$ must be a velocity equal to 2.9979 $\times$ 10^8 m/sec and $\sqrt{\mu_0/\kappa_0}$ must be equal to 376.7 ohms.[1] When, as above, the coulomb is used as an independent unit, we find the never-vanishing reference values $\mu_0 = 4\pi 10^{-7} = 1.257 \times 10^{-6}$ henry/m and from $c = 1/\sqrt{\mu_0\kappa_0} = 2.9979 \times 10^8$ m/sec the value $\kappa_0 = 8.854 \times 10^{-12}$ farad/m.

With respect to the classic systems of units, such as the es-cgs system, the em-cgs system, and the Gaussian system, reference is made to a table in the Appendix, which gives a comparison with mks units. In the rationalized mks system, the force is in newtons, energy in joules, power in watts, charge in coulombs, current in amperes, charge density in coulombs per cubic meter, current density in amperes per square meter, resistance in ohms, electric potential in volts, electric field intensity in volts per meter, dielectric constant in farads per meter, electric dipole moment in coulombs · meters, magnetic flux in webers, magnetic flux density in webers per square meter, magnetomotive force in ampere turns, magnetic field intensity in ampere turns per meter, self- and mutual inductance in henrys, magnetic permeability in henrys per meter, magnetic dipole moment in ampere · square meters, etc.

3. Remarks on Elementary Radiators, Radiation Effects, Electric and Magnetic Field Vectors, and Wave Front. This discussion deals primarily with effects in the distant radiation region, *i.e.*, the electromagnetic field energy that detaches itself from the localized induction field interlinked with the radiator.

The simplest radiator is a point source, also called an "isotropic ele-

[1] This is similar to circuit theory, where CL satisfies the oscillation constant and L/C is operative in the characteristic impedance. It also occurs in the classic transmission-line theory, where $1/\sqrt{CL}$ denotes the phase velocity and $\sqrt{L/C}$ the surge impedance.

mentary radiator." In unobstructed space, such a source radiates equally well in all directions, and electromagnetic field energy is propagated essentially with the velocity of 3×10^8 m/sec. This means that at the end of 1 sec, this field energy would be effective at a distance of 300,000 km. This distance is altogether too great as far as distances on our earth are concerned. At the end of $\frac{1}{100}$ sec, the radiation field would have traveled through 3,000 km, *i.e.*, $0.6214 \times 3{,}000 =$ 1,864.2 miles. If such a point source were available, had sufficient energy, and were located in unobstructed free space, the wave front would be the surface of a sphere with a radius of 1,864.2 miles. The infinitesimally small point source would be at the center of this sphere. At the end of 1/1,000 sec, the wave front would be the surface of a sphere with a radius of 186.42 miles. In each case, 1 sq m of the spherical wave front would be essentially a plane and both the electric field intensity $\mathcal{E}$ in volts per meter and the magnetic field intensity H in amperes per meter would be vectors normal to each other and located in the plane wave front. The direction of wave propagation is normal to this front, *i.e.*, radial. The relative magnitudes of $\mathcal{E}$ and H are such that the ratio $\mathcal{E}/H$ is about 377 ohms. This would also be approximately the case if the unobstructed medium through which radiation takes place were not empty space but the ordinary atmosphere as experienced in the troposphere, for instance.

Such an isotropic point source is not practical but a most suitable reference source when expressing radiation gains of directive radiators and of antenna arrays.

The next radiation source, which is also elementary, is the Hertzian doublet, a two-point source consisting of two very small spheres connected by a thin conducting filament. The filament is of an infinitesimally short length or, at least, very short and exhibits practically the same current along its entire length (exhibits uniform current distribution), since the minute spherical ends act like relatively large capacitances as compared with the capacitance of the filament connection. Such a Hertzian doublet is also of no practical use but is a good reference source for computing radiation effects not only of such a radiator but also of linear antennas. The same can also be done by means of an infinitesimally short current-carrying conductor, since for such a reference radiator, which is taken as a very thin current filament, we also have practically a fixed current distribution along the infinitesimally short length. For the Hertzian doublet, it is also the infinitesimally short filament length that determines the unobstructed-space radiation characteristic. For the doublet as well as the infinitesimal current filament radiator, we have the same radiation conditions. The radiation pattern around each

elementary filament radiator is circular with a maximum effect along the equatorial plane of the filament. The direction effects in any of the meridian planes (with the current filament as the pole axis) are sinusoidal functions (sin θ function) if θ denotes the angle that the direction makes with the filament axis. There can be no radiation in either direction along the elementary linear radiator, since we have sin 0 deg = 0 = sin 180 deg.

When determining the radiation characteristic of an antenna of finite length, we imagine the length as being made up of elementary radiators and add up all radiation effects with respect to the magnitude and the relative phase. This is discussed in detail in Secs. 102–105, 109, and 111.

When we have several linear radiators, we proceed exactly in the same way, by first dividing each individual antenna into infinitesimally short lengths and summing up (called "integrating") all the effects at a distant field point with respect to amplitude and phase. Doing this also for all other individual antennas and adding in proper relative phase relationship give the resultant radiation pattern due to all individual antennas. This assumes that the excitation current in each individual antenna and its relative phase are known. By changing the relative phases, or magnitudes, of exciting currents, the radiation pattern can be changed. This can also be done by altering the relative dispositions of the individual antennas.

There are also the effects of parasitic radiators. Such radiators may be metal towers, guy wires, or special conductors that are not directly excited by a source. Such radiators are then excited by mutual field effects due to a radiator that is driven by a source and will reradiate and, as such, affect the resultant radiation pattern. Such parasitic radiators may be used as reflectors in order to give pronounced radiations effects in a preferred direction.

In the radiation field, the measured field intensity is the combined effect of all elementary portions of an antenna or of several antennas. Owing to wave interference, we may find, in space, regions of no field intensity. Suppose we have a single isolated linear radiator that is 1 wavelength long. At the extreme ends as well as at the mid-point of the radiator, there is no current flow at all for a sinusoidal wave distribution. At a certain instant, a maximum positive current then exists 25 per cent from one free end and a maximum negative current flow 25 per cent from the other free end. Such a condition will exist during a half cycle of the exciting current; and during the half period, the extreme maxima and minima fluctuate from zero over an optimum value back to zero. We note that the magnetic field circulations about one half of the total radiator length must then be clockwise while they are counterclockwise for the remaining half radiator length. We have then, in effect, two collinear linear half-wave radiators joined together and excited in antiphase. We are interested mainly in the distant radiation field. At a certain field point of the distant region, we must then sum up the radiation effects of all elementary portions of the entire radiator length with respect to magnitude as well as relative phase. The relative phase

is partially due to the phase of antiphase differential antenna lengths and partially due to the path differences between the various antenna elements to the particular field point. Since antiphase current excitations exist along a linear radiator that is a full wavelength long, field points must exist where complete field cancellation occurs. There are also field points where only partial field cancellation occurs and points where increased resultant fields exist. As is shown in Sec. 105, whenever a linear radiator is longer than half of a wavelength of the operating wavelength λ* along the radiator, we shall, owing to wave interference, produce also secondary radiation lobes with regions of zero reception between lobes. Such effects can also be produced by metallic electromagnetic mirrors, dielectric as well as complex mirrors. Ordinary earth acts as a complex mirror, whether it is dry or wet soil, rock or ocean surface. Regions of zero field intensity and decreased and reinforced field intensity can then be caused even for radiators that are not longer than a half wavelength. This is treated in detail in Chap. VII and results from multiple-path wave propagation. In the simplest case, we have a direct ray and only one indirect ray arriving at a particular location in the distant radiation field. The indirect ray comes from the image antenna due to the particular mirror and brings along a field intensity that depends on the reflection coefficient of the particular electromagnetic mirror. This reflection coefficient is, at best, only of unity value and generally also has a phase angle that is not zero or 180 time degrees. There is also a phase angle due to the path difference between the direct and indirect ray. Hence, there must be places where the direct field may just cancel the indirect field or may do this only partially or in such a way that the net effect is a stronger field than expected from the free-field value. The free-field value is the field intensity that is due only to the direct radiation effect. It occurs in unobstructed radiation regions or at places where the effect of the indirect rays either is too weak or is screened off. For short waves corresponding to VHF and UHF excitations, a relatively small physical obstacle may act as a partial reflector to waves. Inasmuch as a wave front is the front of an advancing electromagnetic wave, it is the seat of field energy transfer. Hence, it is also the seat of energy and can be used as a radiator. Radiation is caused by diffraction from slots, around sharp wedges, from circular holes or other apertures when occupied by a portion of a wave front, which are then such sources. In other words, *space itself can be made to act as a radiator.* For instance, the open end of a wave guide can be used as such a radiating source.

* This wavelength along an antenna is about $\lambda = 0.95 \times 300/f$ meters, if f is in megacycles per second.

Radiators of this type may be called "wave-front" radiators. They are based on the fact that any cubic-meter space of the distant radiation region (with the front face of the cube in the wave front) contains an electric field energy of $\frac{1}{2}\kappa\mathcal{E}^2$ joules and a magnetic field energy of $\frac{1}{2}\mu H^2$ joules and releases a radiation energy in watts that is equal to the time rate of decrease of $\frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)$ through 1 sq m of the cubical space forming a wave front. The electric vector $\mathcal{E}$ volts per meter and the magnetic vector H amperes per meter then both lie in the plane wave front. They are normal to each other in the distant radiation field and in time phase. We have no medium dissipation ($\sigma = 0$) in this region.

4. Physical Interpretations of Mathematical Expressions Used in Radiation Phenomena. Often it is imperative to use electromagnetic equations and give additional derivations, in order to substantiate numerical applications as well as evaluations and to arrive at useful working formulas. *Mathematics is not used (and should not be used) to obscure facts but to augment them and to explain as well as project things that may escape in experimentation.*

To start with the very foundation underlying electromagnetic phenomena we have the *main Maxwellian equations*

$$\left.\begin{aligned}
\text{curl } H &= \underbrace{J}_{\substack{\text{Ampère's}\\ \text{conduction}\\ \text{current density}}} + \underbrace{\frac{\partial(\kappa\mathcal{E})}{\partial t}}_{\substack{\text{Maxwell's electric}\\ \text{displacement}\\ \text{current density}}} \quad \text{amp/m}^2 \\
\text{curl } \mathcal{E} &= \underbrace{-\frac{\partial(\mu H)}{\partial t}}_{\substack{\text{Faraday's induction}\\ \text{law (Magnetic displacement}\\ \text{current density)}}} \quad \text{volts/m}^2
\end{aligned}\right\} \begin{array}{l}\text{differential form}\\ \text{(curl relations).}\\ \text{for a \textit{dissipative}}\\ \text{medium for which}\\ \sigma \neq 0 \text{ mhos/m}\end{array} \qquad (1)$$

These relations simply tell that when we have magnetic field circulations, we must expect corresponding electric current flows and for electric field circulations we must expect corresponding magnetic current flows.

The curl H refers strictly to a current content, and the curl $\mathcal{E}$ to a voltage content *per unit area with zero area as the limit.* As shown on p. 151, curl H, that is the curl of H, can also be written as $\nabla \times H$ and represents the "cross" product of two vectors, since a cross ($\times$) is used between the differential operator ∇, which behaves like a vector, and the magnetic field vector H. Similarly some writers use $\nabla \times \mathcal{E}$ in place of curl $\mathcal{E}$, where $\mathcal{E}$ is the electric field vector in volts per meter while the associated magnetic field vector H is in amperes per meter. Hence, both $\mathcal{E}$ and H are gradients (voltage and current gradients). We have then a ready means of producing voltage and current variations, respectively, by means of variable $\mathcal{E}$ and H fields. For instance, if we have a voltage of $E = 1{,}000$ volts acting between two large insulated metal plates that are 0.1 m apart, the electric field between the plates will have a gradient of $\mathcal{E} = E/0.1 = 10^3/10^{-1} = 10{,}000$ volts/m.

Since $\kappa\mathcal{E}$ is the same as the electric displacement D in coulombs per square meter, *i.e.*, the electric flux density, we have the curl H also equal to $J + \partial D/\partial t$. Hence, the curl of the magnetic field intensity H is nothing but the sum of two current densities J and $\partial D/\partial t$, which for harmonic fields must be added in proper phase relationship (under 90 time degrees). We have then simply a current of $J + j\omega\kappa\mathcal{E}$ amperes across the same square-meter area.

At this point, it should be realized that the conduction current density J amperes per square meter is used to cover the joulean heat losses of a medium through which an electromagnetic wave passes. This happens only for the general wave medium that also exhibits a certain noticeable conductivity σ mhos per meter across square-meter parallel planes 1 m apart. We then have a current flow of $\sigma\mathcal{E} = J$ amperes across 1 sq m area, which is normal to the direction of the $\mathcal{E}$ vector, plus an electric displacement current $\partial(\kappa\mathcal{E})/\partial t$ amperes per square meter also along the $\mathcal{E}$ direction. Hence, the general medium of wave propagation is not only the seat of electric and magnetic fields but, according to the first of Eqs. (1), also the seat of the mentioned electric current flows. These flows are called "current densities," since we deal with a unit volume of the medium, *i.e.*, a cube of the medium with sides each 1 m in length. In the case as in Fig. 1, at a certain instant, we have the indicated orthogonal field association $\mathcal{E}$ and H with a polarity and magnitude as shown. Inasmuch as we also assume a certain measurable value for the specific conductivity σ, we have the case where an electromagnetic wave passes, for instance, through a wall of a building or penetrates a mountain or the like. The first of Eqs. (1) then tells us that a certain conduction current $J = \sigma\mathcal{E}$ amperes flows from the bottom of the cube to the upper square-meter shaded face. It also tells us that a displacement current of $\partial(\kappa\mathcal{E})/\partial t = j\omega\kappa\mathcal{E}$ amperes flows from the bottom face toward the shaded top face for the indicated polarity of $\mathcal{E}$. If each side of the cube were 3 m in length, both the bottom and the top faces would have an area of 9 sq m each and we should have nine times as much conduction current ($9J$ amperes) and also $9\partial(\kappa\mathcal{E})/\partial t$ amperes displacement current. The

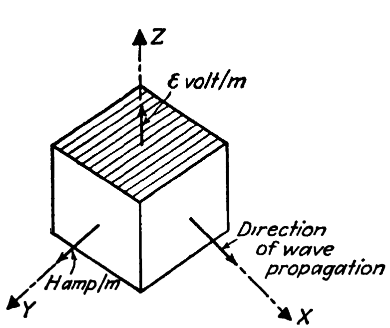


FIG. 1. One cubic meter of the general wave medium (κ farads/m, μ henrys/m, and σ mhos/m). Conditions for a plane electromagnetic wave. Current flow is along the $\mathcal{E}$ direction. Conduction as well as displacement currents flow normally to the shaded area. σ mhos/m means the conductivity between 1-sq-m parallel surfaces separated 1 m.

first of Eqs. (1) in the mks system of units expresses that the work required to carry a unit magnetic pole around a closed path is equal[1] to the total current I in amperes across the surface that has the particular path for its boundary. The Ampère-Maxwell law is expressed also by the closed-line integral $\oint_l H\,dl = I$ amperes, where dl is in meters and l denotes the periphery (closed boundary) in meters. For a circular path of radius r meters, we should then have $l = 6.28r$ meters and the area across which respective average current densities J and $\partial(\kappa\mathcal{E})/\partial t$ are operative is $3.14r^2$ square meters. The total current flow would, therefore, be $3.14[J + \partial(\kappa\mathcal{E})/\partial t]r^2$ amperes.

The orthodox statement of the Ampère-Maxwell law accounting for the total current I referring to moving a unit magnetic pole along a closed path etc., may appear somewhat indirect to a student just taking up electromagnetic propagation. Normally, we do not move a unit magnetic pole, we have also never isolated such a pole, but we have an H field. For instance, for a vertical radiator, we have horizontal circular H fields around the radiator in unobstructed space. In the case of wave propagation in the distant radiation region, we have the indicated H field of Fig. 1. Since the assumed cubical portion of the wave medium has the upper and lower 1-sq-m faces separated by 1 m, we have a dielectric constant of κ farads per meter because $\kappa\mathcal{E}$ is the electric flux density across the bottom and top surfaces of the cube. For harmonic excitations, we have the instantaneous associated field variations $\mathcal{E} = \mathcal{E}_0\epsilon^{j\omega t}$ and $H = H_0\epsilon^{j\omega t}$. They[2] are in time phase and space quadrature. We have for the maximum electric field energy a value of $\frac{1}{2}\kappa\mathcal{E}_0^2$ joules within the cubical medium. Similarly, we have for the maximum magnetic field energy within the cubic-meter medium the expression $\frac{1}{2}\mu H_0^2$ joules. The expression $\frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)$ is the combined instantaneous field energy, which oscillates according to the frequency of the exciting current of the radiator, causing the electromagnetic field effect in the cubical medium of Fig. 1. It is the time rate of decrease of $\frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)$ that accounts for the instantaneous electromagnetic field energy in watts escaping at the square-meter front face of the cubical medium, while the conduction current J amperes causes joulean heat losses within the cube. Hence, a medium used for electromagnetic wave propagation should have a vanishingly small specific conductivity σ. This is fortunately true for ordinary radiation space of the atmosphere. We then deal practically with only the displacement term of the first of Eqs. (1).

[1] In electromagnetic cgs units this law would read "equal to 4π times the current."

[2] In many cases in this text, $\mathcal{E}$ and H denote the maximum values and at times the effective values. But here we must distinguish between any instantaneous field value and its maximum value.

The second of Eqs. (1) satisfies Faraday's induction law for the induced electromotive force around a circuit through which the magnetic flux varies. The induced voltage in a closed circuit, according to the experimental Faraday law, is then equal to the *negative* time rate of change of the magnetic flux linking the closed circuit. This means that the relative motion of the magnetic field flux Φ in webers with respect to the closed path in which a voltage of e volts is induced is in a direction such as to change the total number of magnetic lines of force interlinked with the closed circuit, and the direction of the induced current is always such as to set up a magnetic field that *opposes* such a relative motion between circuit and field. The corresponding curl relation of (1) refers to an electric field intensity $\mathcal{E}$ which is a voltage e divided by a distance, since $\mathcal{E}$ is in volts per meter. We then have the closed-line integral $\oint_l \mathcal{E}\, dl = -\partial\Phi/\partial t = e$ volts, since l represents the length of the closed path in meters This path is linked by the total magnetic flux in webers We have $\mu H = B = \Phi/A$, where A denotes the area in square meters crossed by the total flux of Φ webers According to Ampère's law, in combination with Maxwell's displacement concept, we have also the integral form $\oint_l H\, dl = I$ amperes for the *total* current in amperes across the area bounded by the closed path l. According to Faraday's experimental induction law, we have similarly, $\oint_l \mathcal{E}\, dl = -\partial\Phi/\partial t$ for the so-called magnetic displacement flow (current) in volts across the area bounded by path l. Hence, if l meters encloses an area of 1 sq m in each case, the quantity I represents the total current *density* J, and Φ denotes the magnetic flux density $B = \mu H$, and respective line integrals read $\oint_l H\, dl = J$ total amperes per square meter and $\oint \mathcal{E}\, dl = -\partial(\mu H)/\partial t$ volts per square meter. This holds because $J_{\text{total}} = J + \partial(\kappa\mathcal{E})/\partial t$. Hence, in each case, we have either a vector H or a vector $\mathcal{E}$ taken along a closed contour which is l meters in length. We have, therefore, a circulation, a whirl, a rotation, or some sort of curl enclosing an area. For an H circulation, we have then simply a corresponding total *electric* current variation perpendicular to the area enclosed by the circulation. For an $\mathcal{E}$ circulation, we have a corresponding *magnetic* current variation across the enclosed area. For the limiting case for which the enclosed area approaches zero value, we have the differential forms expressed in (1). We have also the relation

$$\lim_{\substack{\Delta S \to 0 \\ \text{enclosed by } l}} \oint_l \mathcal{E} \cdot dl = (\text{curl } \mathcal{E}) \cdot \Delta S$$

where $\mathcal{E}$ acts always *along* the element dl and curl $\mathcal{E}$ acts *perpendicular to* the *area* ΔS and such that $\Delta S \to 0$. We have exactly the same relation for the magnetic H vector and $\lim \oint_l H \cdot dl = (\text{curl } H) \cdot \Delta S$. This is just an application of Stokes's theorem, expressing that the line integral of a vector taken about a *closed* path can be converted into a surface integral, extended over the surface enclosed by the path.

For a nondissipative medium ($\sigma = 0$ mhos/m), as is the case for customary atmosphere, for a closed contour of l meters surrounding an area S of square meters, we find

$$\underbrace{\int\int_S \text{curl}_n\, H \cdot dS}_{\text{surface integral}} = \underbrace{\oint_l H \cdot dl}_{\text{line integral}}$$

$$= \int\int_S \frac{\partial(\kappa\mathcal{E})}{\partial t} \cdot dS \qquad \textit{electric} \text{ displacement current } I_d \text{ amp}$$

$$I_d = j\omega k \int\int_S \mathcal{E}\, dS \text{ amp for a harmonic excitation}$$

integral form of curls

$$\int\int_S \text{curl}_n\, \mathcal{E} \cdot dS = \oint_l \mathcal{E} \cdot dl$$

$$= -\int\int_S \frac{\partial(\mu H)}{\partial t}\, dS \qquad \textit{magnetic} \text{ displacement current } I_m \text{ volts}$$

$$I_m = E_l = -j\omega\mu \int\int_S H\, dS \text{ volts}$$

$$\text{curl}_n\, F = \frac{\oint_l F\, dl}{S \to 0} \tag{2}$$

definition for the curl of a field vector F. The curl of F is normal to the area elements dS and $\int\int dS = S$ sq m is the area inside the closed boundary of l m

for a nondissipative medium ($\sigma = 0$)

since the conduction current $I = \sigma \int\int_S \mathcal{E}\, dS$ amperes vanishes for $\sigma = 0$ mhos/m. The circulation of H and $\mathcal{E}$ around the boundary of each element is $\text{curl}_n\, F \cdot dS$ for $F = H$ or $F = \mathcal{E}$, respectively, and the subscript n expresses that the curl of F is perpendicular (*normal*) to dS, just as the field vector F acts always *along* the line element dl of the closed contour of l meters. We have, therefore, two symmetrical relations, except for the minus sign for the curl $\mathcal{E}$ expression, and can call the negative time rate of the magnetic flux density $B = \mu H$, a *magnetic displacement* flow or a magnetic current[1] density in volts per square meter.

[1] The dimension volts per square meter for $\partial(\mu H)/\partial t$ is found as follows: We have

This can be realized just as much as an *electric displacement* current density $\partial(\kappa\mathcal{E})/\partial t$.

It is difficult to think of a *magnetic conduction* current density in volts per square meter. So far, we have been unsuccessful in separating the magnetic north pole from the magnetic south pole; *i.e.*, we cannot isolate magnetic charges as we can electric charges with respect to polarity. This, with present-day experimental knowledge, would exclude magnetic conduction as well as such convection currents. Nevertheless, we can introduce in the electromagnetic relations so-called "nonexperimental facts," in order also to secure for the dissipative medium [Eq (1)] symmetric expressions (by adding in the curl $\mathcal{E}$ relation also a magnetic conduction current density, say J_m volts per square meter) for both the curl $\mathcal{E}$ and the curl H and in some theoretical speculations make good use of such a hypothetical symmetry.

In the differential expressions (1) for the dissipative medium and the integral expressions (2) for a pure dielectric wave medium ($\sigma = 0$), both the electric $\mathcal{E}$ and the magnetic H fields appear. *They belong together;* and if only one field in a particular medium is restricted in magnitude, the electromagnetic field is likewise restricted. Suppose an arriving electromagnetic wave enters a medium that is impartial to an H field but decreases rapidly the magnitude of an $\mathcal{E}$ field. This happens when an advancing wave meets the surface of a conductor, since the electric lines of force tend to terminate normal to the surface. At very high and ultrahigh frequencies, the electric $\mathcal{E}$ fields then pierce such a medium hardly more than skin depth. As is shown on page 43, the $\mathcal{E}$ and H vectors then also exhibit a relative phase within this skin even though the magnitude of the H field is relatively large. Suppose we have a copper plate, for which we have about a specific conductivity as high as $\sigma = 5.8 \times 10^7$ mhos across parallel square-meter planes 1 m apart. If $\mathcal{E}$ stands now for the effective electric field, the conduction current density $J = \sigma\mathcal{E} = 5.8 \times 10^7\mathcal{E}$ amperes per square meter prevails. It would be difficult to measure the dielectric displacement current effect $J_d = j\omega\kappa\mathcal{E}$ amperes per square meter in the presence of the magnitude of J. Even if we assume that the dielectric constant of copper is the same as for a vacuum, namely, $\kappa_0 = 8.854 \times 10^{-12}$ farad/m, we should have even at 1,000 Mc/sec only a displacement current $J_d = j6.28 \times 1{,}000 \times 10^6 \times 8.854 \times 10^{-12}\mathcal{E} = j0.0556\mathcal{E}$ amperes per square meter compared with $5.8 \times 10^7\mathcal{E}$ amperes per square meter.

μ in henrys per meter; H in amperes per meter, t in seconds, hence the dimension of the curl of $\mathcal{E}$ is $|(\text{henrys/m}) \cdot (\text{amp/m})/\text{sec}|$. But from $e^{(\text{volts})} = L^{(\text{henrys})} \dfrac{di^{(\text{amp})}}{dt^{(\text{sec})}}$ we find $|\text{henry}| = (\text{volt} \cdot \text{sec})/\text{amp}$ and the dimension of curl $\mathcal{E}$ is

$$\left|\frac{(\text{volt} \cdot \text{sec/amp})/\text{m} \cdot \text{amp/m}}{\text{sec}}\right| = |\text{volts/m}^2|$$

Such mediums will, therefore, practically reduce the $\mathcal{E}$ field to a vanishingly small value, while a magnetic H field by itself experiences hardly any decrease in its value. But since both relations in (1), namely, the curl H and the curl $\mathcal{E}$ relation, contain also the other field vector, namely, both $\mathcal{E}$ and H, respectively, a reduction of any one of these vectors, whether $\mathcal{E}$ or H, means also a reduction of the transferred electromagnetic field energy. This can be seen when we realize that the ratio $\mathcal{E}/H$ is a measure for the characteristic impedance Z_0 in a medium toward wave propagation. In a general medium, this impedance is $Z_0 = \sqrt{\mu/\kappa_e}$ ohms for the *equivalent* dielectric constant

$$\kappa_e = \kappa - j\frac{\sigma}{\omega} \qquad \text{farads/m} \tag{3}$$

if κ denotes the *actual* dielectric constant of the medium in farads per meter, σ is again in mhos per meter, and $\omega/6.28$ is the frequency in cycles per second. For plane waves in the distant radiation field, we have $Z_0 = 377$ ohms. The characteristic wave impedance is then *pure ohmic*, and the effective values $\mathcal{E}_e$ and H_e of $\mathcal{E}$ and H are in time phase and space quadrature within the advancing wave front. The electromagnetic field energy is then $\mathcal{E}_e H_e$ watts per square meter streaming forward with the plane-wave-front portion. If such a wave front arrives right next to a conducting medium (whether a good conductor or a semiconductor), the $\mathcal{E}$ vector will be greatly diminished and the ratio $\mathcal{E}/H = Z_0 = \sqrt{\mu/\kappa_e}$ will be complex (an impedance), with a corresponding time phase angle. It must also be realized that the angle of wave incidence and the type of polarization of the arriving[1] wave has a great deal to do with the entrance wave impedance Z_0 into such a medium. It should also be understood that a metal screen with apertures acts somewhat differently at very high and ultrahigh frequencies from a solid screen. This is especially true for slot apertures with a properly polarized wave arriving on one side of a slot. Such openings coinciding with an arriving wave front may reradiate at the other side of the screen and may act as a desirable wave-front radiator based on Huygens' wavelet principle. Such an opening may also cause diffraction effects behind the screen so that wave propagation into the geometrical shadow region is possible as is shown in Chap. VIII.

Equations (1) are all that there is to the Maxwellian theory, since the $\mathcal{E}$ and H vectors as well as the current densities are interrelated. It is

[1] It is customary in radio engineering to define the polarization of an electromagnetic wave by means of the lines of the $\mathcal{E}$ field. By vertical polarization is meant that the $\mathcal{E}$ vector of the wave is vertical and for horizontally polarized waves the $\mathcal{E}$ vector is horizontal. It is understood that the $\mathcal{E}$ vector is in one case parallel with the plane of wave incidence, and in the other case $\mathcal{E}$ is normal to this plane. This is followed up on pp. 549–552, 785–788, 796–824, and 882–885.

true that the mks system of units no longer directly involves in these expressions the velocity of propagation. Inasmuch as $\mathcal{E}$, H, and respective current densities have definite directions and stand for the resultant vectors, it is customary to deal with component vectors such as $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$ along the X, Y, and Z directions of a Cartesian coordinate system, where F stands for $\mathcal{E}$, H, or any other vector of concern.

We can also split the resultant general field vector F into cylindrical or into spherical coordinates as well as into curvilinear coordinates as is shown in detail in Secs. 31 and 43. The difference between Cartesian and curvilinear coordinates is that a particular space point is located in the former by means of rectangular coordinates; the equations x = constant, y = constant, z = constant each determine a plane. The intersection of two of these planes is then a straight line that intersects the third plane at the particular space point. In the curvilinear coordinate system, we do not use planes but *curved* surfaces to fix the particular space point.

The curvilinear coordinate system has the advantage (see Sec. 31) that we can express electromagnetic field relations generally and readily transform them into Cartesian, cylindrical, or spherical coordinates as well as other coordinates, respectively. Therefore, if we know a certain expression in curvilinear coordinates, we do not have to go to the inconvenience of deriving the corresponding expression in spherical coordinates, for instance.

Space points may be located in the medium in which wave propagation occurs or mediums that act as radiators, whether customary antennas using a conductor, an array of such conductors, or space radiators such as suitable openings in a screen, which may have a slot shape or may be circular holes.

It is necessary to use one of the three fundamental coordinate systems because the shape of the particular body to be mathematically investigated must be satisfied. For ordinary linear antennas, we can use the Cartesian system, since we deal essentially only with the length dimension of the antenna. If, however, we deal with cylindrical radiators whose circular cross section is not negligible in comparison with the length dimension, cylinder coordinates are in order. If we deal with the space radiation characteristic, even of a thin linear radiator, spherical coordinates are used so that we can find the fields in the various meridian planes.

Referring to such vectors as the electric field intensity $\mathcal{E}$, the magnetic field intensity H, the corresponding electric and magnetic current flows across a unit area and the like, we must realize that their values vary with respect to time as well as with respect to position in space. Ignoring for the present the time function $\epsilon^{j\omega t}$ for harmonic excitations, it must be remembered that the above quantities are functions of position. The

electric field intensity $\mathcal{E}$ has not only an absolute value in volts per meter at a certain space point but also a definite direction. The same is also true for the magnetic field intensity H with a magnitude in amperes per meter and also a definite orientation in space and with respect to the $\mathcal{E}$ orientation. This, too, is true for the real value of the complex Poynting vector, whose magnitude in watts per square meter gives the amount of total field power transmitted. Since it is a vector, it must also give the direction in which the power flow occurs. For instance, in the distant radiation region, the effective electric field $\mathcal{E}_e$ in volts per meter may be along the vertical $+Z$ direction (vertical polarization), and the associated effective magnetic field H_e in amperes per meter along the horizontal $+Y$ direction. The Poynting power flow $p = \mathcal{E}_e H_e$ watts per meter is then along the horizontal $+X$ direction (right-hand screw motion). The quantity p then represents the absolute amount of the Poynting vector $\mathbf{p}$.

Theory shows that the radiation field is roughly reached when we are about 1.5 wavelengths away from the radiator. This is, indeed, a very short distance at ultrahigh frequencies and especially so when we deal with microwaves. Suppose we have a 100-Mc wave excitation; then $\lambda = 300/100 = 3$ m and $1.5\lambda = 4.5$ m, which is the same as $3.281 \times 4.5 = 14.5$ ft. For a 3,000-Mc wave excitation, we have then $\lambda = 300/3{,}000 = 0.1$ m, $1.5\lambda = 0.15$ m $= 39.37 \times 0.15 = 5.9$ in. Hence, in either case and generally for the entire VHF and UHF range, we need not go very far from a transmitting aerial to be essentially in the radiation region. This text repeatedly mentions the *distant* radiation region or field because field intensity formulas are derived with the assumption that the distance to a space point, for the inverse distance factor, is the same from all points of a radiator. This is probably essentially true for space points at least 10 wavelengths away from the sender aerial. Hence, measurements, if possible, should not be carried on much closer than this minimum distance. Moreover, it is important that we be far enough away to have at least for a surface of 1 sq m a plane wave front in effect, *i.e.*, that the area of the spherical surface about the symmetry center of a sender antenna or of an antenna array is practically flat for surfaces of at least 1 sq m. This may require distances, *i.e.*, a radius of the sphere, of more than 10λ in many cases.

The example showing that in the distant radiation region both associated field vectors $\mathcal{E}$ and H lie within the flat wave front and are normal to the Poynting vector, and space quadrature and time phase hold only for such a region. Suppose we take a region close to the boundary of 1.5λ from the radiation symmetry center. Assume that we use a small loop for determining the received field intensity. We know that when still

in the true radiation field (which is for 1.5λ or more distance from the radiator), the plane of the loop pickup points toward the radiation center for maximum reception effect. But as we move closer and closer toward the radiation center, we note that the loop must be turned more and more normal with respect to the direction of radiation to reception centers.[1] The reason is that we move more and more into the region of the induction field and finally meet a condition where the magnetic H lines of force of the near field (next to the radiator) must cross normal to the plane of the pickup loop in order to cause maximum reception effect.

5. Remarks on Scalars, Vectors, Gradients, Curls, Divergence with Respect to Point Functions. Inasmuch as space is three-dimensional, at any space point, a function such as $\mathcal{E}$ for the electric field intensity or H for the associated field intensity depends on the position of this point[2] which is given by the xyz coordinates in the Cartesian XYZ system. It is also fixed by the $r\varphi z$ coordinates in the cylindrical system, where r lies in the XY plane or parallel thereto given by $r = \sqrt{x^2 + y^2}$, since $x = r \cos \varphi$, $y = r \sin \varphi$, and $z = z$. It is also fixed by the spherical $r\varphi\theta$ coordinates, where r points from the origin of an XYZ system anywhere into space to a point $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, and $z = r \cos \theta$, since $r = \sqrt{x^2 + y^2 + z^2}$.

We must distinguish between scalars and vectors. Magnitudes such as the electric charge density ρ in coulombs per cubic meter, potential V in volts, temperature in degrees centigrade, mass in kilograms, work and energy (force times distance) in joules or newton · meters are known as "scalar magnitudes," called briefly "scalars." They are also known as "invariant" scalars, since they are completely characterized by magnitude *alone*, which is a number with an appropriate dimension, such as 5 coulombs/m³, where 5 is the number and coulombs/m³ is the dimension in the mks system of units. When a quantity has both magnitude and direction, it is a vector magnitude, called briefly a "vector." A current, for instance, is a vector, since it has a magnitude expressed in so many amperes and besides a definite direction of flow. The electric field intensity $\mathcal{E}$ is a vector, just as is the magnetic field intensity H, since we have magnitudes of so many volts per meter and so many amperes per

[1] By such centers is meant, for instance, the center of the receiving loop, the mid-point of a linear antenna, the mid-point of number 3 antenna when five equal linear antennas numbered 1, 2, 3, 4, 5 are either collinearly lined up and equally spaced along the common axis or form an equal-spaced parallel array in the order 1, 2, 3, 4, 5. If we have a mosaic of like antennas (usually dipoles, half-wave radiators in a plane), the cross point of the diagonals is taken as the symmetry center.

[2] For details consult Figs. 108, 110–113, 123–129, and 131–133

meter, respectively, and each quantity has its particular space direction. For the complete characterization of vectors, three quantities must be specified, which may be the three projections of the resultant directed magnitude along the chosen coordinate directions. The three quantities may also be made up of two direction cosines and a third number expressing magnitude only. It is also possible that the direction as well as the magnitude may be expressed by means of three numbers. Even though every vector generally has three scalar components, however, this does not mean at all that any three scalars always result in a vector quantity when combined. Unless three scalars can be changed like coordinates defining a point in space, we cannot call these scalars components of a vector.

Scalar quantities such as electric charges, vector quantities such as field intensities, and current densities can be distributed in space. Surfaces including only scalars or vectors, respectively, of fixed absolute value are called "contour surfaces" or "level surfaces." Sometimes in case of field intensities, they are surfaces of fixed decibels with respect to a reference value. Thus if we compute solid field patterns (three-dimensional patterns) from formulas, we can draw space surfaces for electric fields of 500, 400, 300 millivolts/m, etc. In the same way, we can also draw equipotential surfaces and equipotential lines if we use only conditions in a plane, such as a horizontal plane or a meridian plane, with a radiator along the vertical pole axis located at the center of a sphere.

A *gradient* is a vector function even though the point function concerned may be a scalar. A gradient expresses the rate of increase or decrease of a point function or of a curve for a two-dimensional system or of a surface for a three-dimensional system that represents the point function. The gradient is then a vector function whose rectangular components are derivatives of xyz of the point function, expressing the rate at which the function varies from point to point in space. Assume an equipotential surface of potential of V_1 volts and a neighboring equipotential surface of potential V_2 volts; then a vector $\partial V/\partial n$ along the normal n of these equipotential surfaces is called the "gradient of V." For the sake of briefness, the unit vector with which this derivative is to be multiplied is omitted. The gradient of V, *i.e.*, grad V, then gives the *maximum* rate of change of potential V between V_1 and V_2 surfaces, where $V_1 - V_2$ is taken small enough so that both equipotential surfaces may be taken as normal to the path of length ∂n along which the gradient holds. Inasmuch as $\partial V/\partial n$ is now a vector quantity, we find the component $(\partial V/\partial n) \cos \theta$ along any direction θ with the normal n to equipoint function surfaces. Hence, if the direction n makes the angles α, β, and γ, respectively, with the XYZ directions of a Cartesian system,

we have the Cartesian gradient[1] components $\partial V/\partial x = (\partial V/\partial n) \cos \alpha$, $\partial V/\partial y = (\partial V/\partial n) \cos \beta$, and $\partial V/\partial z = (\partial V/\partial n) \cos \gamma$.

In Secs. 23 and 24, pages 104–111, relations are given for the divergence and the curl of a vector. If we use the electric flux density, $D = \kappa\mathcal{E}$ as a vector, where κ is the dielectric constant in farads per meter and $\mathcal{E}$ has an absolute value of so many volts per meter as well as direction, *i.e.*, orientation in space instead of arbitrary distribution. If D is the magnitude in coulombs per square meter of the vector $\underset{\cdot}{D}$, we find that the divergence of $\underset{\cdot}{D}$ is a *scalar* while the curl of $\underset{\cdot}{D}$ is a *vector* given by

$$\left.\begin{aligned} \operatorname{div} \underset{\cdot}{D} &= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{a scalar} \\ \operatorname{curl} \underset{\cdot}{D} &= i\left(\frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial z}\right) + j\left(\frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x}\right) \\ &\qquad + k\left(\frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y}\right) \qquad \text{a vector} \end{aligned}\right\} \tag{4}$$

where D_x, D_y, and D_z are absolute amounts of vector $\underset{\cdot}{D} = \kappa\underset{\cdot}{\mathcal{E}}$ along the XYZ directions and i, j, and k again are line vectors each of unit length along the $+X$, $+Y$, and $+Z$ directions. The same relation holds if $D = \kappa\mathcal{E}$ coulombs per square meter is replaced by the magnetic flux density $B = \mu H$ webers per square meter with the exception that div $\underset{\cdot}{D}$, *i.e.*, the divergence of the electric flux density, is equal to the electric charge density ρ coulombs per cubic meter according to the Gauss *theorem*. The divergence of the magnetic flux density vanishes, *i.e.*, div $\underset{\cdot}{B} = 0$, since magnetic lines of force are closed and just as many magnetic lines of force enter any cubic meter of volume[2] as leave it.

6. Physical Meaning of the Divergence of a Vector and the Curl of a Vector Quantity. In order to get a physical feeling of what is meant by *divergence* and a *curl* of a vector, let us assume that we have an electric charge density of ρ coulombs per cubic meter available in a medium.

[1] The resultant vector is

$$\operatorname{grad} V = i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}$$

where V is a scalar, grad V is a vector, and $i(\partial V/\partial x)$, $j(\partial V/\partial y)$, $k(\partial V/\partial z)$ are component vectors, since they are multiplied by the unit vectors i, j, and k along the respective X, Y, and Z directions. Hence,

$$\frac{\partial V}{\partial n} = \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2}$$

[2] As shown on p. 197, div D can also be written as the dot product, so that we have $\nabla \cdot D = \rho$ where, as in most of the text, D and B are written without a dot below, since it is understood that they are directed quantities. We have then also $\nabla \cdot B = 0$. We have $\nabla \cdot D = \rho = 0$ when there are *no* sources or sinks in the field. The operator ∇ is the Hamiltonian nabla.

Let us also assume that owing to a field force, certain velocities $\mathbf{v}$ are imparted to the electric charges so that they can move away from their arbitrary space distribution. The product $\rho\mathbf{v}$ is then the momentum per unit volume (1 cu m); and if v is in meters per second along the $+X$ direction, this would constitute a convection current of ρv amperes per square meter across a YZ plane. Figure 2 shows what is meant by div $(\rho\mathbf{v})$ and curl $\mathbf{v}$, since the divergence of $\rho\mathbf{v}$ measures the rate at which electric charges leave a space region per unit volume, owing to the vectorial quantity $\mathbf{v}$. When we think in terms of a space point, this also holds true, except that the charge which escapes from the space point or

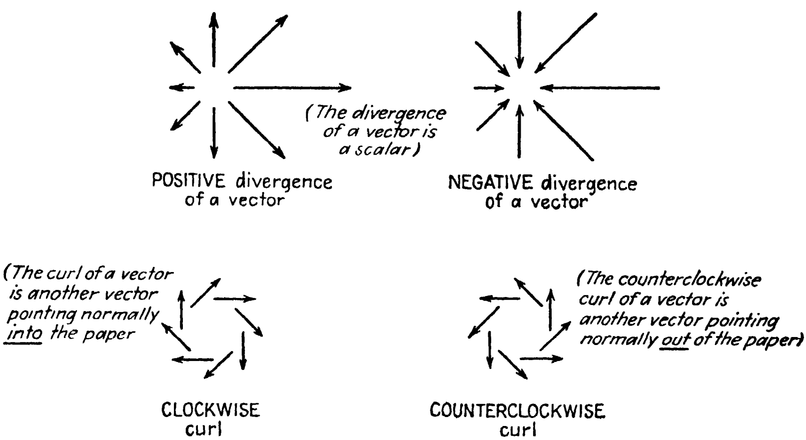


FIG. 2. The divergence and the curl of a vector.

enters it is then only a fractional portion of the value of ρ, holding for a volume as large as 1 cu m. A space point satisfies the requirement also that the surface enclosing the volume should approach zero value. The curl of the velocity vector $\mathbf{v}$, on the other hand, is a measure of a rotation, *i.e.*, a vortex motion. For this reason, it is a vector normal to an area closed by a path that can be due to a clockwise or a counterclockwise rotation. For a rigorous mathematical expression for a divergence of a vector, we deal with a space point and therefore with an enclosed volume that approaches zero value. In the same way, the curl refers strictly to a closed path whose enclosed area likewise becomes infinitely small so that the rotation per unit area causes, by virtue of proper orientation of the vortex, maximum curl effect. For the case of a divergence, we no longer deal with a vector, because it is an amount of something that originates or diverges from a unit volume, since the operation $\nabla\cdot$ on a vector refers to the net outward flux of the particular vector per unit volume. As Fig. 2 shows, we can have a positive as well as a negative divergence,

since an amount of something can either escape or enter per second, a unit volume. In one case, we have a source and in the other case a sink (an absorber) in the region under consideration. If there are no sources or sinks in a region, then the total outward flux through any closed surface must vanish and the divergence is zero. But if div $\neq 0$, the divergence gives the rate at which something is being supplied to a region or removed from it and gives a measure of a point sink or a point source, respectively.

7. Application of the Divergence and the Curl Concept of a Vector. If the magnetic field H in amperes per meter of an electromagnetic field represents the vector quantity, according to Eqs. (1), we have curl $H = J + j\omega\kappa\mathcal{E}$ amperes per square meter. This is a geometric addition of two current densities with a current flow normal across the plane of the H vortex from top to the bottom for a clockwise curl H and from the bottom to the top for a counterclockwise curl H. This is the right-hand screw rule. In a similar way, we may have curls of the electric field $\mathcal{E}$ that are either clockwise or counterclockwise and yield a magnetic displacement current density in volts per square meter, also known as the "negative time rate" of the magnetic flux density $B = \mu H$, where B is in webers per square meter.

Since we have regions with distributed H and $\mathcal{E}$ point functions, we must also have regions of curl H and curl $\mathcal{E}$ distributions. If throughout a particular region div $F = 0$ or curl $F = 0$, where F denotes $\mathcal{E}$ or H, we have a *solenoidal* vector distribution, also called a "lamellar distribution." If div $F = 0$, it is also possible to derive F from a vector potential function Λ with coordinates given by $F =$ curl Λ. It can be shown that a vector quantity is determined by its divergence and curl. Also an arbitrary vector may be expressed in terms of a solenoidal vector and a potential vector (page 219 and Sec. 239 of the Appendix.)

The case of div $B =$ div $(\mu H) = 0$ is *always* true, but div $D =$ div $(\kappa\mathcal{E}) = 0$ is true *only* in *free* space, where there are no charges. It is also true that in the interior of a conducting medium, however, generally we have div $D = \rho$, since it requires solenoidal or sourceless conditions for a divergence to vanish. Hence, for the general case, the divergence of the electric flux density is equal to the volume density ρ in coulombs per cubic meter. This can also be expressed by $\int_S D\,dS = \rho\tau$, where S denotes any surface that completely surrounds τ cubic meters of space with a volume density of ρ coulombs per cubic meter. These relations satisfy the Gauss theorem that the electric flux from a point charge is equal to the magnitude of the point charge; *i.e.*, for ρ coulombs enclosed in 1 cu m of space, the electric flux diverging from this space

is numerically equal to ρ. In the same way, the relation div $B = 0$, which always holds, can be expressed by $\int_S B\,dS = 0$, where S again denotes any closed surface in square meters.

For the divergence of the cross product of the electric and magnetic field vectors in a general medium, we find

$$\text{div}\,(\mathcal{E} \times H) = -\frac{\partial[\frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)]}{\partial t} - \sigma\mathcal{E}^2 \qquad (5)$$

where $-\sigma\mathcal{E}^2$ for the conductivity σ in mhos per meter and $\mathcal{E}$ in volts per meter gives the joulean heat loss in watts in 1 cu m of the general medium (rate at which energy is used up by a leaky dielectric medium). We note that the divergence of the *electromagnetic* flux is equal to the negative time rate at which the *electromagnetic* density increases within 1 cu m and the heat loss in 1 cu m.

On pages 108–111, it is shown that the surface integral, taken over a closed surface S, can be made to express what the divergence of a quantity does for a space completely enclosed by S (Gauss theorem). Similarly, a line integral taken around a closed path l, enclosing an area S (by definition an area approaching zero value in the limit), can be made to express what a curl of a vector does, since according to (2), we have $\int\int_S \text{curl}\,F\,dS = \oint_l F\,dl$ for any field vector F (Stokes's theorem). This also holds for a dissipative wave medium ($\sigma \neq 0$). A conduction current must then be added in Eq. (2) in the $\int\int_S \text{curl}_n\,H\,dS$ relation.

For a line integral, we have a force (like $\mathcal{E}$ volts per meter or H amperes per meter) multiplied by a line element dl, which is expressed in meters even though vanishingly small. This product means force times distance, *i.e.*, work performed by a force in moving something through a distance of dl meters. Hence, for $\int_l H\,dl$, where the total path length is closed, we have the result Hl amperes, since the unit becomes (amperes per meter) meters;[1] and for $\int_l \mathcal{E}\,dl$, if l is an open path, we find $\mathcal{E}l$ volts. We speak then of an *open*-line integral, which would refer to a potential in the latter case. For the result of Hl, we may call it strictly ampere-path, which is a magnetomotive force (work

[1] For line integrals, the vector $\mathcal{E}$ or H, is a tangential force along the path l; otherwise if $\mathcal{E}$ and H make an angle θ with the line element dl, we have $\int_l \mathcal{E} \cos\theta\,dl$ and $\int_l H \cos\theta\,dl$, respectively.

required to carry a unit magnetic pole along the path of l meters). However, we are concerned with relations that mean the same as the curl or cross-product relations, *i.e.*, curl $H = \nabla \times H$ and curl $\mathcal{E} = \nabla \times \mathcal{E}$, respectively, for which the path l must be a whirl enclosing an area (a *closed* path), and relations $\oint_l H \cdot dl = \int\int_S \text{curl } H \cdot dS$ and $\oint_l \mathcal{E} \cdot dl = \int\int_S \text{curl } \mathcal{E} \cdot dS$ hold. We then deal with a circulation for which the enclosed area approaches zero value in the limit (defines a curl). Suppose we have a magnetic field intensity of H amperes per meter acting around a very small circular loop of l meters in length, say that of the circumference of a fine copper wire of radius r expressed in meters. Then the closed-line integral of the H force becomes $\oint_l H \cdot dl = 2\pi r H$ ampere turns circulation, since $l = 2\pi r$ meters for one complete circulation. The enclosed area of πr^2 square meters is not unit area or of vanishing magnitude. We have, however, $\int\int_S \text{curl}_n H \cdot dS = 2\pi r H = I$ amperes, where I denotes the total conduction current in the wire of circumference $2\pi r$ causing the tangential H circulation. The displacement current in the copper conductor is negligibly small. The quantity $\left[\oint_l H\, dl\right]/S = 2\pi r H/S = 2\pi r H/(\pi r^2) = 2H/r$ amperes per square meter represents the H circulation per unit area. For the actual area πr^2 approaching 0 yields the curl of H.

Figure 3 shows a curl H with Cartesian, cylindrical, and spherical components in the case of harmonic wave excitation. The expressions given are the results of the solutions shown in Secs. 23 and 41. In a similar way, we can show the curl $\mathcal{E}$ components in respective coordinate systems. In this case, the factor k, holding for the general medium, with μ henrys per meter, κ farads per meter as well as σ mhos per meter, is no longer the complex quantity $k = \sigma + j\omega\kappa$ of Fig. 3 but a negative imaginary quantity $k = -j\mu\omega$, where $\omega/6.28$ is again the frequency in cycles per second. The *negative* time rate of the magnetic flux density $B = \mu H$ in the curl $\mathcal{E}$ relation of Eq. (1) accounts for the minus sign. We find, according to Eq. (1) of Sec. 4, when operated by the time function $\epsilon^{j\omega t}$, the above factor $k = -j\mu\omega$ and

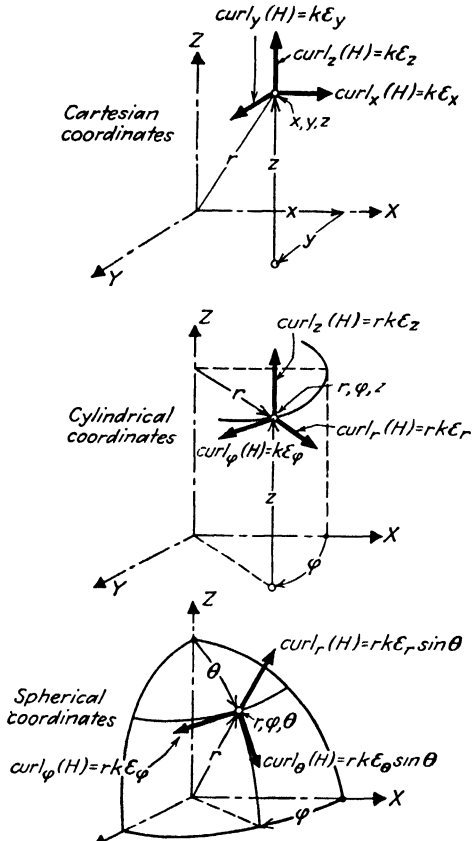
$$\text{curl}_x \mathcal{E} = kH_x; \qquad \text{curl}_y \mathcal{E} = kH_y; \qquad \text{curl}_z \mathcal{E} = kH_z$$
$$\text{for } H = \sqrt{H_x^2 + H_y^2 + H_z^2} \qquad (6)$$

for the Cartesian components. It is necessary to show only the above $\text{curl}_x \mathcal{E}$, $\text{curl}_y \mathcal{E}$, and $\text{curl}_z \mathcal{E}$ components in Fig. 3 in place of the corresponding $\text{curl}_x H$, $\text{curl}_y H$, and $\text{curl}_z H$ components. For cylindrical com-

ponents from solutions 135 and 136 of Sec. 31, we find

$$\mathrm{curl}_r\, \mathcal{E} = rkH_r; \quad \mathrm{curl}_\varphi\, \mathcal{E} = kH_\varphi; \quad \mathrm{curl}_z\, \mathcal{E} = rkH_z; \quad k = -j\mu\omega \qquad (6a)$$

which can again be shown in Fig. 3 in the cylindrical $r\varphi z$ system in place



All curls are in amp/m²
$k = \sigma + j\omega\kappa$; σ mhos/m;
κ farads/m; $\omega/6.28$ in cycles/sec
$r = \sqrt{x^2 + y^2 + z^2}$;
$|\mathcal{E}| = \sqrt{|\mathcal{E}_x|^2 + |\mathcal{E}_y|^2 + |\mathcal{E}_z|^2}$.
$\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$ vectors are along the $\mathrm{curl}_x\,(H)$, $\mathrm{curl}_y\,(H)$ and $\mathrm{curl}_z\,(H)$ directions

$\mathcal{E}_r$, $\mathcal{E}_\varphi$, and $\mathcal{E}_z$ are along $\mathrm{curl}_r\,(H)$, $\mathrm{curl}_\varphi\,(H)$, and $\mathrm{curl}_z\,(H)$ directions
Laplace transformation:
$x = r\cos\varphi$; $y = r\sin\varphi$; $z = z$
$r = \sqrt{x^2 + y^2}$

$\mathcal{E}_r$, $\mathcal{E}_\varphi$, and $\mathcal{E}_\theta$ are along the $\mathrm{curl}_r\,(H)$, $\mathrm{curl}_\varphi\,(H)$, and $\mathrm{curl}_\theta\,(H)$ directions
Laplace transformation:
$x = r\cos\varphi\sin\theta$; $y = y\sin\varphi\sin\theta$
$z = r\cos\theta$; $r = \sqrt{x^2 + y^2 + z^2}$

FIG. 3. The components of curl H in the Cartesian, cylindrical, and spherical coordinate systems.

of the $\mathrm{curl}_r\, H$, $\mathrm{curl}_\varphi\, H$, and $\mathrm{curl}_z\, H$ vectors. For spherical components from solutions 135 and 136 of Sec. 31, we find

$$\mathrm{curl}_r\, \mathcal{E} = rkH_r \sin\theta; \quad \mathrm{curl}_\varphi\, \mathcal{E} = rkH_\varphi; \quad \mathrm{curl}_\theta\, \mathcal{E} = rkH_\theta \sin\theta; \quad k = -j\mu\omega \qquad (6b)$$

where in Fig. 3 we can again substitute those values in place of the corresponding $\mathrm{curl}_r\, H$, $\mathrm{curl}_\varphi\, H$, and $\mathrm{curl}_\theta\, H$ vectors of the $r\varphi\theta$ system. Then instead of the $\mathcal{E}$ vectors the H vectors with their components are also

along the same directions as the respective and corresponding curl components.

8. Application of the Components of a Curl. A knowledge of the components of a curl is of advantage for some applications. For instance, from the general curl relations (1), we note that the curl of the magnetic field intensity H in amperes per meter actually refers to a conduction and displacement current density. The conduction current density $J = \sigma\mathcal{E}$ amperes per square meter does not play an important part in customary atmospheric wave propagation, since $\sigma = 0$ mhos/m for air. If it does play a part, the dissipation of field energy is the joulean heat loss $\sigma\mathcal{E}^2$ watts per cubic meter, if $\mathcal{E}$ volts per meter denotes the average effective electric field value as the wave passes through 1 m distance.

The curl H relation of Eq. (1) refers to the resultant space value. Figure 3 gives the Cartesian components $\text{curl}_x H$, $\text{curl}_y H$, and $\text{curl}_z H$, corresponding to the electric field components $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$. Such field components are known, since a certain transmitter antenna initiates a particular type of polarization.

Vertically polarized waves are produced, for instance, with a linear radiator that is vertical, just as a linear horizontal antenna produces horizontally polarized waves. Other methods of doing this are described elsewhere in this text.

In Fig. 4, 1 cu m of a general medium (σ mhos per meter also present) is shown in positions (a), (b), and (b'). The source for vertical polarization is a linear vertical radiator, and respective medium positions are shown so that they are at exactly the same distance of r meters from the radiation center of the source.

First, let us investigate position (a) for which an arriving electromagnetic wave ($\mathcal{E}$, H) arrives "head-on" and pierces the cubic meter in the square-meter back face of the medium "squarely." The $\mathcal{E}$ vector is vertical, *i.e.*, also along the Z axis of the cubical medium of position (a). We are concerned with the $\mathcal{E}$ vector, since for plane waves in an unobstructed medium, the magnetic H vector is along the Y axis because the wave front that advances is normal to the positive X direction in the true radiation region.

For such a condition, we have the $\text{curl}_x H$ component only in the Cartesian coordinate system of Fig. 4 and position (a), since $\mathcal{E}_x = \mathcal{E}_y = 0$ and $\mathcal{E} = \mathcal{E}_z$ volts per meter. Hence, our general curl H relation of Eqs. (1) reduces to

$$\text{curl}_z H = (\sigma + j\omega\kappa)\mathcal{E}_z = \underbrace{\underbrace{\sigma\mathcal{E}_z}_{\text{conduction}} + \underbrace{j\omega\kappa\mathcal{E}_z}_{\text{displacement}}}_{\text{current density}} \quad \text{amp/m}^2$$

From this we see that the total current density becomes

$$J_{\text{total}} = \mathcal{E}_z \sqrt{\sigma^2 + (\omega\kappa)^2}\,\underline{/\varphi} \qquad \text{amp/m}^2 \tag{7}$$

with a phase angle φ of $\tan^{-1}(\omega\kappa/\sigma)$ degrees, if J and $\mathcal{E}$ are taken as effective values. Let us see what this means numerically. Assume that the cubic meter is a portion of a rock. Such material has at average temperature, for example, a conductivity σ as low as 10^{-6} mho/m and a dielectric constant κ of $8\kappa_0$ farads per meter, where κ_0 is the value for

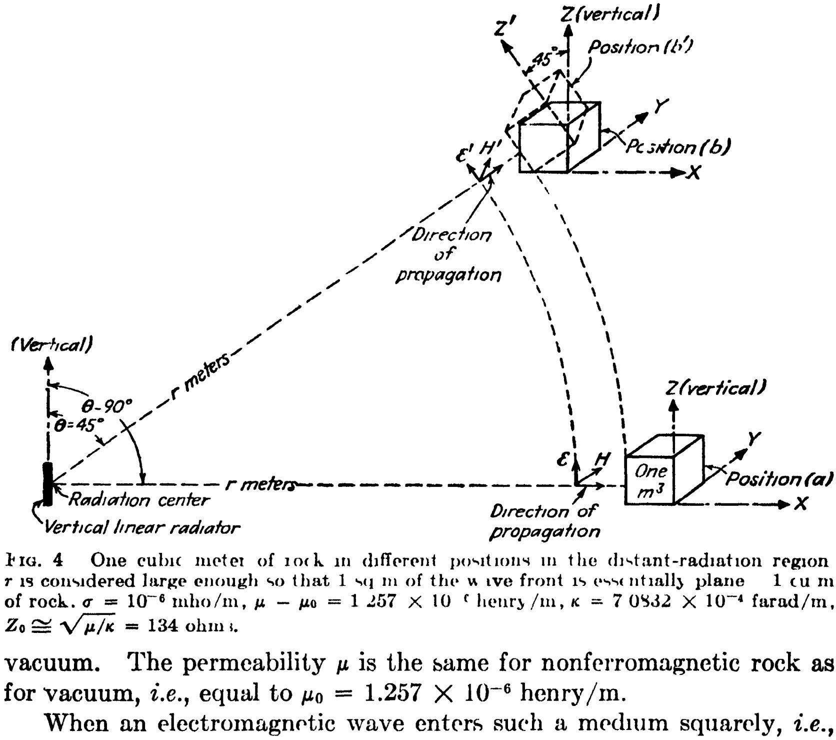


FIG. 4 One cubic meter of rock in different positions in the distant-radiation region r is considered large enough so that 1 sq m of the wave front is essentially plane 1 cu m of rock. $\sigma = 10^{-6}$ mho/m, $\mu - \mu_0 = 1.257 \times 10^{-6}$ henry/m, $\kappa = 7.0832 \times 10^{-4}$ farad/m, $Z_0 \cong \sqrt{\mu/\kappa} = 134$ ohms.

vacuum. The permeability μ is the same for nonferromagnetic rock as for vacuum, *i.e.*, equal to $\mu_0 = 1.257 \times 10^{-6}$ henry/m.

When an electromagnetic wave enters such a medium squarely, *i.e.*, at normal incidence, first a portion goes straight into the medium at a reduced speed with some energy reflected backward into the less electromagnetically dense medium (air in this case). The speed of wave propagation in atmosphere (since $\sigma = 0$ mhos/m) is essentially $c = 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec, since $\kappa_0 = 8.854 \times 10^{-12}$ farad/m is the dielectric constant for empty space, which holds practically for atmosphere also. The phase velocity of propagation in the cube is now

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{1}{2}\mu\kappa[1 + \sqrt{1 + (\sigma/\kappa\omega)^2}]}} \qquad \text{m/sec} \tag{8}$$

since we have now a propagation constant

$$\Gamma = j\omega\sqrt{\mu\kappa_e} = j\omega\sqrt{\mu\left(\kappa - j\frac{\sigma}{\omega}\right)} = \sqrt{\frac{1}{2}\mu\omega[\sqrt{(\kappa\omega)^2 + \sigma^2} - \kappa\omega]} + j\sqrt{\frac{1}{2}\mu\omega[\sqrt{(\kappa\omega)^2 + \sigma^2} + \kappa\omega]}$$

$$= \underbrace{\omega\sqrt{\tfrac{1}{2}\mu\kappa[\sqrt{1 + (\sigma/\kappa\omega)^2} - 1]}}_{\substack{\alpha \\ \text{in nepers/m}}} + j\underbrace{\omega\sqrt{\tfrac{1}{2}\mu\kappa[\sqrt{1 + (\sigma/\kappa\omega)^2} + 1]}}_{\substack{\beta \\ \text{in circular radians/m*}}}$$

Suppose the frequency f is 100 Mc/sec, *i.e.*, $100 \times 10^6 = 10^8$ cycles/sec; then $\omega = 6.28 \times 10^8$, $\kappa = 8 \times 8.854 \times 10^{-12} = 7.0832 \times 10^{-11}$ farad/m, and $\sigma/(\kappa\omega) = 10^{-6}/(7.0832 \times 10^{-11} \times 6.28 \times 10^8) = 2.26 \times 10^{-5}$. Now the quantity $[\sigma/(\kappa\omega)]^2 = (2.26 \times 10^{-5})^2 = 5.108 \times 10^{-10}$ is surely negligible in comparison with unity. Therefore, (8) reduces to $v = 1/\sqrt{\mu\kappa}$. Hence, at the frequency of 100 Mc/sec, the effect of the specific conductivity σ on the product $\omega\kappa$ can be ignored. This is still more evident as we go to still higher frequencies. We can therefore write $\kappa_e \cong \kappa$ farads per meter. According to Eq. (8), this gives a phase velocity $v = 1/\sqrt{\mu_0\kappa}$ meters per second, since $\mu = \mu_0$ henrys per meter for nonferromagnetic rock. But the velocity in vacuum is $c = 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec and $\kappa = 8\kappa_0$ or $v = 1/\sqrt{8\kappa_0\mu_0} = 1/(2.83\sqrt{\mu_0\kappa_0}) = c/2.83 = 1.06 \times 10^8$ m/sec. The speed of electromagnetic waves in this rock is almost one-third of the velocity of light.

Inasmuch as the resultant electric $\mathcal{E}$ vector for position (*a*) is entirely along the vertical Z direction, both conduction and displacement currents are also along this direction (positive and negative, depending on the instantaneous polarity of the $\mathcal{E}$ vector). According to Eq. (7), we find the currents across any of the square-meter cross sections normal to the Z axis as $\mathcal{E}_z\sqrt{\sigma^2 + (\omega\kappa)^2}$ amperes. But $\sigma^2 = 10^{-12}$, $(\omega\kappa)^2 = 6.28^2 f^2 \times (8\kappa_0)^2$ for the frequency f in cycles per second. For $f = 100$ Mc/sec $= 10^8$ cycles/sec, we find $(\omega\kappa)^2 = 39.44 \times 10^{16} \times 64 \times 78.32 \times 10^{-24} = 1.96 \times 10^{-3}$. Therefore,

$$\text{curl}_z H = J_{z\,\text{total}} = \mathcal{E}_z\sqrt{10^{-12} + 1.96 \times 10^{-3}} \cong \mathcal{E}_z\sqrt{19.6 \times 10^{-4}}$$
$$= 4.44 \times 10^{-2}\mathcal{E}_z \quad \text{amp/m}^2$$

where the phase angle $\varphi = \tan^{-1}(\omega\kappa/\sigma) = \tan^{-1}(6.28 \times 10^8 \times 8 \times 8.854 \times 10^{-12}/10^{-6}) = \tan^{-1}(44{,}500) \cong 90$ time degrees also shows that the term $(\omega\kappa)^2$ in formula (7) determines the square-root value. This can also be seen from the complex value form $\text{curl}_z H = (\sigma + j\omega\kappa)\mathcal{E}_z$

* We have $\Gamma = \alpha + j\beta$ and find α and β from Γ^2 and equating real and j terms, from which $\alpha\beta = \frac{1}{2}\mu\sigma\omega$ and $\beta^2 - \alpha^2 = \mu\kappa\omega^2$.

leading to formula (7), since the value $\sigma = 10^{-6}$ mho/m is practically negligible compared with the absolute value $\omega\kappa = 6.28 \times 10^8 \times 8 \times 8.854 \times 10^{-12} = 44{,}500 \times 10^{-6}$ mho/m. For a frequency still higher than 100 Mc/sec, the displacement factor $\omega\kappa$ would be still more pronounced, for instance, thirty times larger than $44{,}500 \times 10^{-6}$ mho/m for a 3,000-Mc wave excitation.

Suppose the same cubical medium is now in position (*b*) and the same distance of *r* meters away from the radiation center. The conditions then differ in two respects. First, the square-meter back face of the cubical medium no longer receives normal wave incidence (under 90 space degrees), but wave incidence under 45 deg occurs. If the radiator were an isotropic elementary point source, the intensity of the arriving electromagnetic ray in the 45-deg direction would be, in unobstructed space, just as great as in any other direction about the point source if the ray is the same distance away. However, we have no point source, but a linear radiator in Fig. 4 in order to bring about vertical polarization. If this radiator were an infinitesimally small Hertzian doublet or an infinitesimally short radiator, the radiation effect in any direction θ of any meridian plane with respect to the length orientation of the radiator would depend on this angle. It can be shown that maximum effect occurs anywhere (in any direction) in the equatorial plane of a single linear radiator. The cubical medium in position (*a*) lies in the equatorial plane, and the electric field intensity $\mathcal{E}$ is then a maximum effect at a distance[1] of *r* meters. It can be shown that the value of the reduced field intensity $\mathcal{E}'$, also at *r* meters distance from the radiation center but in the direction θ, can be computed from the corresponding equatorial $\mathcal{E}$ value by multiplying this value by a direction factor D_θ. Such a direction factor depends upon the configuration of the sender antenna, whether linear or not, whether an antenna array or a wave-front radiator, such as a slot radiator, an electromagnetic horn, the plane wave front in the opening of a paraboloid, and the like. For an Hertzian doublet, as for an elementary linear radiator and as for a linear radiator whose very short length approximates the same current distribution as for the $I\,dl$ radiator, where I is the current and dl the infinitesimally short length, the directivity in any meridian plane of the radiator is $D_\theta = \sin\theta$, proving that along any of the directions in the equatorial plane of the radiator, we have $D_\theta = \sin 90 \text{ deg} = 1$, which is the maximum possible value that a sine function can produce. Such a space directivity distribution law

[1] We have only to specify either the value of $\mathcal{E}$ volts per meter or the value of *H* amperes per meter if we deal with waves in the distant radiation region (plane waves), since for ordinary atmosphere $\mathcal{E}/H$ is essentially equal to 377 ohms and, if $\mathcal{E}$ is known, we can compute *H* from $\mathcal{E}/377$, and vice versa.

holds, of course, for the free-space field (unobstructed field) only. Hence, for such radiators, we should have for position (*b*) of the cubical medium the electric vector of the arriving electromagnetic wave as $\mathcal{E}' = \mathcal{E} \cdot D_\theta = \mathcal{E} \sin 45 \text{ deg} = 0.7071\mathcal{E}$ volts per meter and the associated magnetic intensity $H' = 0.7071\mathcal{E}/377 = 1.87 \times 10^{-3}\mathcal{E}$ amperes per meter just before entering the cube from atmosphere (for which $Z_0 \cong \sqrt{\mu_0/\kappa_0} = 377$ ohms). We have only 70.71 per cent of the equatorial field intensity $\mathcal{E}$ available at the same distance of *r* meters from the radiation center in the 45-deg direction. Since $D_\theta = D_0 = \sin 0 \text{ deg} = 0$, there can be no radiation effect in the cubical medium in the distant radiation field along either direction of a linear radiator (even if of finite length).

An elementary radiator is hypothetical; a very short radiator that may also practically satisfy the $D_\theta = \sin \theta$ law would not offer enough powerful radiation effect in the distant radiation region. We need a radiator that can, so to speak, be loaded with energy and can, as such, produce powerful space radiation. A half-wave radiator (length ½ wavelength) is a linear antenna of this kind. It is of small dimensions at very high and ultrahigh frequencies. For instance, at $f = 100$ Mc/sec, we have a full free-space wavelength of $300/100 = 3$ m; at 1,000 Mc/sec, we have only 0.3 m; at 10,000 Mc/sec, we have only 0.03 m. Somewhat less than half of these respective values would be the radiator length (1 m = 3.281 ft = 39.37 in.). Hence, taking the free-space wavelength for a 10,000-Mc wave excitation, we have $\frac{1}{2}\lambda = 0.015 \text{ m} = 0.015 \times 39.37 = 0.59$ in. Therefore, the radiator length would be somewhat less than this length (about 5 per cent less). For a 1,000-Mc wave excitation the length would be somewhat less than 5.9 in.

It is shown in this text that the meridian directivity pattern of such a useful and efficient half-wave radiator, in unobstructed radiation space, is

$$D_\theta = \frac{\cos (90^\circ \cos \theta)}{\sin \theta} \tag{9}$$

where θ denotes the angle that the direction makes with the linear radiator. For position (*b*) referring to $\theta = 45$ deg with the radiator length, we find then a directivity

$$D_\theta = \frac{\cos (90^\circ \cos 45)}{\sin 45} = \frac{\cos (0.7071 \times 90^\circ)}{0.7071} = \frac{\cos 63.7^\circ}{0.7071}$$
$$= \frac{0.4431}{0.7071} = 0.628$$

Hence, $\mathcal{E}' = 0.628\mathcal{E}$ volts per meter and $H' = 0.628\mathcal{E}/Z_0$ amperes per meter, where Z_0 denotes the wave impedance in the cubical medium. Hence, for a half-wave radiator, the effect in a 45-deg direction in any

vertical plane through the radiator is only 62.8 per cent of the maximum value in the equatorial plane. The maximum effect occurs for position (*a*) of the cubical medium.

For the cubical medium in position (*b*), we must also take into consideration the *orientation* of this cubic meter of space with respect to the angle of incidence of the arriving wave. Also the type of polarization of this wave plays a part. We note that the resultant $\mathcal{E}'$ and H' field associations are no longer vertical and horizontal, but we can resolve these vectors into Cartesian components satisfying the cubical medium of position (*b*). We can then resolve $\mathcal{E}'$ into a vertical $\mathcal{E}'_z = \mathcal{E}' \cos\theta$ and a horizontal $\mathcal{E}'_x = \mathcal{E}' \sin\theta$ component, respectively, making both components for $\theta = 45$ space degrees equal, since $\mathcal{E}'_z = \mathcal{E}'_x = 0.707 \times 0.628\mathcal{E} = 0.442\mathcal{E}$ volts per meter, where $\mathcal{E}$ is the value anywhere in the equatorial plane a distance r meters away from the radiation center.

For the (*b*) position of the cubical medium, we no longer deal with only one $\text{curl}_z\, H$ vector as for position (*a*), but with two such components, namely, a $\text{curl}_z\, H$ and a $\text{curl}\, H$, since two electric field components $\mathcal{E}'_z$ and $\mathcal{E}'_x$ are operative in the respective Z and X directions. Hence, we must, according to Fig. 3, generally have conduction and displacement currents in such directions also, *i.e.*, across XY and YZ planes, but not normal to XZ planes, since we have no driving field component along the Y axis ($\mathcal{E}'_y = 0$). Therefore, also $\text{curl}_y\, H = 0$, since $(\sigma + j\omega\kappa)\mathcal{E}'_y = 0$.

On account of the orientation of the (*b*) position of the cube we have the reduced total current densities

$$\left.\begin{aligned} \text{curl}_z\, H &= \mathcal{E}'_z \sqrt{\sigma^2 + (\omega\kappa)^2} \\ \text{curl}_x\, H &= \mathcal{E}'_x \sqrt{\sigma^2 + (\omega\kappa)^2} \end{aligned}\right\} \qquad \text{amp/m}^2$$

Since $\mathcal{E}'_z = \mathcal{E}'_x = 0.442\mathcal{E}$ volts per meter, we find the total current density along the Z as well as along the X directions for position (*b*)

$$J_z = J_x = 0.442\mathcal{E}\sqrt{\sigma^2 + (\omega\kappa)^2} \cong 0.442\omega\kappa\mathcal{E} \qquad \text{amp/m}^2$$

since we have already found that σ^2 is negligible in comparison with $\omega^2\kappa^2$ in our numerical example. Using the numerical result for $\mathcal{E}\sqrt{\sigma^2 + \omega^2\kappa^2} = 4.44 \times 10^{-2}\mathcal{E}$ amperes per square meter holding for $f = 100$ Mc/sec, we find $J_z = J_x = 0.442 \times 4.44 \times 10^{-2}\mathcal{E} = 1.96 \times 10^{-2}\mathcal{E}$ amperes per square meter along the Z and X directions. This assumes that when the electromagnetic wave arrives at the cube, no back reflection takes place as it enters the cube. Hence, this numerical result is true only when the wave always travels in the general medium ($\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m, $\kappa = 8 \times 8.854 \times 10^{-12}$ farad/m, $\sigma = 10^{-6}$ mho/m). It would not hold if the arriving wave came from an atmospheric medium and just

entered under 45 space degrees the rock where the cube in position (*b*) forms a flat part of the rock's front portion with its rear face. In such a case, we should have also to subtract the field energy due to wave back reflection for a 45-deg incidence. This wave reflection as described in Chap. VII depends not only on the relative index of refraction but also on the angle of wave incidence and the kind of polarization. Moreover, the value of $1.96 \times 10^{-2}\mathcal{E}$ amperes per square meter for the current densities along either the Z or the X direction is less now than the value of $4.44 \times 10^{-2}\mathcal{E}$ amperes per square meter occurring for position (*a*) only along the Z direction. This is justified, since, first, we have less radiation effect along a 45-deg direction in vertical planes through the radiator and, second, the oscillating field energy arriving at position (*b*) resolves in two distinct directions.

If the position (*b'*) of the cubical medium is now investigated, we note that the new Z' axis is no longer vertical but turned back by 45 space degrees so that the arriving $\mathcal{E}'$, H' wave pierces the cubic meter of medium squarely. In a case of this kind, we have one polarization only, namely, the resultant $\mathcal{E}'$ polarization, and one curl component *only*, namely, a $\text{curl}_{z'} H$ component, with a total current density of

$$J_{z'} = \mathcal{E}' \sqrt{\sigma^2 + \omega^2\kappa^2} \qquad \text{amp/m}^2$$

across $X'Z'$-square-meter areas of the cubical medium in position (*b'*). But $\mathcal{E}' = \mathcal{E} \cdot D_0 = \mathcal{E} \cdot D_{45°}$, where, for a vertical half-wave radiator, $D_{45} = 0.628$. Hence, $J_{z'} = 0.628J_z$, where J_z denotes the vertical total current density for the cubical medium in the equatorial location [the (*a*) position]. Therefore, $J_{z'} = 0.628 \times 4.442 \times 10^{-2}\mathcal{E} = 2.76 \times 10^{-2}\mathcal{E}$ amperes per square meter This is the only total current density possible for the (*b'*) position of the cubical medium. For an effective field intensity $\mathcal{E} = 1$ volt/m at a distance of r meters from the radiation center in the equatorial plane, we then find 0.0276 amp flowing across 1-sq-m $X'Y'$ planes for the (*b'*) position, while the same cube in position (*a*) would experience 0.0442 amp across the same 1-sq-m planes. We see, therefore, that by the components of the curl H as shown in Fig. 3 we can compute without any difficulty current densities of mediums placed anywhere in space.

Thus far, we have found that the displacement current density was so predominant for our rock medium that any component curl effects are due mostly to the displacement component. Let us see what other effects the conductivity $\sigma = 10^{-6}$ mho may have across parallel square-meter planes of the rock, if separated by 1 m. For position (*a*), current flows only along Z directions, depending, according to Eq. (7), upon the polarity of the $\mathcal{E}_z$ electric field vector. This vector happens to be iden-

tical for the (*a*) position with the arriving resultant $\mathcal{E}$ vector. The quantity $J = \sigma\mathcal{E}_z$ represents the only conduction current in amperes per square meter that is possible with the cube at position (*a*). Since $\mathcal{E}_z J$ represents the inphase product, it must be the power lost in the cubical medium in watts. But $\mathcal{E}_z = \mathcal{E}$ for the (*a*) position, and $J = \sigma\mathcal{E}$, and, therefore, $P = \mathcal{E}J = \sigma\mathcal{E}^2 = 10^{-6}\mathcal{E}^2$ is the joulean heat loss in the cubical medium (for 1 cu m). This means that if a field intensity of 1 volt/m were effective, exactly 1 μw of power would be lost in every cubic-meter medium located around a circle of r meters distance from the radiator and lying in the equatorial plane. For 10 volts/m, the loss would be one hundred times more, or 0 0001 watt/m³. This implies that $\mathcal{E}$ stands for the average of the effective values within the cube. On account of the losses due to the conductivity σ of the general medium, the field $\mathcal{E}$ is actually decreased as it travels between the square-meter faces of the cubic meter.

9. Application of the Poynting Vector and Its Components. The total field energy in the medium of 1 cu m is $\frac{1}{2}(\mu H^2 + \kappa\mathcal{E}^2)$ joules. This means that we have the field energy in watt · seconds. For the (*a*) position, the magnetic contribution $\frac{1}{2}\mu H^2$ would oscillate along the $\pm Y$ directions, since we have only magnetic polarizations along the Y axis (only H_y actions). The electric field contribution $\frac{1}{2}\kappa\mathcal{E}^2$ joules occurs only along the $\pm Z$ directions, since space quadrature prevails in the radiation region between electric and magnetic field associations.

If we have no energy dissipation in the medium ($\sigma = 0$ mhos/m), and neither sources nor sinks (generation or absorption within the cube), then the *increase* of $\frac{1}{2}(\mu H^2 + \kappa\mathcal{E}^2)$ per second gives the rate of energy entering the cubical medium along the X direction for position (*a*) of Fig. 4. The divergence of energy ($\operatorname{div} p = \operatorname{div} \mathcal{E} \times H$) must then be the rate of *decrease* in the cubic-meter medium during 1 sec. Hence, for a pure dielectric medium,

$$\operatorname{div} p = -\frac{1}{2}\frac{\partial(\mu H^2 + \kappa\mathcal{E}^2)}{\partial t} \qquad \text{watts/m}^3 \qquad (10)$$

This is the watt energy leaving the front face of the cubical medium for $\sigma = 0$. This is obvious if we realize that the Poynting vector $p = \mathcal{E} \times H$ watts per square meter is the amount of energy crossing 1-sq-m area normal to p every second and $\frac{1}{2}(\mu H^2 + \kappa\mathcal{E}^2)$ represents the energy density per unit volume (per cubic meter). Hence, we can say that the divergence of p is the rate at which the amount $\frac{1}{2}(\mu H^2 + \kappa\mathcal{E}^2)$ joules expands per cubic meter. For a conductivity σ mhos per meter, we have also a conduction current $J = \sigma\mathcal{E}$ amperes per square meter. The quantity $\mathcal{E}J$ represents the rate at which energy is lost within the cubic meter

owing to joulean heat loss, and $-\mathcal{E}J$ is then the rate at which energy is created.

The instantaneous electric and magnetic field energies per unit volume are

$$P_e = \tfrac{1}{2}\kappa\mathcal{E}_0^2 \cos^2(\omega t - \beta r) \qquad \text{and} \qquad P_m = \tfrac{1}{2}\mu H_0^2 \cos^2(\omega t - \beta r) = P_e$$

since $H = \mathcal{E}\sqrt{\kappa/\mu}$, if r is the distance moved through. Since

$$\mathcal{E}_0 \cos(\omega t - \beta r) \qquad \text{and} \qquad H_0 \cos(\omega t - \beta r) = \mathcal{E}_0 \sqrt{\frac{\kappa}{\mu}} \cos(\omega t - \beta r)$$

are the instantaneous field associations, we find the Poynting vector

$$p = \mathcal{E} \times H = \mathcal{E}_0^2 \sqrt{\frac{\kappa}{\mu}} \underbrace{\cos^2(\omega t - \beta r)}_{\substack{\text{average value} \\ = \frac{1}{2}}} = \frac{1}{2}\frac{\mathcal{E}_0^2}{Z_0} = \frac{1}{2} Z_0 H_0^2 \qquad \text{watts/m}^2 \quad (11)$$

for the average rate and $Z_0 = \sqrt{\mu/\kappa}$ ohms for the medium of μ henrys per meter and κ farads per meter.

For a dissipative wave medium, we have the wave impedance

$$Z_0 = \sqrt{\frac{\mu}{\kappa_e}} \qquad \text{complex ohms} \qquad (12)$$

where the equivalent dielectric constant κ_e is, even for as low a frequency as 100 Mc/sec ($=10^8$ cycles/sec), only

$$\kappa_e = \kappa - j\frac{\sigma}{6.28 \times 10^8} = 8 \times 8.854 \times 10^{-12} - j\frac{10^{-6}}{6.28 \times 10^8}$$
$$\cong \kappa = 7.0832 \times 10^{-11} \qquad \text{farad/m}$$

Therefore, the wave impedance in the cubical rock, as specified in this section, is practically purely resistive and equal to

$$Z_0 \cong \sqrt{\frac{\mu}{\kappa}} \cong \sqrt{\frac{\mu_0}{\kappa}} = \sqrt{\frac{1.257 \times 10^{-6}}{7.0832 \times 10^{-11}}} = 134 \text{ ohms}$$

as against 377 ohms in customary atmosphere. We have then for the Poynting vector, according to (11),

$$p = \frac{1}{2}\frac{\mathcal{E}_0^2}{Z_0} = \frac{\mathcal{E}_0^2}{268} \qquad \text{watts/m}^2$$

With respect to the div p relation, as given by (10), and holding for a nondissipative medium, we have

$$\frac{1}{2}\frac{\partial[\mu H_0^2 \cos^2(\omega t - \beta r) + \kappa \mathcal{E}_0^2 \cos^2(\omega t - \beta r)]}{\partial t}$$
$$= \frac{1}{2}\frac{\partial\left[\mu \frac{\kappa}{\mu} \mathcal{E}_0^2 \cos^2(\omega t - \beta r) + \kappa \mathcal{E}_0^2 \cos^2(\omega t - \beta r)\right]}{\partial t}$$
$$= \frac{\partial[\kappa \mathcal{E}_0^2 \cos^2(\omega t - \beta r)]}{\partial t}$$

a result that would be expected, since in an electromagnetic field, the magnetic as well as the electric instantaneous fields fluctuate in synchronism and are in the ratio equal to $\sqrt{\kappa/\mu}$. As far as the differentiation is of concern, we can operate with cos ωt only, instead of the cos function with the retarded argument $\omega t - \beta r$, since only the term ωt contains the time function and also the respective average values of $\cos^2 \omega t$ and $\cos^2(\omega t - \beta r)$, are both equal to ½. We find then

$$\frac{\partial[\kappa \mathcal{E}_0^2 \cos^2(\omega t)]}{\partial t} = \kappa \mathcal{E}_0^2 \frac{d[\cos(\omega t)\cos(\omega t)]}{dt}$$
$$= \kappa \mathcal{E}_0^2(-\omega \sin \omega t \cos \omega t - \omega \cos \omega t \sin \omega t)$$
$$= -2\omega\kappa \mathcal{E}_0^2 \sin \omega t \cos \omega t = -\omega\kappa \mathcal{E}_0^2 \sin(2\omega t) \qquad \text{watts/m}^3$$

Hence, the maximum instantaneous value of the available power in 1 cu m is for a 100-Mc/sec field excitation

$$\omega\kappa \mathcal{E}_0^2 = 6.28 \times 100 \times 10^6 \times 7.0832 \times 10^{-11} \mathcal{E}_0^2 = 4.45 \times 10^{-2} \mathcal{E}_0^2$$
$$= 6.25 \times 10^{-2} \mathcal{E}_e^2 \qquad \text{watts/m}^3$$

where $\mathcal{E}_0$ volts per meter denotes the maximum electric field value at the center of the cube and the effective value $\mathcal{E}_e = \mathcal{E}_0/\sqrt{2}$. Since, according to (5), in presence of a specific conductivity $\sigma = 10^{-6}$ mho/m, we have also joulean heat losses within the cubic meter, and we actually must deal with a complex divergence of p, because p has a real and imaginary part. The real part represents the average power passing perpendicularly to a square-meter area in the direction of wave propagation. The imaginary part is reactive. This is dealt with in Sec. 27.

The divergence of a vector is a *scalar* (only magnitude). This means that the quantity div p has only magnitude. This does not necessarily convey that we cannot resolve the total amount which is either supplied or released from a certain volume, into definite directions. Formula (5) (pages 25 and 197) gives an expression for the divergence of the "cross" product $\mathcal{E} \times H$, where the Poynting vector $p = \mathcal{E} \times H$ represents the Poynting vector flow of energy across unit areas normal to p and in a general medium. The cross product $p = \mathcal{E} \times H$ is a new vector like any cross product and refers, in this case, to an energy stream across 1 sq m in a

unit of time. The divergence of p, *i.e.*, div p, can also be written as $\nabla \cdot p$ or $\nabla \cdot (\mathcal{E} \times H)$. It is, therefore, a "dot" product of the operator ∇ and p and consequently a mere scalar quantity, *i.e.*, no longer a vector. Even though div $(\mathcal{E} \times H)$ is no longer a directed quantity, the amount of available field energy can escape across 1-sq-m surface of the cube if the respective perpendicular $\mathcal{E}$ and H vectors, *i.e.*, the plane wave front, are in the plane of this square meter. But if the resultant $\mathcal{E}$ vector is oriented generally, *i.e.*, not along any of the 1-m edges of the cubical medium (Fig. 4), then $\mathcal{E}$ can be resolved into $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$ components, with corresponding perpendicular H vectors for each such component.

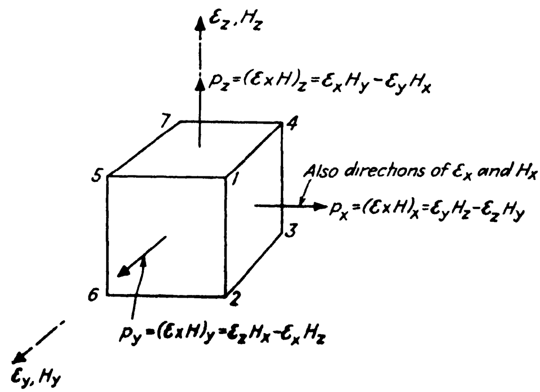


FIG. 5. Surface divergences of Poynting flux Electromagnetic energy flux: p_x is watts/m² across area 12341, p_y is watts/m² across area 12651; p_z is watts/m² across area 15741.

For such a condition, three[1] 1-sq-m surfaces of the cube will release electromagnetic field energy. The energy will stream out of the cube in three definite directions, normal to each other.

Electromagnetic field energy can be radiated only from regions (volumes) over which p is directed away from a point that is the seat of the energy origin. It is possible that a source, *i.e.*, a generator, receives just as much energy in return as it emits. This would be the case of a short-circuited transmission line, assumed ideal, with no joulean heat losses and no dielectric and other field losses.

By divergence of p, we deal then with an absolute amount of something that originates or diverges from 1 cu m of a medium and div p or $\nabla \cdot p$ means net outward flux of vector p per cubic meter. Hence, even though div p is a scalar, the quantity p has magnitude and direction. We can then resolve p along the X, Y, and Z directions with components

[1] There are six 1-sq-m surfaces to a cube, but two such surfaces are always parallel. Hence, in the case of div p, only one of respective two parallel surfaces can release excess of energy flow.

p_x, p_y, and p_z, respectively, where the resultant vector $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$. The components are indicated in Fig. 5. The component p_x stands for the electromagnetic energy flux in watts per square meter across planes parallel with the YZ plane; p_y is the Poynting component normal to 1 sq m in the XZ plane; and p_z stands for watts per square meter in the XY plane with p_z normal to it. By using such components, we change from an *average* divergence div p of electromagnetic field energy released from 1 cu m across the entire enclosing surface of 6 sq m (from a volume divergence) to *surface* divergences. This is obvious if we realize that the energy flux p_x watts flows normal to square-meter surfaces that are perpendicular to p_x. Mathematically, this is written as div $p_x = (\int p_x\,dx)/(dy\,dz)$, since we must express the surface divergence div p_x per unit surface, which in the limit becomes infinitesimally small. In the theoretical part of this text, it is shown that we can compute the Cartesian components p_x, p_y, and p_z for a resultant Poynting vector $p = \mathcal{E} \times H$, where $\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2 + \mathcal{E}_z^2}$ and $H = \sqrt{H_x^2 + H_y^2 + H_z^2}$. We have then the energy-flow components

$$p_x = \mathcal{E}_y H_z - \mathcal{E}_z H_y; \qquad p_y = \mathcal{E}_z H_x - \mathcal{E}_x H_z; \qquad p_z = \mathcal{E}_x H_y - \mathcal{E}_y H_x \quad \text{watts/m}^2 \tag{12a}$$

The orientations of the various field vectors are shown in Fig. 5.

With respect to Fig. 4, for the position (*a*) of the cubic meter of a general medium, we deal with only *one* surface divergence, since the resultant $\mathcal{E}$ and H vectors of the arriving wave are polarized along the vertical Z and horizontal Y directions. At the front face of the cube, we have a plane wave front across which (along the X direction) the electromagnetic field energy $p_x = \frac{1}{2}\mathcal{E}_z H_y$ watts per square meter escapes, since we have neither an $\mathcal{E}_y$ nor an H_z component. The factor $\frac{1}{2}$ enters, since $\mathcal{E}$ and H are maximum time values and the effective values $\mathcal{E}_z/\sqrt{2}$ and $H_y/\sqrt{2}$ must be used. It is assumed that the entire medium outside the cube is also the specified rock, so that no reflection losses can occur upon emergence from the shaded area. Because $\mathcal{E}_z/H_y = 134$ ohms, we have an electromagnetic power transfer of

$$p_x = \frac{1}{2}\mathcal{E}_z\left(\frac{\mathcal{E}_z}{134}\right) = 0.00374\mathcal{E}_z^2 \qquad \text{watts/m}^2$$

across the square-meter area. It is understood that the power entering the front face of the cube must be somewhat larger in order to supply the joulean heat loss within the cube due to the finite conductivity $\sigma = 10^{-6}$ mho/m.

10. Electric and Magnetic Field Magnitudes. In alternating-current networks, for a sinusoidal current flow I in amperes, we have the voltage

$E = ZI$ volts, if $Z = R \pm jX$ represents the impedance in ohms through which the current is driven by the impressed voltage. If an alternating voltage is impressed across a transmission line of great length (theoretically infinite in length), for any point of the line the relation $E = Z_0 I$ volts holds, where Z_0 denotes the characteristic impedance $\sqrt{Z/Y}$ of the line for $Z = R + j\omega L$ and $Y = G + j\omega C$, R and L are the resistance and inductance along the line, and G and C the conductance and capacitance across the line per unit length. This relation holds also for any length of the line when the far end is terminated into the characteristic impedance Z_0 ohms. It also holds for any length of the line whether open or loaded at the far end, as far as the outgoing or the reflected wave-like disturbance, respectively, is concerned.

For electromagnetic waves, in unobstructed radiation space, the ratio $\mathcal{E}/H$ of the electric and associated magnetic field intensities is expressible by the three-dimensional characteristic impedance Z_0, given by

$$Z_0 = \sqrt{\frac{\mu}{\kappa_e}} \quad \text{ohms}$$

$$\kappa_e = \kappa - j\frac{\sigma}{\omega} \quad \text{farads/m}$$

holds for the unbounded medium (but is different for the same medium in a wave guide)

for $\mathcal{E}$ in volts per meter and H in amperes per meter. This holds for the general medium of σ mhos per meter, μ henrys per meter, and κ farads per meter.

Generally, the numerical magnitude of E volts in $E = Z_0 I$ is equal to the numerical magnitude of the current I amperes only when the characteristic line impedance Z_0 is 1 ohm. This is seldom the case. This statement also holds for the space impedance of the expression $\mathcal{E} = Z_0 H$, and generally Z_0 is complex with a corresponding phase angle.

It is shown on pages 125, 289, 291, 316–317, 374, and 470 that for plane electromagnetic waves with $\mathcal{E}$ along the X axis and H along the Y axis the rate of change $\partial\mathcal{E}_x/\partial z$ of the electric field $\mathcal{E}_x$ along the Z axis (direction of electromagnetic energy transfer) and the rate of change $\partial H_y/\partial z$ of the associated magnetic field are

$$\frac{\partial \mathcal{E}_x}{\partial z} = -j\omega\mu H_y = -ZH_y; \qquad \frac{\partial H_y}{\partial z} = -(\sigma + j\omega\kappa)\mathcal{E}_x = Y\mathcal{E}_x$$

for harmonic fields. From transmission-line theory, we know that $\sqrt{ZY}$ yields the propagation constant and $\sqrt{Z/Y}$ the characteristic impedance. We find then

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\kappa}} \quad \text{ohms} \tag{13}$$

which leads to the relation given before.

It is of interest to have numerical values of this outcome. For a nonconducting medium, such as customary atmosphere below the ionized region, we have $\sigma = 0$ and

$$Z_0 = \sqrt{\mu/\kappa} \cong \sqrt{\mu_0/\kappa_0} = \sqrt{1.257 \times 10^{-6}/(8.854 \times 10^{-12})} \cong 377 \text{ ohms}$$

as for vacuum. We can then say that for plane electromagnetic waves in unobstructed troposphere, the electric field in volts per meter at any space point in the distant region is numerically 377 times larger than the numerical value of the associated magnetic field in amperes per meter at the same space point. These fields are in time phase with each other but in space quadrature (perpendicular to each other and normal to the energy flow).

For a plane electromagnetic wave passing through ideal distilled water, for which the dielectric constant κ in farads per meter is no longer equal to κ_0 but about eighty times larger, the medium is no longer so "stiff," since the characteristic space impedance is only $Z_0 = \sqrt{\mu_0/(80\kappa_0)} = 377/\sqrt{80} \cong 42$ ohms. Hence, a greater ampere per meter magnetic field association exists for a corresponding electric gradient in volts per meter in the case of distilled water. The power transferred across each square meter of the $\mathcal{E}H$ plane is then $\mathcal{E}H$ watts if $\mathcal{E}$ and H are taken as the effective field values. We have then the Poynting radiation vector in watts per square meter as

$$\left.\begin{aligned} p &= \mathcal{E}H = \frac{\mathcal{E}^2}{377} = 377H^2 \qquad \text{for air} \\ p &= \mathcal{E}H = \frac{\mathcal{E}^2}{42} = 42H^2 \qquad \text{for distilled water} \end{aligned}\right\} \tag{14}$$

In either case, H is in time phase with $\mathcal{E}$, since Z_0 is purely resistive. On page 17, we found that for the specified dry rock, we practically have Z_0 also resistive and equal to 134 ohms, which means that $\mathcal{E}$ volts per meter is equal to $134H$, where H is in amperes per meter.

Matters are somewhat more involved for a pronounced conductive medium, where σ mhos per meter is no longer negligible in comparison with $\omega\kappa$ mhos per meter. The three-dimensional impedance Z_0 is then complex according to (13), and $\mathcal{E}$ and H are no longer in time phase. Suppose we deal with ocean water as the medium and with a 796-Mc wave excitation. Hence, $\omega = 6.28f = 6.28 \times 796 \times 10^6 = 5 \times 10^9$ radians/sec; $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m; $\kappa \cong 80\kappa_0 \times 80 \times 8.854 \times 10^{-12} = 7.1 \times 10^{-10}$ farad/m. Assuming that the salinity is such that $\sigma = 5$ mhos/m, formula (13) yields

$$Z_0 = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\kappa}} = \sqrt{\frac{j5 \times 10^9 \times 1.257 \times 10^{-6}}{5 + j5 \times 10^9 \times 7.1 \times 10^{-10}}} = \sqrt{\frac{j1{,}257}{1 + j0.71}}$$
$$= \sqrt{\frac{j1{,}257(1 - j0.71)}{1 + 0.71^2}} = \sqrt{\frac{890 + j1{,}257}{1.5041}}$$
$$= 0.257 \sqrt{\sqrt{890^2 + 1257^2} \underline{/\delta}}$$
$$= 0.257 \sqrt{1{,}540 \underline{/\delta}} = 10.1 \underline{/\frac{\delta}{2}};$$
$$\delta = \tan^{-1} \frac{1{,}257}{890} = \tan^{-1} 1.407 = 54.5°$$

The medium has then a power factor $\cos \varphi = \cos (\delta/2) = \cos 27.25$ deg $= 0.889$, where φ is the phase difference between $\mathcal{E}$ and H. The Poynting vector is now only

$$p = \mathcal{E}H \cos \varphi = 0.889\mathcal{E}H = 10.1 \times 0.889H^2$$
$$= 8.95H^2 = \frac{\mathcal{E}^2}{8.95} \qquad \text{watts/m}^2$$

as far as the active component is concerned, since the magnetic vector H amperes per meter lags $\mathcal{E}$ volts per meter by 27.25 deg. In the expression for p, the quantities $\mathcal{E}$ and H are taken again as the effective field values. This result can also be understood when transforming (13) as follows:

$$\left.\begin{aligned} Z_0 &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\kappa}} = \sqrt{\frac{j\omega\mu(\sigma - j\omega\kappa)}{\sigma^2 + \omega^2\kappa^2}} = \sqrt{\frac{\omega^2\mu\kappa + j\omega\mu\sigma}{\sigma^2 + \omega^2\kappa^2}} \\ &= \sqrt{\frac{\omega\mu(\omega\kappa + j\sigma)}{\sigma^2 + \omega^2\kappa^2}} \\ &= \omega\sqrt{\frac{\mu\kappa}{\sigma^2 + \omega^2\kappa^2}\left(1 + j\frac{\sigma}{\omega\kappa}\right)} \\ &= \omega\sqrt{\frac{\mu\kappa}{\sigma^2 + \omega^2\kappa^2}} \sqrt{1 + j\tan\delta} \qquad \text{ohms} \\ \delta &= \tan^{-1}\left(\frac{\sigma}{\omega\kappa}\right) \end{aligned}\right\} \qquad (15)$$

for δ as the loss angle of the medium. The ratio $\sigma/(\omega\kappa) = 5/(5 \times 10^9 \times 7.1 \times 10^{-10}) = 1.407$ yields, as before, $\delta = 54.5$ deg. We note, therefore, that the ratio $\sigma/\omega\kappa$ determines to what extent the value of the characteristic impedance is complex. For $\sigma/(\omega\kappa) \ll 1$, we have essentially a real characteristic impedance (pure resistive), since

$$Z_0 \cong \omega\sqrt{\frac{\mu\kappa}{\sigma^2 + \omega^2\kappa^2}} = \omega\sqrt{\frac{\mu\kappa/\omega^2\kappa^2}{(\sigma^2/\omega^2\kappa^2) + 1}} \cong \sqrt{\frac{\mu}{\kappa}}$$

For the general medium (μ, κ, and σ effective), we have a three-dimensional impedance $|Z_0|\epsilon^{j\varphi}$, and

$$\mathcal{E} = |Z_0|\epsilon^{j\varphi}H = Z_0(\cos\varphi + j\sin\varphi)H \qquad \text{volts/m} \qquad (16)$$

Applied to the above numerical example, we have

$$\mathcal{E}^{(\text{volts/m})} = 10.1H\cos\varphi + j10.1H\sin\varphi = \underbrace{8.95H}_{\substack{\text{watt}\\ \text{component}}} + \underbrace{j0.4578H}_{\substack{\text{wattless}\\ \text{component}}}$$

The factors 8.95 and 0.4578 are quite small compared with 377 holding for atmosphere. It is to be realized, however, that ocean water is not a medium into which appreciable wave penetration can be expected. The conduction current losses are so pronounced that the field amplitudes decrease very rapidly. This is brought out in the next section.

When we take a medium of dry ground such that $\sigma \cong 10^{-5}$ mho/m, $\kappa = \sigma\kappa_0 = 6 \times 8.854 \times 10^{-12} = 53 \times 10^{-12}$ farad/m, and $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m, we find for a frequency $f = 796$ Mc/sec again $\omega = 5 \times 10^9$ radians/sec and a loss angle $\delta = \tan^{-1}(\sigma/\omega\kappa) = \tan^{-1}(3.77 \times 10^{-5})$, which is essentially zero degrees, so that $Z_0 = \sqrt{\mu/\kappa}$ applies, which results in practically a pure resistive characteristic medium impedance of $\sqrt{1.257 \times 10^{-6}/53 \times 10^{-12}} = 154$ ohms. Therefore, $\mathcal{E} = 154H$, *i.e.*, somewhat less than half as "stiff" a medium as air. This is then an application for which $\tan\delta \ll 1$. Let us now investigate what happens when $\tan\delta \gg 1$, which is, for instance, the case of a good conductor, such as a medium of copper. We find from (13), since $\tan\delta = \sigma/(\omega\kappa)$ and $\sigma/(\omega\kappa) \gg 1$ or $[\sigma/(\omega\kappa)]^2 >>> 1$

$$Z_0 = \omega\sqrt{\frac{\mu\kappa/\omega^2\kappa^2}{(\sigma^2/\omega^2\kappa^2) + 1}}\sqrt{1 + j\frac{\sigma}{\omega\kappa}} \cong \omega\sqrt{\frac{\mu\kappa/\omega^2\kappa^2}{\sigma^2/\omega^2\kappa^2}}\sqrt{j\frac{\sigma}{\omega\kappa}}$$

Therefore,

$$Z_0 = \underbrace{\sqrt{j}}_{\frac{1+j}{\sqrt{2}}}\sqrt{\frac{\mu\omega}{\sigma}} = \left(\sqrt{\frac{\mu\omega}{2\sigma}} + j\sqrt{\frac{\mu\omega}{2\sigma}}\right) \underline{/45^\circ} \qquad \text{ohms} \qquad (17)$$

a result that is also arrived at, in a different manner, in the problem on page 48. The three-dimensional impedance Z_0 in a good conductor has then the form $|Z_0|\mathcal{E}^{j45^\circ}$ ohms, which means that there is a phase difference of 45 time degrees between the electric and magnetic field vector for mediums in which $\sigma/(\omega\kappa) \gg 1$. For such a condition the dielectric constant κ has only negligible effect. This is obvious, since for copper $\sigma \cong 5.8 \times 10^7$ mhos/m and $\kappa = \kappa_0 = 8.854 \times 10^{-12}$ farad/m. The conduction current density $5.8 \times 10^7\mathcal{E}$ amperes per square meter will

then essentially outbalance the displacement current density $j\omega \cdot 8.854 \times 10^{-12}\mathcal{E}$ amperes per square meter for all useful VHF's and UHF's of $\omega/6.28$ cycles/sec. In the case of copper and $f = 796$ Mc/sec, the real component of the space impedance Z_0 becomes

$$\sqrt{\mu\omega/(2\sigma)} = \sqrt{1.257 \times 10^{-6} \times 5 \times 10^9/(2 \times 5.8 \times 10^7)}$$
$$= \sqrt{0.54 \times 10^{-4}} = 0.00735 \text{ ohm.}$$

This result shows that the electric field is exceedingly small within the copper medium as compared with the magnetic field if compared with the conditions that would prevail in air. This can be felt if we realize that the amplitudes of the electric and the associated magnetic vector are $\mathcal{E}_{x_0} = |Z_0|\mathcal{E}^{j45}H_0$ and $H_{y_0} = H_0$, which for the active energy components are $\mathcal{E}_{x_0} = 0.00735H_0$ and $H_{y_0} = H_0$, showing that the amplitude $\mathcal{E}_x$ is indeed very small compared with amplitude H_y. We must realize that H_y large and $\mathcal{E}_x$ small are just as unfavorable as H_y small and $\mathcal{E}_x$ large. It is the *product* of $\mathcal{E}$ and H and the *relative phase* that counts in wave propagation, just as it is the product of E and I that counts in electric networks in addition to the power factor. In an electromagnetic field, we have *always* both an electric and a magnetic field contribution; and when one field becomes vanishingly small, we also arrive at a very small electromagnetic field.

Generally, for a wave propagation along the Z axis, in the case of plane electromagnetic waves, we have

$$\left.\begin{aligned} \mathcal{E}_x &= \epsilon^{j\varphi}|Z_0|H_0\epsilon^{j(\omega t - z\Gamma)} \quad \text{volts/m} \\ H_y &= H_0\epsilon^{j(\omega t - z\Gamma)} \quad \text{amp/m} \\ \Gamma &= \sqrt{YZ} = \sqrt{j\omega\mu(\sigma + j\omega\kappa)} = j\omega\sqrt{\mu\kappa}\left(1 - j\frac{\sigma}{\omega\kappa}\right) \\ &= \alpha + j\beta \quad \text{generalized hyperbolic radians/m} \end{aligned}\right\} \tag{18}$$

where $|Z_0|$ represents the scalar value of the three-dimensional impedance of a medium whose electromagnetic properties are σ mhos per meter, μ henrys per meter, and κ farads per meter. Since Γ denotes the propagation constant in an unobstructed medium (unbounded) and Z_0 the space impedance for such a medium, we have for air $\mathcal{E}_x$ and H_y in time phase ($\varphi = 0$), rendering Z_0 pure resistive, since $\epsilon^{j\varphi} = 1$ and $Z_0 = \sqrt{\mu/\kappa} \cong \sqrt{\mu_0/\kappa_0} \cong 377$ ohms. The propagation constant simplifies to $\Gamma = j\omega\sqrt{\mu\kappa} \cong j\omega\sqrt{\mu_0\kappa_0}$ and

$$\left.\begin{aligned} \mathcal{E}_x &= 377H_0\epsilon^{j(\omega t - \omega\sqrt{\mu_0\kappa_0}z)} \quad \text{volts/m} = 377H_0\epsilon^{j\left(\omega t - \frac{\omega}{c}z\right)} \\ H_y &= H_0\epsilon^{j\left(\omega t - \frac{\omega}{c}z\right)} \quad \text{amp/m} \\ c &= \frac{1}{\sqrt{\mu_0\kappa_0}} \cong 3 \times 10^8 \quad \text{m/sec} \end{aligned}\right\} \tag{19}$$

For a dissipative medium, the quantity $\epsilon^{j\varphi}|Z_0|$ gives a means of finding whether the electric or the magnetic amplitude is numerically larger. For a medium of copper,

$$Z_0 = \sqrt{j\,\frac{\mu\omega}{\sigma}} = \sqrt{\epsilon^{j90^\circ}\,\frac{\mu\omega}{\sigma}} = \epsilon^{j45^\circ}\sqrt{\frac{\mu\omega}{\sigma}} = \epsilon^{j\varphi}|Z_0| = \sqrt{\frac{\mu\omega}{2\sigma}} + j\sqrt{\frac{\mu\omega}{2\sigma}}$$

From the above computations, we have already found the absolute value $|Z_0| = 0.0104$ ohm, since $\sqrt{\mu\omega/\sigma} = 0.00735 \times 1.41 = 0.0104$. Since in a medium in which the conduction current component is very large compared with the displacement current component,

$$\Gamma = \alpha + j\beta = \alpha + j\alpha = \beta + j\beta = \sqrt{\frac{\mu\sigma\omega}{2}}\,(1 + j) \tag{20}$$

the numerical value of the attenuation constant α in nepers per meter or the numerical value of the phase constant β in circular radians per meter may be used. We find then

$$\sqrt{\frac{\mu\sigma\omega}{2}} = \sqrt{\tfrac{1}{2}\,1.257 \times 10^{-6} \times 5.8 \times 10^{7} \times 5 \times 10^{9}} = 4.3 \times 10^{5}$$

for the numerical value of either the attenuation constant α or the phase constant β for a 796-Mc excitation and a medium of copper. We then have $\Gamma = 4.3 \times 10^5 + j4.3 \times 10^5$, which, when used in (16) and (18), yields

$$\mathcal{E}_x^{(\text{volts/m})} = \epsilon^{j45^\circ} \cdot 0.0104 H_0 \epsilon^{-4.3\times10^5 z} \epsilon^{j(\omega t - 4.3\times10^5 z)}$$
$$H_y^{(\text{amp/m})} = H_0 \epsilon^{-4.3\times10^5 z} \epsilon^{j(\omega t - 4.3\times10^5 z)}$$

But $\epsilon^{j45^\circ} = \cos 45 \text{ deg} + j \sin 45 \text{ deg}$, which for the watt component again yields $0.0104 \cos 45 \text{ deg} = 0.0104 \times 0.7071 = 0.00735$ ohm as found before, showing that we deal almost entirely with a magnetic field. Because $\exp(-4.3 \times 10^5 z)$ accounts for the attenuation along the direction of wave propagation, the electromagnetic wave will not make much headway, since absorption reduces the amplitude of the wave exceedingly fast. For $z = 1$ m, we had a negative exponent of 4.3×10^5; while for a wave travel of only 1 mm (about a pencil line's thickness), we should have $z = 10^{-3}$ m and a negative exponent still as large as $4.3 \times 10^5 \times 10^{-3} = 430$, giving still a very great amplitude reduction, since even a negative exponent of only 4 already gives a decay factor $\epsilon^{-4} = 0.01832$.

11. Notes on the Wave Penetration into a Partially Conductive Medium, Phase Velocity, and Values of Electromagnetic Constants. The general case of a conductive medium is one that gives rise to conduction as well as displacement currents. We have a relative numerical index of refraction

$$n = \sqrt{\kappa_r - jq} \tag{21}$$

for

$$\kappa_r = \frac{\kappa}{\kappa_0}; \qquad q = 1.8 \times 10^4 \frac{\sigma}{f} \tag{21a}$$

where κ_r is the relative numerical dielectric constant of a medium, referred to the dielectric constant $\kappa_0 = 8.854 \times 10^{-12}$ farad/m which holds for vacuum and practically so for air. The conductivity σ of the medium is in mhos per meter, and the operating frequency f in megacycles per second. Since $\kappa_r - jq = (\kappa - j\sigma/\omega)/\kappa_0 = \kappa_e/\kappa_0 = \kappa'_e$, the quantity κ'_e denotes the *relative* equivalent dielectric constant, which is *numerical*, and $\omega = 6.28 \times 10^6 f$, if f is in megacycles per second. The relative index of refraction is, therefore, $n = \sqrt{\kappa'_e}$ for a complex medium that is not ferromagnetic. This is followed up on pages 56–63. The propagation constant in the general medium may be written in the convenient form $\Gamma = j\omega\sqrt{\mu(\kappa - jk)}$, if μ is the permeability in henrys per meter and $k = \sigma/\omega$ for the frequency $\omega/6.28$ in cycles per second. From this expression, we find

$$\left.\begin{aligned} &\Gamma = j\omega\sqrt{\mu\kappa} \qquad \text{for } \sigma = 0 \text{ (a } \textit{pure} \text{ dielectric medium)} \\ &\Gamma = j\omega\sqrt{\mu\kappa\left(1 - j\frac{k}{\kappa}\right)} = j\omega\sqrt{\mu\kappa}(1 - j\tan\delta) \quad \text{for } \sigma \neq 0 \text{ (general medium)} \end{aligned}\right\} \tag{22}$$

if δ denotes the loss angle $\tan^{-1}[\sigma/(\omega\kappa)]$ of the medium due to attenuation.

Let us see what these expressions mean numerically. We have two extreme conditions. For one condition, the equivalent dielectric constant κ_e is essentially equal to the actual dielectric constant κ in farads per meter; and for the other condition, it is equal to $-j\sigma/\omega$. If we deal with the earth crust, κ has an average value of about $13\kappa_0$ farads per meter, *i.e.*, a value thirteen times the dielectric constant holding for empty space (vacuum). For dry rock, κ may be as low as $5\kappa_0$ farads per meter; and for wet ground, it may be as high as $30\kappa_0$ farads per meter, where $\kappa_0 = 8.854 \times 10^{-12}$ farad/m. For ocean water as well as lakes and rivers, κ is about $80\kappa_0$ farads per meter under normal conditions; while for ice, a quite low dielectric constant exists again. Hence, for the average earth surface, we shall cover a range of hardly more than $(5–80)\kappa_0$ farads per meter. As far as the specific conductivity σ across parallel square-meter end surfaces of a cubic meter are concerned, dry ground shows values of σ from 10^{-4} to 10^{-5} mho/m, wet ground from 10^{-2} to 10^{-3} mho/m, tap water when fresh about 10^{-3} mho/m, lakes with spring water about 7.5×10^{-3} mho/m, and ocean water from 3 to 6 mhos/m, depending upon the salinity. Hence, the average earth surface will cover hardly

more range than about $\sigma = 10^{-5}$ to 6 mhos/m. The condition for which the equivalent dielectric constant κ_e is essentially equal to the actual dielectric constant κ farads per meter requires, according to $\kappa_e = \kappa - j\sigma/\omega$, that $k = \sigma/\omega$ is negligible in comparison to κ. Since k depends upon the frequency, the earth crust cannot behave electromagnetically the same for the entire frequency range used in communication. Let us then first find out how ground behaves. The most severe criterion exists when we choose the lowest value $\kappa = 5 \times 8.854 \times 10^{-12} = 4.427 \times 10^{-11}$ farad/m, since this is the lowest value occurring in the earth crust (for dry rock). Hence, when σ/ω is negligible compared with this value, it must vanish all the more for all other κ values occurring in ground. Suppose we have $f = 100$ Mc/sec; then $\omega = 6.28 \times 100 \times 10^6$. The severest criterion for the imaginary term jk occurs when we choose the largest possible value of σ for ground. We have for it $\sigma = 10^{-2}$. Therefore, $k = \sigma/\omega = 10^{-2}/(6.28 \times 10^8) = 1.592 \times 10^{-11}$ and $\kappa_e = 4.427 \times 10^{-11} - j1.592 \times 10^{-11}$ farad/m, showing that for the most severe ground condition, the conduction effect of the medium still apparently decreases the actual dielectric constant of 4.427×10^{-11} farad/m by about 30 per cent. It is actually less than this approximate percentage, since k acts in quadrature with respect to κ, because $\kappa_e = \kappa - jk$. An average conductivity as high as 10^{-2} mho/m is not likely to happen most of the time, and surely not for the severe limit of $\kappa = 5\kappa_0$ farads per meter, since this value refers to *dry* rock and the 10^{-2} mho/m severe limit to *wet* soil. Now, if we choose the average ground dielectric constant of about, say, $\kappa = 15\kappa_0 = 13.281 \times 10^{-11}$ farad/m and take also $\sigma = 10^{-3}$ mho/m as a compromise value, we find k equal to 0.1592×10^{-11} farad/m and $\kappa_e = 132.81 \times 10^{-12} - j1.592 \times 10^{-12}$ farad/m. The quadrature jk term can, therefore, be neglected for ground above 100 Mc/sec. Inasmuch as for ocean surface of the earth surface, we deal with quite a conductivity ($\sigma = 3$ to 6 mhos/m), we have to go to much higher ultrahigh frequencies before the jk term becomes negligible. Using $\sigma = 4$ mhos/m as a likely condition, $\kappa = 80\kappa_0$ farads per meter $= 7.0832 \times 10^{-10}$ farad/m. Now for a 100-Mc wave excitation, we find $k = 4/(6.28 \times 10^8) = 0.637 \times 10^{-8}$ farad/m. We have then

$$\kappa_e^{(\text{farads/m})} = 7.0832 \times 10^{-10} - j63.7 \times 10^{-10} \qquad \text{for 100 Mc/sec}$$
$$\kappa_e^{(\text{farads/m})} = 7.0832 \times 10^{-10} - j6.37 \times 10^{-10} \qquad \text{for 1,000 Mc/sec}$$
$$\kappa_e^{(\text{farads/m})} = 7.0832 \times 10^{-10} - j0.637 \times 10^{-10} \qquad \text{for 10,000 Mc/sec}$$

and it takes at least 10,000 Mc/sec before the jk term can be neglected, since then the hypotenuse $\sqrt{7.0832^2 + 0.637^2}$ is essentially equal to the long side 7.0832. We have then for ocean above 10,000 Mc/sec essentially a dielectric medium just as in the case for ordinary ground above

100 Mc/sec. Other numerical data are given throughout this text on this subject. We also note that below 100 Mc/sec, ocean water behaves more or less like a conductor and all the more so if we choose, for instance, $\sigma = 5$ mhos/m as the specific conductivity for which $k \geqq 79.5 \times 10^{-10}$ for frequencies of 100 Mc/sec and lower. For the most severe limit (at 100 Mc/sec), we have a hypotenuse $\sqrt{79.5^2 + 7.083^2} \cong 79.8$, which is essentially equal to the long side of 79.5; the factor 10^{-10} is common to all sides of the vector triangle and of no concern. The phase angle is $\tan^{-1}(79.5/7.083) \cong -85$ deg. Hence, if we take about 80 Mc/sec, we can write for the propagation constant

$$\Gamma = j\omega\sqrt{\mu(\bar{\kappa} - j\bar{k})} \simeq j\omega\sqrt{-j\bar{k}\mu}.$$

But $-jk\mu$ under the square root means that we deal with a vector that is 90 space degrees behind the real axis direction, or $\Gamma = j\omega\sqrt{k\mu\,/\!-90 \text{ deg}} = j\omega\sqrt{k\mu}\,/\!-45$ deg, since the effect of the square root halves the angle. Therefore, the hypotenuse that is the resultant vector is, for

$$\sqrt{k\mu}\,/\!-45 \text{ deg},$$

along the -45-deg direction and can be written as $\epsilon^{-j45}\sqrt{k\mu}$ with equal components $\sqrt{k\mu}/\sqrt{2}$ and $-j\sqrt{k\mu}/\sqrt{2}$. Hence,

$$\Gamma = j\omega\sqrt{k\mu}\,\epsilon^{-j45^\circ} = j\omega(1-j)\sqrt{\frac{k\bar{\mu}}{2}} = (1+j)\sqrt{\frac{k\mu\omega^2}{2}}$$

$$= (1+j)\sqrt{\frac{(\sigma/\omega)\mu\omega^2}{2}} = (1+j)\sqrt{\frac{\sigma\mu\omega}{2}}$$

which checks the result of (20) and shows that for ocean water as a medium below 80 Mc/sec, we have $\Gamma = \alpha + j\alpha = \beta + j\beta$, since the attenuation constant α is numerically equal to the phase constant β and of value $\sqrt{\sigma\mu\omega/2}$. When, therefore, a plane electromagnetic wave passes perpendicularly into the ocean, we have a propagation factor $\epsilon^{-\alpha r}\epsilon^{j(\omega t - \alpha r)}$, where r is the distance in meters from the surface of the ocean to a field point perpendicular below it. If $r_0 = 1/\alpha$ denotes such a distance to a field point at which the electric and magnetic fields are attenuated to $1/\epsilon$ of their surface values (at $r = 0$), we have, for this reference penetration depth,

$$r_0 = \frac{1}{\alpha} = \frac{2}{\mu\sigma\omega} = \frac{5.68 \times 10^{-4}}{\sqrt{\mu\sigma f}} \tag{23}$$

if the frequency is expressed in megacycles per second. Since for ocean water $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m and taking $\sigma = 5$ mhos/m for a

80-Mc wave, we find a reference penetration of

$$r_0 = 5.68 \times 10^{-4}/\sqrt{1.257 \times 10^{-6} \times 5 \times 80} = 0.0251 \text{ m}$$

This is $0.0251 \times 3.281 = 0.0825$-ft penetration depth for an amplitude reduction to $1/\epsilon = 1/2.7183 = 0.367$ of the surface value. Hence, at such a depth the respective field values are only 36.7 per cent of the surface values. Inasmuch as at such a frequency the phase constant β is numerically equal to the attenuation constant α, we have for the phase velocity $v = \omega/\beta = \omega/\alpha = \omega r_0 = 6.28 \times 80 \times 10^6 \times 0.0251 = 0.1265 \times 10^8$ m/sec as against 3×10^8 m/sec in vacuum. This is about 1/23.7 of the velocity of light. Therefore, the wavelength in ocean water at 80 Mc/sec is $\lambda_0/23.7$ long, where λ_0 stands for the wavelength in empty space. But $\lambda_0 = {}^{300}\!/_{80} = 3.75$ m, and $\lambda_{\text{ocean}} = 3.75/23.7 = 0.158$ m $= 0.158 \times 3.281 = 0.52$ ft, which is very short indeed. For frequencies for which $\kappa_e \cong \kappa$, *i.e.*, jk negligible in comparison with κ, we have the phase velocity

$$c' = v \cong \frac{1}{\sqrt{\mu_0 80 \kappa_0}}; \qquad \frac{v}{c} = \frac{\sqrt{\mu_0 \kappa_0}}{\sqrt{\mu_0 80 \kappa_0}} = \frac{1}{\sqrt{80}}; \qquad v = \frac{c}{8.944} \quad \text{m/sec}$$

with a wavelength in ocean that is about one-ninth of the free-space wavelength, the exact value being

$$\lambda_{\text{ocean}} = \frac{\lambda_0}{8.944} = \frac{300/f}{8.944} = \frac{33.5}{f^{(\text{Mc})}} \quad \text{m} \tag{24}$$

This relation, according to speculations already made, applies then to frequencies above 10,000 Mc/sec in the case of suboceanic wave propagation.

It is now of interest to learn what portion of a wavelength can be useful for suboceanic intelligence at very high and ultrahigh frequencies. We found that for an 80-Mc wave excitation, we have an oceanic wavelength of 0.158 m. The reference penetration is, for such an excitation, only 0.0251 m. The oceanic propagation has, therefore, completed only $0.0251/0.158 = 0.159$ of its wavelength (of its cycle) before the field intensity has lost 63.4 per cent of the value it had at the surface of the ocean. It must also be realized that we assumed that the arriving electromagnetic wave enters the ocean normal to its surface. A portion of the arriving field power is, of course, reflected back at the surface.

We have so far discussed the conditions of the general medium, when the equivalent dielectric constant $\kappa_e = \kappa - j\sigma/\omega$ farads per meter is either essentially equal to κ or essentially equal only to $-j\sigma/\omega$. When both terms κ and $j\sigma/\omega$ play an active part, we have $\Gamma = \alpha + j\beta$ and find the important formulas

$$\left.\begin{aligned}
\Gamma &= j\omega\sqrt{\mu\kappa_e} = \alpha + j\beta \qquad \text{generalized hyperbolic radians/m}\\
Z_0 &= \sqrt{\frac{\mu}{\kappa_e}} = R_0 + jX_0 \qquad \text{ohms}\\
\alpha &= \sqrt{0.5\mu\omega(\sqrt{\sigma^2 + \kappa^2\omega^2} - \kappa\omega)} \qquad \text{nepers/m}\\
\beta &= \sqrt{0.5\mu\omega(\sqrt{\sigma^2 + \kappa^2\omega^2} + \kappa\omega)} \qquad \text{circular radians/m}\\
R_0 &= \frac{\beta}{\sqrt{\sigma^2 + \kappa^2\omega^2}} \qquad \text{ohms}\\
X_0 &= \frac{\alpha}{\sqrt{\sigma^2 + \kappa^2\omega^2}} \qquad \text{ohms}
\end{aligned}\right\} \tag{25}$$

These expressions have the general form of the expressions used for transmission lines. We can add to these expressions the formulas

$$f_e = 1.591 \times 10^{-7}\frac{\sigma}{\kappa} \quad \text{Mc/sec}; \qquad c' = \frac{\omega}{\beta} \quad \text{m/sec};$$
$$c'' = \frac{d\omega}{d\beta} \quad \text{m/sec} \tag{26}$$

where f_e is the frequency of a medium for which just as much conduction as displacement current flows and c' and c'' are the phase and group velocities, respectively. The concept of group velocity is limited as can be seen from Secs. 70–73A. Applications of (25) and (26) appear throughout this text. The f_e formula is based on the fact that the conduction current density is $\sigma\mathcal{E}$ amperes per square meter and the displacement current density $j\kappa\omega\mathcal{E}$ amperes per square meter. Equality then occurs for $\sigma = \omega\kappa = 6.28 \times 10^6 f\kappa$, where f is in megacycles per second. Taking $\sigma = 4$ mhos/m for a certain salinity of the ocean and $\kappa = 7.083 \times 10^{-10}$ farad/m, we find a frequency $f_e = 1.591 \times 10^{-7} \times 4/(7.083 \times 10^{-10})$ $= 900$ Mc/sec at which the ocean causes a displacement current density equal to the conduction current density. We have for $\omega\kappa > \sigma$ a general medium that behaves more like a dielectric (displacement current exceeds the conduction current); and for $\omega\kappa < \sigma$, the medium behaves more like a conductor. For an efficient wave medium, we need a quasi-dielectric condition $\omega\kappa \gg \sigma$ for the general medium. For average ground, we have $\kappa \cong 15\kappa_0 = 15 \times 8.854 \times 10^{-12} = 1.33 \times 10^{-10}$ farad/m and $\sigma \cong 5 \times 10^{-4}$ mho/m. The equality frequency of such earth crust then becomes $f_e = 1.591 \times 10^{-7} \times 5 \times 10^{-4}/(1.33 \times 10^{-10}) \cong 0.6$ Mc/sec. For a quasi-dielectric ($\omega\kappa/\sigma \gg 1$), we have essentially a phase velocity $v \cong 1/\sqrt{\mu\kappa}$ m/sec, which is independent of the operating frequency as long as μ and κ are not affected by f. We have then a nondispersive medium. For a pronounced conductive medium, we have the phase

velocity $v = \omega/\beta$ meters per second. We have for $k = \sigma/\omega \gg \kappa$ a phase velocity

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{1}{2}\mu\sigma\omega}} = A\sqrt{f}$$

for the proportionality factor $A = \sqrt{6.28/(\frac{1}{2}\mu\sigma)}$, and the medium becomes *dispersive* (depends upon the frequency f). Hence, for ocean water below 80 Mc/sec, for which such a condition prevails, distortion would take place in a modulated propagation, since the speed of propagation is different for different frequencies. Unless we go very much above 80 Mc/sec, we could not expect any distances, on account of the high attenuation effect. But we found that above 10,000 Mc/sec, the condition $k \ll \kappa$ prevails with essentially a fixed speed $v = 1/\sqrt{\mu\kappa}$ of propagation and no dispersion. A copper conductor represents a medium that may be called the extreme condition for $k \gg \kappa$. We find then $\sqrt{\mu\sigma f} = \sqrt{\mu_0\sigma f}$, which for a 100-Mc excitation yields

$$\sqrt{1.257 \times 10^{-6} \times 5.8 \times 10^{7} \times 100} = 85.4$$

According to (23), the reference penetration depth becomes $r_0 = 5.68 \times 10^{-4}/85.4 \cong 6.6 \times 10^{-6}$ m. For a medium of copper, formula (23) reads

$$r_0 = \frac{6.6 \times 10^{-5}}{\sqrt{f^{\mathrm{Mc}}}} \qquad \text{meters} \tag{27}$$

for the penetration that decreases respective field amplitudes by 63.4 per cent of the value at the surface of the copper medium. Hence, the reference penetration depth r_0 is inversionally proportional to $\sqrt{f}$, where f is in megacycles per second. The wavelength in copper is found from the velocity relation $v = f\lambda$ meters per second, for f in cycles per second and λ in meters. But $v = \omega/\beta = \omega/\alpha$ for $k \gg \kappa$ and $r_0 = 1/\alpha$, or

$$v = \omega r_0 = 6.28 \times 100 \times 10^6 \times 6.6 \times 10^{-6} = 4{,}150 \text{ m/sec}$$

This is, indeed, a very small velocity compared with 3×10^8 m/sec in free space and 0.1268×10^8 m/sec in ocean at 80 Mc/sec. Moreover, since we have $v = \omega/\alpha = B\sqrt{f}$ for the proportionality factor $B = 1/\sqrt{\mu\sigma/(4\pi)}$, we note that for a fixed conductive medium, the phase velocity is directly proportional to $\sqrt{f}$. The above evaluations show also that the wavelength in copper $\lambda = v/f = 4150/10^8 = 4.15 \times 10^{-5}$ m as against 3 m in air. Since the reference depth $r_0 = 6.6 \times 10^{-6}$ m, the ratio $6.6 \times 10^{-6}/(4.15 \times 10^{-5}) = 0.159$ shows that only 0.159 cycle occurs for the penetrating wave into copper before respective field amplitudes have lost 63.4 per cent of the values they had when just entering the copper medium normal to its surface. Therefore, only 15.9 per cent

of the wavelength in copper is the path length during such an amplitude decays.

12. Notes on the Quality (Electromagnetic Merit) of a Medium. According to Maxwell's equations, the total current density at any point of a medium is

$$\left.\begin{aligned} J_{\text{total}} &= \sigma\mathcal{E} + \kappa\frac{\partial\mathcal{E}}{\partial t} && \text{for any variation} \\ J_{\text{total}} &= \underbrace{\sigma\mathcal{E}}_{\substack{\text{conduction}\\ \text{component}}} + \underbrace{j\omega\kappa\mathcal{E}}_{\substack{\text{displacement}\\ \text{component}}} && \text{for a sinusoidal variation} \end{aligned}\right\} \text{amp/m}^2 \qquad (28)$$

The quality Q of a medium is the ratio of the displacement to the conduction current density, or

$$Q = \frac{\kappa\,\partial\mathcal{E}/\partial t}{\sigma\mathcal{E}} = \frac{\omega\kappa}{\sigma} \qquad (29)$$

where the frequency $\omega/6.28$ is in cycles per second, $\mathcal{E}$ in volts per meter, σ in mhos per meter, and κ in farads per meter. For ocean water, for a conductivity $\sigma = 5$ mhos/m and $\kappa = 7.083 \times 10^{-10}$ farad/m, we have at 100 Mc/sec a value $Q = 6.28 \times 10^8 \times 7.083 \times 10^{-10}/5 = 0.0885$. This is a very low value, since $Q = 1$ corresponds to the equality frequency f_e at which the medium is effectively just as much a conductor as it is a dielectric. At 1,000 Mc/sec, we find $Q = 0.885$, which shows that the conductive effect is still somewhat larger. Since $\mathcal{E} = \sqrt{2}\,\mathcal{E}_e$, where $\mathcal{E}_e$ denotes the effective electric field intensity and $\mathcal{E}$ its maximum value, we can write Q as

$$Q = \frac{j\omega\kappa}{\sigma} = \frac{j\omega\kappa(\mathcal{E}/\sqrt{2})}{\sigma(\mathcal{E}/\sqrt{2})} = \frac{j(J_d/\sqrt{2})}{J/\sqrt{2}} = \frac{\mathcal{E}/\sqrt{2}\,j(J_d/\sqrt{2})}{\mathcal{E}/\sqrt{2}\,J/\sqrt{2}}$$

$$= \frac{\text{volt}\cdot\text{amp/m}^3}{\text{watts dissipated/m}^3}$$

since $jJ_d/\sqrt{2}$ denotes the effective displacement current in amperes per square meter, which is 90 time degrees out of phase with respect to the driving effective electric field intensity $\mathcal{E}/\sqrt{2}$ volts per meter, and $J/\sqrt{2}$ amperes per square meter is the effective inphase conduction current driven by the same electric field intensity. We may also arrive at the merit factor Q of the general medium by the definition

$$Q = \frac{\omega(\text{field energy in joules stored/m}^3)}{\text{average watt loss/m}^3}$$

Now the average power loss per cubic meter is the specific conductivity in mhos per meter multiplied by the square of the effective field intensity $\mathcal{E}/\sqrt{2}$ in volts per meter, or $P_{\text{avg}} = \sigma\mathcal{E}^2/2$ watts. The energy stored in a

cubic meter must be equal to the maximum electric field energy in joules contained in a cubic meter, *i.e.*, equal to $\frac{1}{2}\kappa\mathcal{E}^2$ joules. Therefore,

$$Q = \frac{\omega\frac{1}{2}\kappa\mathcal{E}^2}{\sigma(\mathcal{E}^2/2)} = \frac{\omega\kappa}{\sigma}$$

confirming the result of (29) and found from Maxwell's equations (28). We can also use this result as follows:

$$Q = \frac{\omega\kappa}{\sigma} = \frac{\pi\kappa}{\sigma/2f} = \frac{\pi\kappa}{(T/2)\sigma} = \frac{2\pi(\kappa/2)}{(T/2)\sigma} = 2\pi\frac{(\kappa/2)(\mathcal{E}^2/2)}{(T/2)\sigma(\mathcal{E}^2/2)}$$
$$= 2\pi\frac{\text{field energy in joules/m}^3\text{ stored per half cycle}}{\text{watts dissipated/m}^3\text{ and per half cycle}}$$

since $T/2 = 1/2f$ denotes the half period in seconds. If we apply the Q concept to dry ground with $\kappa = 10 \times 8.854 \times 10^{-12} = 8.854 \times 10^{-11}$ farad/m and $\sigma = 10^{-4}$ mho/m, we find at 100 Mc/sec a value

$$Q = 6.28 \times 10^8 \times 8.854 \times 10^{-11} \times 10^4 = 553$$

and a value $Q = 5{,}530$ at 1,000 Mc/sec.

Many of the formulas used in the preceding sections can now be expressed to advantage in terms of $Q = \omega\kappa/\sigma$, $q = \sqrt{1 + 1/Q^2}$ and $r = 1 - j/Q$. We find then

$$\underbrace{Q = \frac{\omega\kappa}{\sigma}; \qquad q = \sqrt{1 + \frac{1}{Q^2}}; \qquad r = 1 - \frac{j}{Q}}_{\text{numeric abbreviations for } \sigma \text{ mhos/m, } \kappa \text{ farads/m, and } \omega = 6.28 \text{ cycles/sec}}$$

$$\left.\begin{aligned}
&Z_0 = \sqrt{\frac{\mu}{r\kappa}} = R_0 + jX_0 \quad \text{ohms}; \qquad \Gamma = j\omega\sqrt{r\mu\kappa} = \alpha + j\beta \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{generalized hyperbolic radians/m} \\
&R_0 = \sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{1+q}{2q^2}} \quad \text{ohms}; \qquad X_0 = \sqrt{\frac{\mu}{\kappa}}\frac{\sqrt{2/(1+q)}}{2\sqrt{1+Q^2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{reactive ohms} \\
&\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{1+q}} \quad \text{nepers/m}; \quad \beta = \omega\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{circular radians/m}
\end{aligned}\right\} \quad (30)$$

for the three-dimensional propagation constant Γ and the characteristic impedance Z_0 of the general medium σ mhos per meter, μ henrys per meter, and κ farads per meter. The quantity R_0 is the resistive (real) component, and X_0 the reactive component of Z_0, while α represents the attenuation constant, and β the phase constant of the medium. In each case, these expressions show the formula holding for a perfect dielectric

medium ($\sigma = 0$), since then $Q = \omega\kappa/\sigma = \omega\kappa/0 = \infty$, $q = \sqrt{1 + 0} = 1$, and $r = 1 - 0 = 1$, yielding $Z_0 = \sqrt{\mu/(r\kappa)} = \sqrt{\mu/\kappa}$ ohms and $\Gamma = j\omega\sqrt{\mu\kappa}$ generalized hyperbolic radians per meter (simplifies to circular radians per meter). The expressions are also written in a form that we can read off directly the phase velocity in a general medium, since

$$v = c' = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\kappa}\sqrt{(1+q)/2}} = \frac{1}{\sqrt{\mu\kappa}\sqrt{(1+q)/2}} \quad \text{m/sec} \tag{31}$$

with a corresponding wavelength of

$$\lambda_{\text{medium}} = \frac{10^{-6}}{f\sqrt{\mu\kappa}\sqrt{1 + q/2}} \quad \text{m/sec} \tag{32}$$

if the frequency is now expressed in megacycles per second. Suppose we again take ocean water and for a 100-Mc wave excitation. Let us again take such a salinity that $\sigma = 5$ mhos/m and $\kappa = 7.083 \times 10^{-10}$ farad m. We have already evaluated the Q value for such electromagnetic properties and found $Q = 0.0885$. According to (30), we have $q = \sqrt{1 + (1/Q)^2} = \sqrt{1 + 1/(78.32 \times 10^{-4})} = \sqrt{1 + 127.8} = 11.36$ and $(1 + q)/2 = 12.36/2 = 6.18$. Therefore, $\sqrt{(1 + q)/2} = \sqrt{6.18} = 2.49$. Since for ocean $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m, (32) yields

$$\lambda = \frac{10^{-6}}{100\sqrt{1.257 \times 10^{-6} \times 7.083 \times 10^{-10}} \times 2.49} = \frac{0.335}{2.49} = 0.135 \text{ m}$$

as against 0.335 m if the medium has zero conductivity (a perfect dielectric). According to (31), the velocity is the perfect dielectric velocity divided by 2.49. For dry ground, specified as $\sigma = 10^{-4}$ mho/m, $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m, and $\kappa = 8.854 \times 10^{-11}$ farad/m, we find, at 100 Mc/sec, a Q value of $\omega\kappa/\sigma = 553$, which means that

$$q = \sqrt{1 + \left(\frac{1}{Q^2}\right)} = \sqrt{1 + \frac{1}{305{,}809}} \cong 1$$

and $\sqrt{(1 + q)/2} = 1$, expressing that at 100 Mc/sec the dry ground behaves like a quasi-dielectric of electromagnetic constants $\mu = 1.257 \times 10^{-6}$ henry/m and $\kappa = 8.854 \times 10^{-11}$ farad/m, with a phase velocity and wavelength of

$$v = \frac{1}{\sqrt{1.257 \times 10^{-6} \times 8.854 \times 10^{-11}}\underbrace{\sqrt{(1+q)/2}}_{=1}} = \frac{10^8}{\sqrt{1.11}}$$

$$= \frac{10^8}{1.053} = 0.95 \times 10^8 \text{ m/sec}$$

$$\lambda = \frac{10^{-6}v}{f_{(\text{Mc/sec})}} = \frac{10^{-6} \times 0.95 \times 10^8}{100} = 0.95 \text{ m}$$

against 0.135 m in ocean and $300/100 = 3$ m in vacuum and practically

so in air. Now let us see what happens at a frequency f_c for which the general medium is just as much a dielectric as it is a conducting medium. For such a condition, we have $Q = 1$, $q = \sqrt{1 + (1/Q^2)} = \sqrt{2} = 1.4142$, $r = 1 - j$, and the following details

$$Z_0 = \sqrt{\frac{\mu}{r\kappa}} = \sqrt{\frac{\mu}{(1-j)\kappa}} = \sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{1+j}{2}} = \sqrt{\frac{\mu}{2\kappa}}\sqrt{\sqrt{2}\underline{/45^\circ}} = \sqrt{\frac{\mu}{\kappa}}\frac{\sqrt{1.4142}}{1.4142}\underline{/22.5^\circ}$$

$$= 0.843\sqrt{\frac{\mu}{\kappa}}\epsilon^{j22.5^\circ} = \sqrt{\frac{\mu}{\kappa}}(0.843\cdot\underbrace{\cos 22.5^\circ}_{0.9239} + j0.843\cdot\underbrace{\sin 22.5^\circ}_{0.3827})$$

$$= \underbrace{0.78\sqrt{\frac{\mu}{\kappa}}}_{R_0} + \underbrace{j0.323\sqrt{\frac{\mu}{\kappa}}}_{X_0} \qquad \text{ohms}$$

These results can also be found from the R_0 and X_0 formulas of (30), and we find

$$R_0 = \sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{1+q}{2q^2}} = \sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{1+\sqrt{2}}{2\times 2}} = \sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2.4142}{4}} = \sqrt{\frac{\mu}{\kappa}}\frac{1.552}{2}$$

$$= 0.776\sqrt{\frac{\mu}{\kappa}} \qquad \text{ohms}$$

$$X_0 = \sqrt{\frac{\mu}{\kappa}}\frac{\sqrt{2/(1+q)}}{2\sqrt{1+Q^2}} = \sqrt{\frac{\mu}{\kappa}}\frac{\sqrt{2/2.4142}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\mu}{\kappa}}\frac{1}{\sqrt{2.4142}} = \frac{\sqrt{\mu/\kappa}}{2\times 1.552}$$

$$= 0.323\sqrt{\frac{\mu}{\kappa}} \qquad \text{reactive ohms}$$

checking the results found above. For the attentuation constant α and phase constant β, we find, according to (30),

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{1+q}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{1+\sqrt{2}}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{2.4142}}$$

$$= 0.455\sigma\sqrt{\frac{\mu}{\kappa}} \qquad \text{nepers/sec}$$

$$\beta = \omega\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}} = \omega\sqrt{\mu\kappa}\sqrt{\frac{2.4142}{2}} = 1.1\omega\sqrt{\mu\kappa} \qquad \text{circular radians/m}$$

$$\Gamma = \alpha + j\beta = 0.455\sigma\sqrt{\frac{\mu}{\kappa}} + j1.1\omega\sqrt{\mu\kappa} \qquad \text{generalized hyperbolic radians/m}$$

This result can also be obtained from the Γ expression of (30), where $r = 1 - j/Q = 1 - j$ for $Q = 1$, and

$$\Gamma = j\omega\sqrt{r\mu\kappa} = j\omega\sqrt{(1-j)\mu\kappa} = j\omega\times\sqrt{\mu\kappa}\sqrt{1-j} = j\omega\sqrt{\mu\kappa}\sqrt{\sqrt{2}\underline{/-45^\circ}}$$

$$= j\omega\sqrt{\mu\kappa}\sqrt{1.4142}\underline{/-22.5^\circ} = j\omega\sqrt{\mu\kappa}\,1.187\epsilon^{-j22.5^\circ}$$

$$= j\omega\sqrt{\mu\kappa}\,1.187(\cos 22.5^\circ - j\sin 22.5^\circ)$$

$$= \omega\sqrt{\mu\kappa}(1.187\sin 22.5^\circ + j1.187\cos 22.5^\circ) = \underbrace{0.455\omega\sqrt{\mu\kappa}}_{\alpha} + \underbrace{j1.1\omega\sqrt{\mu\kappa}}_{\beta}$$

checking the above results, since for $Q = 1$, we have $\omega\kappa/\sigma = 1$, or $\omega = \sigma/\kappa$ and $\omega\sqrt{\mu\kappa} = (\sigma/\kappa)\sqrt{\mu\kappa} = \sigma\sqrt{\mu/\kappa}$ as used in the α term above. Since the phase

velocity $v = c' = \omega/\beta$, we find for $Q = 1$

$$v = \frac{1}{\sqrt{\mu\kappa}}\frac{1}{\sqrt{(1+q)/2}} = \frac{1}{\sqrt{\mu\kappa}}\frac{1}{\sqrt{(1+\sqrt{2})/2}} = \frac{1}{\sqrt{2.4142/2}}\frac{1}{\sqrt{\mu\kappa}} = \frac{1}{1.1\sqrt{\mu\kappa}}$$

$$= \frac{0.91}{\sqrt{\mu\kappa}} \quad \text{m/sec}$$

showing that we have a 9 per cent velocity reduction due to the conductive effect of the medium, since $1/\sqrt{\mu\kappa}$ would be the phase velocity if the pure dielectric alone were active.

13. Possible Order of Magnitude of the Complex Index of Refraction and the Effect on the Radiation Beam. The relative index of refraction of the earth crust is generally not real as in optics but complex. According to (21), the index of refraction is given by

$$n = \sqrt{\kappa_r - j\,\frac{1.8 \times 10^4\sigma}{f}} \qquad \text{numerical} \tag{33}$$

where κ_r denotes the numerical relative dielectric constant κ/κ_0 with respect to vacuum, σ the conductivity of the medium in mhos per meter, and f the operating frequency in megacycles per second

If we deal with two homogeneous mediums 1 and 2, as in Fig. 6, of conductivities σ in mhos per meter, permeabilities μ in henrys per meter, and the dielectric constants κ in farads per meter, radio propagation occurs along the path $ABCD$ if the planes of separation (boundaries) are parallel. The beam CD that emerges is, therefore, displaced with respect to the incident beam AB, while the transmitted beam BC, in medium 2, is refracted with respect to beam AB, which starts the propagation. This assumes that we deal with plane waves for which the electric and magnetic vectors form a plane wave front perpendicular to the direction of propagation. The parallel planes of such wave fronts are often called the "equiphase" planes. This does not necessarily mean that $\mathcal{E}$ volts per meter is in time phase with H amperes per meter. This is the case only when the medium exhibits zero conductivity ($\sigma = 0$). For instance, for a medium that is a quasi-conductor (essentially a conductor), the electric $\mathcal{E}$ and magnetic H vectors differ 45 deg in time phase; while for wave propagation through dry rocky earth crust, the fixed time phase angle is about 27 deg. Hence, when medium 1 represents air and a medium 2 is a mountain slab, the homing radio beam DC would be shifted with respect to the true direction (course) $D'BA$. Such ideal slabs normally do not occur often in nature, but what holds for the ideal case also holds for the general case if we deal with boundaries that are reasonably flat, even though not plane parallel as indicated in Fig. 7.

In step 10 of Table 44 it is shown that for two mediums 1 and 2,

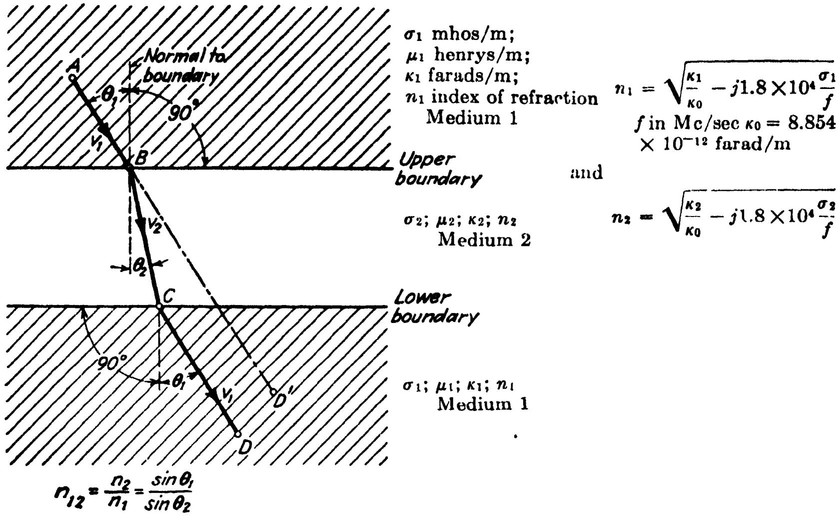


FIG. 6. Course of transmitted radio beams for plane parallel boundaries. AB is *incident* beam on the upper boundary, BC is the *refracted* beam in medium 2, and CD is the *refracted* beam below the lower boundary. BC and CD are also known as the transmitted beams of the primary incident beam AB. σ, μ, and κ are the electromagnetic properties of the general medium.

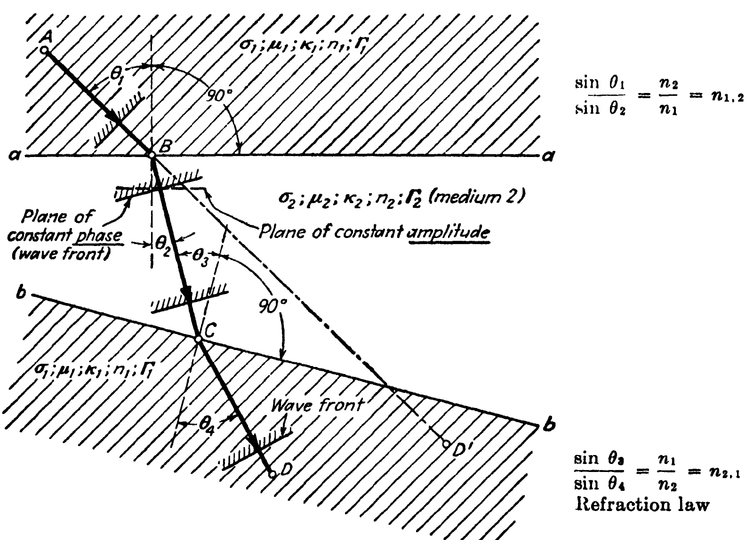


FIG. 7. Transmitted radio beams for plane boundaries which are *not* parallel. The wave front is a plane of constant phase (the plane of constant amplitude is parallel with the boundary). θ_1 and θ_3 are angles of wave *incidence* on boundaries *aa* and *bb*, respectively. θ_2 and θ_4 are angles of wave *refraction* on boundaries *aa* and *bb*, respectively. AB is primary incident beam; BC and CD are refracted (transmitted) beams.

as in Figs. 6 and 7, for the incident beam AB and the refracted beam BC, the "complex" refraction law plays a part. For respective propagation constants Γ_1 and Γ_2, we have then $\Gamma_1 \sin \theta_1 = \Gamma_2 \sin \theta_2$, instead of $v_2 \sin \theta_1 = v_1 \sin \theta_2$, where v_1 and v_2 stand for the phase velocities in mediums 1 and 2, respectively. Therefore, when n_1 and n_2 are the refractive indexes of medium 1 and medium 2, respectively, we have for the *mutual* index of refraction *from medium* 1 *into medium* 2, in Fig. 6,

$$n_{1\,2} = \frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} = n \tag{34}$$

This means that n is the relative index of refraction of medium 2 with respect to medium 1. This is evident from the following: For $\mu_1 = \mu_2 = \mu_0$, holding for nonferromagnetic mediums, we have the propagation constants

$$\Gamma_1 = j\omega \sqrt{\mu_0 \left(\kappa_1 - j\frac{\sigma_1}{\omega}\right)} \qquad \text{and} \qquad \Gamma_2 = j\omega \sqrt{\mu_0 \left(\kappa_2 - j\frac{\sigma_2}{\omega}\right)}$$

where $\omega/6.28$ is in cycles per second, σ in mhos per meter, and κ in farads per meter. Applying the generalized refraction law to wave propagation from medium 1 into medium 2, we find

$$n_{1,2} = \frac{\sin \theta_1}{\sin \theta_2} = \frac{\Gamma_2}{\Gamma_1} = \sqrt{\frac{\kappa_2 - j(\sigma_2/\omega)}{\kappa_1 - j(\sigma_1/\omega)}} = \sqrt{\frac{\sigma_2}{\sigma_1}}\sqrt{\frac{Q_2 - j}{Q_1 - j}}$$

for respective figures of merit $Q_1 = \omega\kappa_1/\sigma_1$ and $Q_2 = \omega\kappa_2/\sigma_2$. But the relative index of refraction n for the general medium (μ, κ, and σ effective), is its propagation constant divided by the propagation constant of empty space (only μ_0 and κ_0 effective); *i.e.*,

$$n = \frac{j\omega \sqrt{\mu\left(\kappa - j\frac{\sigma}{\omega}\right)}}{j\omega \sqrt{\mu_0\kappa_0}} = \sqrt{\frac{\kappa}{\kappa_0} - j\frac{\sigma}{\omega\kappa_0}}$$

$$= \sqrt{\kappa_r - j\frac{\sigma}{6.28 \times 10^6 f^{\mathrm{Mc}} \times 8.854 \times 10^{-12}}}$$

$$= \sqrt{\kappa_r - j\frac{18 \times 10^3 \sigma}{f^{\mathrm{Mc}}}}$$

since for a nonferromagnetic medium we have $\mu = \mu_0$. Taking a practical application, for which medium 1 is air and medium 2 is a portion of a mountain, we have for medium 1 essentially $\mu_1 = \mu_0$, $\kappa_1 = \kappa_0$, and $\sigma = 0$, yielding $\Gamma_1 = j\omega\sqrt{\mu_0\kappa_0}$, and the relative index of refraction of medium 1 is unity, since

$$n_1 = \frac{\Gamma_1}{\Gamma_0} = \frac{j\omega\sqrt{\mu_0\kappa_0}}{j\omega\sqrt{\mu_0\kappa_0}} = 1$$

We find then

$$n_2 = \sqrt{\kappa_r - j\frac{1.8 \times 10^4 \sigma_2}{f^{Mc}}} = n_{12}$$

since $n_{12} = n_2/n_1 = n$. Suppose that we take a medium 2, consisting

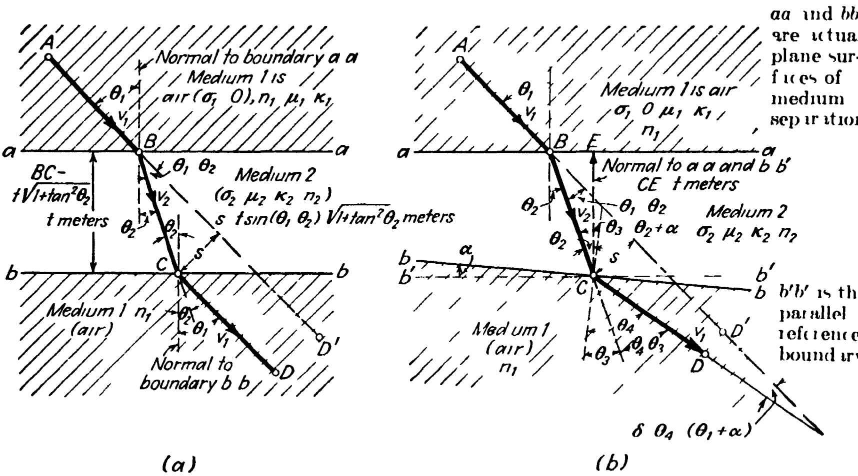


FIG. 8 Detailed analysis for boundaries *aa* and *bb* which are parallel and oblique planes. Propagation constant in general medium 2 is $\Gamma = j\omega\sqrt{\mu\kappa}\left(1 - j\frac{\sigma}{\omega}\right) = \alpha_2 + j\beta_2$ $= \alpha_2 + \frac{\omega}{v}$; α is the attenuation constant in nepers/m; β is the phase constant in circular radians/m. Propagation constant in medium 1 is $\Gamma_1 = j\omega\sqrt{\mu_1\kappa_1} = j\beta_1 = j\frac{\omega}{v_1}$, since $\alpha_1 = 0$.

Velocity of propagation $= v_1 = \frac{\omega}{\beta_1} = \frac{1}{\sqrt{\mu_1\kappa_1}}$ m/sec in medium 1 for μ_1 henrys/m and κ_1 farads/m.

Velocity of propagation $= v = \frac{\omega}{\beta_2} = \frac{1}{\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}}}$; $q = \sqrt{1 + \frac{1}{Q^2}}$; $Q = \frac{\omega\kappa}{\sigma_2}$; $\omega = 2\pi$ cycles/sec; σ in mhos/m for medium 2.

of rocky ground, for which $\kappa_r = \kappa/\kappa_0 = 5$ and $\sigma_2 = 10^{-5}$ mho/m and $\mu_2 = \mu_0$ if we deal with no iron deposits. We then find

$$n_2 = \sqrt{5 - j\frac{1.8 \times 10^4 \times 10^{-5}}{f}} = \sqrt{5 - j\frac{0.18}{f}}$$

For a 10-Mc wave, we have $n_2 = \sqrt{5 - j0.018/\underline{-2\psi}}$, where $2\psi = \tan^{-1}$ $(0.018/5) = \tan^{-1}(0.0036) = 0.2$ deg. Hence,

$$n_2 = \sqrt{25 + 0.000324}\ \angle{-0.1}\ \text{deg} = n_{12}$$

The mutual index of refraction from medium 1 into medium 2 is then

$$n_{1,2} = 2.24\epsilon^{-j0.1^\circ} = 2.24(\cos 0.1^\circ - j \sin 0.1^\circ) \cong 2.24$$

which is practically a quasi-real value. Since $\sin \theta_1/\sin \theta_2 = n_{1,2}$, we find, for instance, for an angle of incidence of $\theta_1 = 45$ deg an angle of refraction of

$$\theta_2 = \sin^{-1}\left(\frac{\sin 45^\circ}{2.24}\right) = \sin^{-1}\left(\frac{0.7071}{2.24}\right) = 18.4^\circ$$

With respect to Figs. 6 and 7, we have the geometry indicated in Fig. 8*a* and *b*. For parallel separation faces, we have $\sin \theta_1/\sin \theta_2 = 2.24$ for the radio beam *AB*, causing the continuation beam *BC* in medium 2. But the refracted beam *BC* is incident on boundary *bb* under an angle θ_2, and we have a continuation beam *CD*. The beam *CD* with respect to beam *BC* is based on the refraction law $\sin \theta_2/\sin \theta_1 = 1/2.24$. Hence, beam *CD* enters the medium below boundary *bb* again under the angle $\theta_1 = 45$ deg. Beam *CD*, even though along the same θ_1 direction, with respect to the normal to faces *aa* and *bb*, is side-shifted by a distance *s*. When the thickness of medium 2 is $t = 10$ ft, we find $s = 10 \sin (45 \text{ deg} - 18.4 \text{ deg}) \sqrt{1 + \tan^2 18.4 \text{ deg}} = 4.48 \sqrt{1 + 0.3327^2} = 4.71$ ft side shift. For no parallel faces (as in Fig. 8*b*), besides a side shift, we also have a bearing error, *i.e.*, an additional direction change. As far as the continuation of beam *BC* is concerned, for an angle of incidence $\theta_1 = 45$ deg, we obtain the same angle $\theta_2 = 18.4$ deg at 100 Mc/sec. The angle of incidence of the refracted beam *BC* on boundary *bb* is no longer equal to θ_2 but is $\theta_2 + \alpha$, where α is the angle that boundary *bb* makes with *aa*. Suppose this angle is 5 deg; then $\sin \theta_3/\sin \theta_4 = \sin (\theta_2 + \alpha)/\sin \theta_4 = n_{2,1} = 1/n_2$. But $\theta_2 + \alpha = 23.4$ deg and $n_2 = 2.24$ at 100 Mc/sec, yielding

$$\theta_4 = \sin^{-1} (2.24 \sin 23.4^\circ) = \sin^{-1} (0.885) = 62.3^\circ$$

From Fig. 8*b*, we note that the continued beam *BC* is off the course *AB* by an angle $-(\theta_1 - \theta_2)$ and the continued beam *CD* is off the course of beam *BC* by an angle $+(\theta_4 - \theta_3)$. Hence, the final beam *CD* is off the original *AB* direction by an angle $\delta = -(\theta_1 - \theta_2) + \theta_4 - \theta_3 = -\theta_1 + \theta_2 + \theta_4 - \theta_2 - \alpha = \theta_4 - (\theta_1 + \alpha) = 62.3 - 45 - 5 = 12.3$ deg. Besides, the course *CD* is side-shifted by an amount

$$s = t \sin (\theta_1 - \theta_2) \sqrt{1 + \tan^2 \theta_2}$$

if *t* stands for the normal from *C* on *aa* and *s* for the normal from *C* on the original incident beam direction *AB*. It is the direction shift that, in beacon work, leads to a bearing error. For the conditions computed, this error is far more pronounced at lower frequencies, say around 300

kc/sec, since the j term in (33) is then more prominent. This frequency range, however, is beyond the scope of this text.

If the inclination α had been only 3 deg steeper, *i.e.*, $\alpha = 8$ deg or even larger, we should have $\theta_3 = \theta_2 + \alpha = 18.4 + 8 = 26.4$ deg and

$$\theta_4 = \sin^{-1}(2.24 \sin 26.4°) = \sin^{-1}(2.24 \times 0.445) = \sin^{-1}(1.0)$$

which, when exceeded, *cannot* be a real angle θ_4, since the sine function can never give values beyond unity. A condition of this kind is treated in detail on pages 804, 805, and 817–820, for a medium that is electromagnetically denser for the incident beam BC than for the medium (air) into which a refracted beam can penetrate. This is also the case in this condition. We then have $\sin \theta_3 = (\sin \theta_4)/2.24$, such that $\theta_4 > 90$ deg, which is physically impossible, since for $\theta_4 = 90$ deg, we already have a tangential propagation for beam CD and beyond 90 deg, *total* reflection back into medium 2, in which medium the incident beam BC is. For such a condition, *no energy* can be propagated into the lower medium 1, although we still have an electromagnetic field in the lower medium, next to the boundary bb, since we can write

$$\cos \theta_4 = \pm j \sqrt{5 \sin^2 \theta_3 - 1}$$

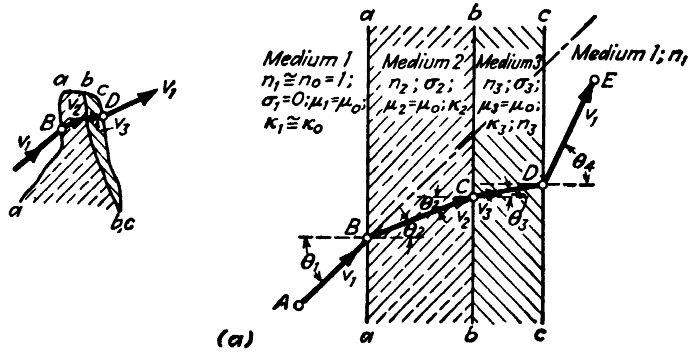
for imaginary angles θ_4. This is the case, because $\sin \theta_4 = n_2 \sin \theta_3$ and

$$\cos \theta_4 = \sqrt{1 - \sin^2 \theta_4} = j \sqrt{\sin^2 \theta_4 - 1} = j \sqrt{5 \sin^2 \theta_3 - 1}$$

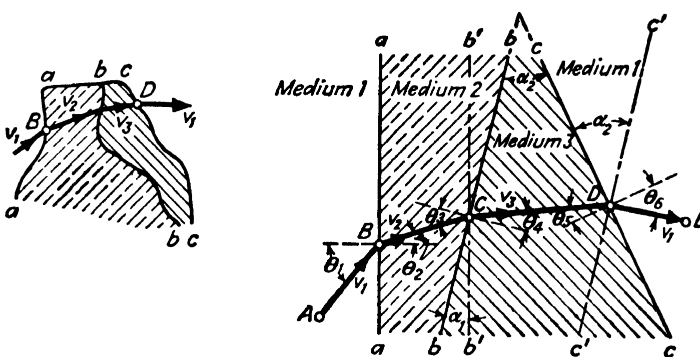
since $n_2^2 = 5$. We have then perpendicular to boundary bb a rapid exponential decay while excited along bb. This is an inhomogeneous electromagnetic wave, which is no longer transverse and of no practical importance.

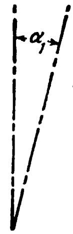
It is to be realized that at UHF's, even for air, we may have a gradient dn/dh of the index of refraction due to small variations in the dielectric constant κ, because of humidity and temperature variations along the direction r. This generally leads to a curved path through the lower atmosphere. It is treated in detail in Sec. 178.

We have also cases where, for instance, a beacon beam passes through a mountain ridge that is formed either by nearly parallel faces as in Fig. 9*a* or by inclined faces (Fig. 9*b*) that are flat enough to act as planes to short waves. Mediums to the left of boundary aa and to the right of boundary cc are then again ordinary atmosphere, for which we have essentially a real index of refraction $n_1 = n_4 = 1$. The medium between the boundary aa and bb may be dry rock, generally of complex index of refraction n_2 as found from Eq. (33) for a particular frequency. The medium between boundaries bb and cc may be sandy earth, which generally has a complex index of refraction. The case of Fig. 9*a* is, there-



$$\frac{\sin\theta_1}{\sin\theta_2} = n_2 \qquad \frac{\sin\theta_2}{\sin\theta_3} = \frac{n_3}{n_2} \qquad \frac{\sin\theta_3}{\sin\theta_4} = \frac{1}{n_3}$$





$$v_1 \cong \frac{1}{\sqrt{\mu_0\kappa_0}} = 3 \times 10^8 \text{ m/sec}$$

$$v_2 = \frac{1}{\sqrt{\mu_2\kappa_2}\sqrt{\frac{1+q}{2}}} \text{ m/sec}; \qquad q = \sqrt{1 + \frac{\sigma_2^2}{\omega^2\kappa_2^2}}$$

$$v_3 = \frac{1}{\sqrt{\mu_3\kappa_3}\sqrt{\frac{1+q}{2}}} \text{ m/sec}; \qquad q = \sqrt{1 + \frac{\sigma_3^2}{\omega^2\kappa_3^2}}$$

σ in mhos/m; $\mu_1 = \mu_0 = 1.257 \times 10^{-6}$ henry/m $= \mu_2 = \mu_3$
$\kappa_1 \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m; κ_2 and κ_3 in farads/m

$\theta_3 = \theta_2 + \alpha_1$; $\theta_5 = \theta_4 - \alpha_2$

$$\frac{\sin\theta_1}{\sin\theta_2} = n_2; \qquad \frac{\sin\theta_3}{\sin\theta_4} = \frac{n_3}{n_2}; \qquad \frac{\sin\theta_5}{\sin\theta_6} = \frac{1}{n_3}$$

FIG. 9. Path of an electromagnetic ray through mountain ridges. All θ angles give the direction of wave propagation with respect to the normal to the surface of medium separation. Refracted beam bends toward the normal when beam passes from a less dense into a denser medium.

fore, an extension of the condition described in connection with Fig. 8*a*, while the case of Fig. 9*b* is an extension of the analysis of Fig. 8*b*. Again applying the generalized refraction law, we find the relations shown in Fig. 9, which are different from Fig. 9*b* because of the inclinations α_1 and α_2, since angles θ_2 and θ_3 are no longer equal and θ_4 and θ_5 also are not equal. Even for the simple case of Fig. 9*a*, we get a direction error, because generally the beam *AB* is no longer parallel with beam *DE*, since the angle θ_3 of beam *CD* incident on boundary *cc* is no longer equal to θ_2 and n_3 is not equal to n_2. Hence, we generally have a side drift of the beam as well as a direction shift. Numerical computation is as for Fig. 8 and completely given by expressions in Fig. 9.

14. Energy Flow and Space Impedance in a Moving Electromagnetic Field. The bottom and rear side of an elementary volume $d\tau = dx\,dy\,dz$ expressed in cubic meters is shown in Fig. 10*a* for a moving electromagnetic plane wave of electric intensity $\mathcal{E}_0 \sin \beta(x - vt)$ along the $+X$ direction and the associated magnetic field intensity $H_0 \sin \beta(x - vt)$. Inasmuch as the $\mathcal{E}_y$ and H_z vectors are in the plane wave front and normal to the direction of propagation, $\mathcal{E}_y$ oscillates along the Y axis and H_z along the Z axis or vice versa. We know from the elementary theory that the energy stored in the dielectric of a condenser is $\frac{1}{2}Ce^2$ just as we have for an inductor the stored magnetic field energy $\frac{1}{2}Li^2$. Therefore, in an electromagnetic field, we have, in each unit volume, a storage of $\frac{1}{2}\kappa\mathcal{E}_y^2 + \frac{1}{2}\mu H_z^2$ joules. The total electromagnetic field energy in the space element $d\tau$ is then

$$dP = \tfrac{1}{2}(\kappa\mathcal{E}_y^2 + \mu H_z^2)\,dx\,dy\,dz \qquad \text{joules} \tag{35}$$

For unobstructed radiation space of electromagnetic properties μ henrys per meter, κ farads per meter, and zero specific conductivity ($\sigma = 0$ mhos per meter), we find from Sec. 25

$$\frac{\partial H_z}{\partial x} = -\kappa\frac{\partial \mathcal{E}_y}{\partial t} = v\beta\kappa\mathcal{E}_0 \cos \beta(x - vt) \tag{35a}$$

since for the conduction current density $J = \sigma\mathcal{E}$ amperes per square meter curl $H = J + \kappa\partial\mathcal{E}/\partial t$ leaves for the $\mathcal{E}_y$ and H_z polarizations shown in Fig. 10 only the component curl relation

$$\underbrace{\frac{\partial H_x}{\partial z}}_{=0} - \frac{\partial H_z}{\partial x} = \underbrace{\sigma\mathcal{E}_y}_{=0} + \kappa\frac{\partial \mathcal{E}_y}{\partial t} \qquad \text{or} \qquad \frac{\partial H_z}{\partial x} = -\kappa\frac{\partial \mathcal{E}_y}{\partial t}$$

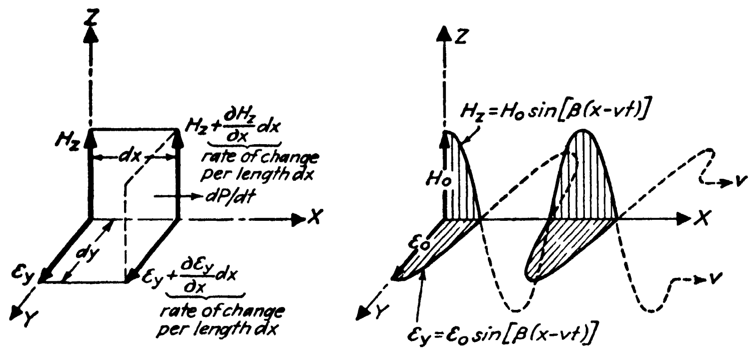
since $H_x = 0$ and $\sigma = 0$. Since $\beta = \omega/v$, we have, in (35), $v\beta\kappa = \omega\kappa$ and find H_z from

$$H_z = \omega\kappa\int \mathcal{E}_0 \cos \beta(x - vt)\,dx = \frac{\omega\kappa}{\beta}\mathcal{E}_0 \sin \beta(x - vt) = \sqrt{\frac{\kappa}{\mu}}\,\mathcal{E}_y \tag{36}$$

since $\omega\kappa/\beta = \omega\kappa v/\omega = \kappa v = \kappa/\sqrt{\mu\kappa} = \sqrt{\kappa/\mu} = 1/Z_0$ mhos, since Z_0 denotes the three-dimensional impedance $\mathcal{E}_y/H_z = Z_0$ ohms. Since for air $\mu \cong \mu_0 = 1.257 \times 10^{-6}$ henry/m and $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m, we find $Z_0 = \sqrt{\mu_0/\kappa_0} = 376.7$ ohms and $v \cong 1/\sqrt{\mu_0\kappa_0} = 2.998 \times 10^8$ m/sec $= c$. Using $H_z = \mathcal{E}_y \sqrt{\kappa/\mu}$, found in (36), in the energy relation (35), we have

$$dP = \frac{1}{2}(\kappa\mathcal{E}_y^2 + \mu H_z^2)\,d\tau = \frac{1}{2}\left(\kappa\mathcal{E}_y^2 + \mu\frac{\kappa}{\mu}\mathcal{E}_y^2\right)d\tau = \kappa\mathcal{E}^2\,d\tau \qquad \text{joules} \quad (37)$$

for the electromagnetic field energy contained in the volume element of $dx\,dy\,dz$ cubic meters. Inasmuch as in Fig. 10 the electromagnetic field



(a) (b)

$\frac{\partial H_z}{\partial x}$ and $\frac{\partial \mathcal{E}_y}{\partial x}$ are respective rates of change per unit length

$\beta = \frac{2\pi}{\lambda}$; $\lambda^{(m)} = \frac{10^{-6}}{\sqrt{\mu\kappa}\,f}$

f in Mc sec, μ henrys/m, κ farads/m, $\sigma = 0$.

For air $\lambda^{(m)} \cong \frac{300}{f^{(\mathrm{Mc})}}$

$v = \frac{1}{\sqrt{\mu\kappa}}$ m/sec

$\cong 3 \times 10^8$ m/sec for air

FIG. 10. Motion of a plane electromagnetic wave in a pure dielectric medium. Wave impedance $Z_0 = \mathcal{E}_y/H_z = \sqrt{\mu/\kappa}$ ohms $= 376.7$ ohms in air is purely resistive, and electric and magnetic fields are in time phase and space quadrature. If the medium exhibits also a certain conductivity, we have no longer inphase condition, since Z_0 becomes complex.

moves along the positive X direction, the quantity dP/dt must represent the rate at which the field energy streams across the end face $dy\,dz$ of the parallelepiped $dx\,dy\,dz$. The power flow per unit area is known as the Poynting vector p, and the instantaneous power flow becomes

$$p = \frac{\partial P/\partial t}{dy\,dz} = \frac{[d(\kappa\mathcal{E}_y^2)/dt]\,dx\,dy\,dz}{dy\,dz} = \kappa\mathcal{E}_y^2\frac{dx}{dt} = v\kappa\mathcal{E}^2 = \frac{\kappa}{\sqrt{\mu\kappa}}\mathcal{E}_y^2$$
$$= \sqrt{\frac{\kappa}{\mu}}\,\mathcal{E}_y^2 = \mathcal{E}_y\sqrt{\frac{\kappa}{\mu}}\,\mathcal{E}_y = \mathcal{E}_y \cdot H_z$$

The average power flow normal across unit area then becomes

$$p_{\text{avg}} = \frac{1}{T}\int_0^T p\,dt = \frac{1}{T}\int_0^T \mathcal{E}_y \cdot H_z\,dt = \frac{1}{T}\int_0^T \mathcal{E}_0 H_0 \sin^2 \beta(x - vt)$$

$$= \mathcal{E}_0 H_0 \underbrace{[\sin^2 \beta(x - vt)]_{\text{avg}}}_{=\frac{1}{2}} = \frac{\mathcal{E}_0 H_0}{2} \quad \text{watts/m}^2 \tag{38}$$

This result can also be written as $(\mathcal{E}_0/\sqrt{2})(H_0/\sqrt{2}) = \mathcal{E}_e \cdot H_e$ watts per square meter, since $\mathcal{E}_e$ and H_e represent the effective field associations in the plane of the wave front and normal to the direction of wave propagation.

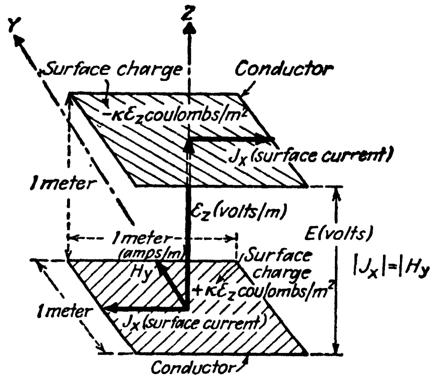


FIG. 11. Relations between surface current and magnetic field and surface charge and electric field. μ is in henrys/m, and κ farads/m is the medium between the square-meter metal plates ($\sigma = 0$ for medium).

15. Relations between Magnetic Field Intensity and Surface Current, Electric Field Intensity, and Surface Charge and Reasons for the Henry per Meter and Farad per Meter Units for the Permeability and Dielectric Constant of a Medium. It is necessary to have an understanding of the relations between charges on conductors and the electric field and currents along conductors and the magnetic field. This is of importance when dealing, for instance, with the principal wave along customary transmission lines. In Fig. 11 we have a cubical space such that the top and bottom boundaries, each of 1 sq m area, are perfect conducting surfaces (or at least quasi-perfect). The cubic meter denotes a perfect dielectric of μ henrys per meter and κ farads per meter. Perfect conductivity of the above-mentioned surfaces means $\sigma = \infty$ mhos per meter. We must now meet boundary conditions, since the shaded parallel surfaces are perfect conductors. The boundary conditions require that we have *continuity* for the *normal B, i.e.,* the μH, component and the *tangential* $\mathcal{E}$ component, but *discontinuity* for the *normal* $\mathcal{E}$ component and the

tangential B component. Equations (1) and (2) for the dissipative and pure dielectric medium hold *only* for field points where the fields cannot change abruptly (discontinuously). But at the shaded top and bottom surfaces of Fig. 11, we have discontinuities, since inside a perfect conductor all $\mathcal{E}$ components as well as all H components must be zero. The electric field $\mathcal{E}_z$ in Fig. 11 is along the positive Z axis. There are no tangential $\mathcal{E}$ components just outside respective metal surfaces and therefore also no such tangential $\mathcal{E}$ fields inside the conductor. The normal μH vector to the conductor surfaces is likewise zero outside as well as inside the conductor. It is the discontinuities that play a part. The tangential magnetic field H_y next to the inside surface of the conductor but in the dielectric changes to a zero tangential magnetic field just inside the conductor, which means that the discontinuity of H_y amperes per meter must be equal to the surface current density. Normal to the conducting surfaces, we have on the dielectric side of the surface an electric flux density $\kappa\mathcal{E}_z$ coulombs per square meter; inside the conductor there is zero electric flux density. Consequently, the discontinuity of this normal flux density at the surface of the conductor must be equal to the surface charge density of $\kappa\mathcal{E}_z$ coulombs per square meter. Inasmuch as H_y is equal to the surface current in amperes per meter and H_y acts along a width of 1 m, the H_y field expresses directly the number of amperes in the conductive layers.

It is now of interest to account for the direction of current flow. Since the electric field is $\mathcal{E}_z$ and the associated perpendicular magnetic field is H_y, the propagation of this plane wave front is normal to the ZY direction, *i.e.*, along the $+X$ direction, and occurs with a velocity $v = 1/\sqrt{\mu\kappa}$ meters per second. For the instantaneous polarities indicated in Fig. 11, the surface charge on the inside skin of the upper conductor is $-\kappa\mathcal{E}_z$ coulombs per square meter, while it is $+\kappa\mathcal{E}_z$ coulombs per square meter on the inside skin of the bottom conductor. Hence, $\kappa\mathcal{E}_z$ coulombs per square meter move, as does the $\mathcal{E}_z$, H_y wave front, along the $+X$ direction with $1/\sqrt{\mu\kappa}$ meters per second and constitute the current flow

$$I = \kappa\mathcal{E}_z \frac{1}{\sqrt{\mu\kappa}} = \mathcal{E}_z \sqrt{\frac{\kappa}{\mu}} = |H_y| \qquad \text{amp} \tag{39}$$

This result, at first sight, may look inconsistent as far as the units of H_y and I are concerned, since H_y is in amperes per meter. But in Fig. 11, we note that I acts over a conductor width of *one meter*, *i.e.*, $H_y \cdot 1$ m is a measure in amperes. In the same way, it is evident that the potential difference between the upper and lower conductive layer is $\mathcal{E}$ volts per meter times the separation of 1 m between these layers, resulting in

E volts $= |\mathcal{E}|$. Therefore, for the cubical medium of Fig. 11, we have across the conductive square-meter surfaces, a voltage of E volts equal to the numerical value $|\mathcal{E}_z|$ of the acting field intensity $\mathcal{E}_z$ in volts per meter, just as the current flow of I amperes is equal to the numerical value $|H_y|$ of the magnetic field intensity H_y in amperes per meter. Therefore, $\mathcal{E}_z$ measures the transverse voltage and H_y the current flow directly.

Inasmuch as in Fig. 11 we have a section of a transmission line that is 1 m in length, the effective area of the "go" and the "return" conductor is 1 sq m with 1-cu-m field space between the conductive layers. The charge on each conductive layer is $\pm\kappa\mathcal{E}_z$ coulombs with a voltage of $|\mathcal{E}_z|$ volts across the layers, as far as the numerical value is concerned. The charge of a condenser with terminal voltage of $|\mathcal{E}_z|$ volts and a capacity of C farads is $C\mathcal{E}_z = \kappa\mathcal{E}_z$ coulombs. This means that the unit of the dielectric constant κ must be equal to the unit of C. In reality, C is the capacitance in farads per meter, since the condenser is a section of a uniform line *one meter* in length Hence, κ is also in farads per meter. In exactly the same way, we can relate the inductance L in henrys per meter of the line that is 1 m in length to the permeability μ in henrys per meter. We have the well-known relation $e = L\,di/dt$ for the induced voltage e volts across a self-inductance of L henrys. Hence, for 1 m length of our line, the inductance is $L = e/(di/dt)$ in henrys per meter. This means that L is equal to the induced voltage when the current changes at unit rate, and 1 henry/m, in our line case, would be 1 volt induced by a change of 1 amp/sec. In our case, the induced voltage is $\partial(\mu H_y)/\partial t$ (time rate of magnetic flux), and the current is numerically equal to H_y. Therefore,

$$L^{(\text{henrys/m})} = \frac{e}{di/dt} = \frac{\mu(\partial H_y/\partial t)}{\partial H_y/\partial t} = \mu \qquad \text{henrys/m}$$

16. Energy Flow, Radiation Resistance, and Characteristic Line Impedance of a Guided Transverse Electromagnetic Wave. The dominant principal TEM wave can exist *only* along a transmission line with a go and return conductor. It is a TEM wave, because both the electric $\mathcal{E}$ vector and the magnetic H vector are transverse to the direction of wave propagation. The TEM wave has the inherent advantage that it is not limited by a cutoff frequency below which field energy transfer is no longer possible. For a very long ribbon transmission line, as in Fig. 12a and c, of l meters length and a width of w meters (large in comparison with the spacing s), we have indicated $\mathcal{E}$ and H lines of force. A uniform $\mathcal{E}$ field is obtained particularly when the guard lines of Fig. 12c are also used. When we compare relations of the classic transmission-

line theory with relations based on Maxwell's equations, we find for perfect conductors ($\sigma = \infty$ mhos per meter)

$$\left.\begin{array}{ll} \dfrac{d\mathcal{E}}{dl} = -j\omega\mu H \quad \text{volts/m}^2; & \dfrac{dH}{dl} = -\sigma\mathcal{E} - j\omega\kappa\mathcal{E} \quad \text{amp/m}^2 \\ & \qquad \text{from Maxwell's equations} \\ \dfrac{dE}{dl} = -j\omega LI \quad \text{volts/m}; & \dfrac{dI}{dl} = -GE - j\omega CE \quad \text{amp/m} \\ & \text{customary transmission-line equations} \end{array}\right\} \tag{40}$$

for the field vectors shown in Fig. 12, since for perfect or nearly perfect conductors, the $\mathcal{E}$ lines are normal to the conductive ribbons and the

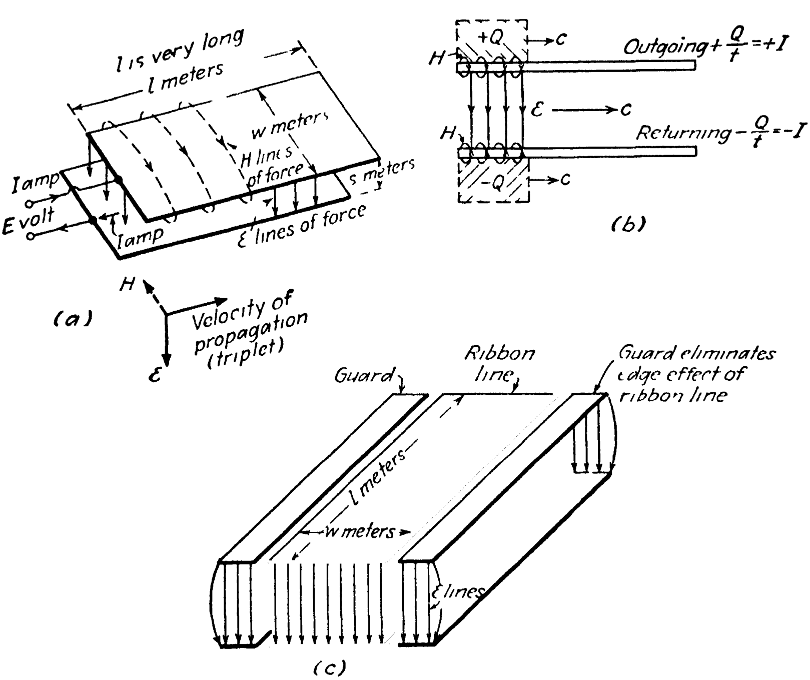


FIG. 12. A long ribbon transmission line of infinite conductivity embedded in an ideal dielectric.

resistance R of the line is practically zero, or at least negligible compared with the reactive component ωL along the go and return per meter length. The quantity dE/dl is the voltage drop per meter length, and the quantity dI/dl the current decrease per unit length, since $(GE + j\omega CE)$ is the current flow across the ribbons (from ribbon to ribbon) per meter length. The quantities E and I are the transverse voltage and longi-

tudinal ribbon current, and L, C, and G denote the inductance, capacitance, and leakance per meter length, as used in the classic transmission-line theory. The quantity $d\mathcal{E}/dl$ is the decrease of the electric field intensity per meter length along the length direction, which is the direction of the TEM wave propagation. The quantity dH/dl is the decrease in the associated magnetic field. For the sake of generality, it is assumed that the medium about the ribbons is general; *i.e.*, it exhibits, besides μ henrys per meter and κ farads per meter, also σ mhos per meter. The latter electromagnetic constant corresponds to the leakance G in mhos per meter length acting across the conductors in the case of a transmission line.

In the last section, we found that the absolute value of the magnetic field vector H, if oriented tangentially to the conductor surface, expresses directly the current flow per meter width. Therefore, for w meters width, the total current flow in the ribbon is w times the H value. We also found that the absolute value of the electric field vector $\mathcal{E}$ measures the transverse voltage E if the separation is 1 m. Therefore, for s meters separation, the transverse voltage is s times the value of $\mathcal{E}$. Hence

$$I = wH \quad \text{amp} \quad \text{and} \quad E = s\mathcal{E} \quad \text{volts} \tag{41}$$

Utilizing these expressions in (40) leads to the solution

$$\frac{dI}{dl} = w\frac{dH}{dl} \quad \text{and} \quad \frac{dE}{dl} = s\frac{d\mathcal{E}}{dl}$$

therefore

$$-j\omega LI = \frac{dE}{dl} = s\frac{d\mathcal{E}}{dl} = -js\omega\mu H = -js\omega\mu\frac{I}{w}$$

Therefore, $L = (s/w)\mu$ henrys per meter length of the go and the return conductor. We also find

$$-(G + j\omega C)E = \frac{dI}{dl} = w\frac{dH}{dl} = -w(\sigma + j\omega\kappa)\mathcal{E} = -w(\sigma + j\omega\kappa)\frac{E}{s}$$

Therefore, $G + j\omega C = (w/s)(\sigma + j\omega\kappa)$. From transmission-line theory, we know that the characteristic impedance is

$$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)} \text{ ohms}$$

and the propagation constant $\Gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ generalized hyperbolic radians per meter. In our case, we have assumed practically perfect conductivity for the conducting ribbons; therefore, $R = 0$ or at least is negligible in comparison with the reactance ωL ohms (no doubt so at VHF's and UHF's), and

$$Z_0 = \sqrt{\frac{j\omega L}{G + j\omega C}} = \sqrt{\frac{j\omega(s/w)\mu}{(w/s)(\sigma + j\omega\kappa)}} = \sqrt{\frac{j\omega\mu s^2}{(\sigma + j\omega\kappa)w^2}}$$
$$= \frac{s}{w}\sqrt{\frac{\mu}{(\sigma/j\omega) + j(\omega\kappa/j\omega)}} = \frac{s}{w}\sqrt{\frac{\mu}{\kappa - j(\sigma/\omega)}} = \frac{s}{w}\sqrt{\frac{\mu}{\kappa_c}} \quad \text{complex ohms} \quad (42)$$

which, compared with the three-dimensional characteristic impedance of Eq. (13) holding for unobstructed radiation space (no guiding metal surfaces present), shows that for guided TEM waves the characteristic wave impedance based on the classic $j\omega L/(G + j\omega C)$ ratio is (s/w) times the value of the unguided wave impedance. For the propagation constant, we have

$$\Gamma = \sqrt{j\omega L(G + j\omega C)} = \sqrt{j\omega \frac{s}{w}\mu(\sigma + j\omega\kappa)\frac{w}{s}} = \sqrt{j\omega\mu(\sigma + j\omega\kappa)}$$
$$= j\omega\sqrt{\frac{\mu}{j\omega}(\sigma + j\omega\kappa)} = j\omega\sqrt{\mu\left(\frac{\sigma}{j\omega} + \kappa\right)} = j\omega\sqrt{\mu\left(\kappa - j\frac{\sigma}{\omega}\right)}$$
$$= j\omega\sqrt{\mu\kappa_c} \quad \text{generalized hyperbolic radians/m} \quad (43)$$

which is exactly the same as for the unguided three-dimensional propagation constant of Eq. (18). We have, therefore, the relations

$$\left.\begin{array}{llll}
\mu = \dfrac{w}{s}L & \text{henrys/m}; & \kappa = \dfrac{s}{w}C & \text{farads/m}; \\
 & & \sigma = \dfrac{s}{w}G & \text{mhos/m} \\
Z_0 = \dfrac{s}{w}\sqrt{\dfrac{\mu}{\kappa_c}} & \text{complex ohms} & \multicolumn{2}{l}{\text{for a } \textit{dissipative} \text{ dielectric between the conducting ribbons of width } w \text{ and spacing } s} \\
Z_0 = \dfrac{s}{w}\sqrt{\dfrac{\mu}{\kappa}} & \text{ohms} & \multicolumn{2}{l}{\text{for a } \textit{perfect} \text{ dielectric } (\sigma = 0 \text{ mhos/m})} \\
\Gamma = j\omega\sqrt{\mu\kappa_c} & \text{complex radians/m} & \multicolumn{2}{l}{\text{for a } \textit{dissipative} \text{ dielectric}} \\
\Gamma = j\omega\sqrt{\mu\kappa} & \text{circular radians/m} & \multicolumn{2}{l}{\text{for a } \textit{perfect} \text{ dielectric}} \\
\kappa_c = \kappa - j\dfrac{\sigma}{\omega} & \text{farads/m} & & \\
\omega = 6.28 \text{ cycles/sec} & & &
\end{array}\right\} \quad (44)$$

Usually, we have air as a dielectric, for which $\sqrt{\mu/\kappa} \cong \sqrt{\mu_0/\kappa_0} \cong 377$ ohms and $\Gamma = j\omega\sqrt{\mu_0\kappa_0} = j\omega/c = j\omega \times 10^{-8}/3 = j2.093 \times 10^{-2}f^{(\mathrm{Mc})}$ circular radians per meter. Hence, when the ribbon conductors are 1 m wide ($w = 1$) and spaced 10 cm ($s = 0.1$ m), then $s/w = 0.1/1$, and the wave impedance is only $Z_0 = 0.1 \times 377 = 37.7$ ohms instead of 377

ohms in the case of unbounded (unguided) space if the classic Z_0 concept of the transmission-line theory is used.

It may now be of interest to learn what we can obtain by using the elementary line theory. Suppose we have air as a dielectric between the conducting ribbons of Fig. 12 and deal with quasi-perfect conductivity for the ribbons. For such conditions, we have practically no losses, and a sudden applied direct voltage causes (Fig. 12*b*) a voltage pulse to travel with the velocity $v = 1/\sqrt{\mu_0\kappa_0} = c = 3 \times 10^8$ m/sec toward the far end without any return from this end, since we assume a very long ribbon line. Figure 12*b* indicates that a positive charge of $+Q$ coulombs and a negative charge of $-Q$ coulombs move with the above-mentioned velocity outward, thus causing the corresponding transient current flows $+I$ and $-I$ amperes in the respective conductors. Associated with these charges are the effective electric $\mathcal{E}$ and magnetic H fields, which move also with the velocity $c = 3 \times 10^8$ m/sec for an ideal guide of this type (without losses). We have then $\mathcal{E} = E/s$ volts per meter and $H = I/w$ amperes per meter. Since E is in volts and I in amperes, the power transferred is $P = EI$ joules per second; *i.e.*,

$$P = sw\mathcal{E}H \qquad \text{watts} \tag{45}$$

The Poynting radiation vector is, therefore, $p = \mathcal{E}H$ watts per square meter. This shows that the entire useful power transfer occurs in the $\mathcal{E}$, H field, and *not* in the conductors. For conductors that are not ideal and exhibit joulean heat losses, *only that portion of energy is in the conductor which is dissipated* as heat. The motion of the Q charges at first causes a magnetic H field, which glides along the direction of propagation along the line, inducing an electromotive force of $-\mu_0 Hcs$ volts across the conductors. The minus sign expresses the experimental fact that the induced emf is always in such a direction as to oppose the motion that causes it. This induced voltage must be counterbalanced by $E = +\mu_0 Hcs$ volts, causing in turn a gliding electric $\mathcal{E}$ field, given by $\mathcal{E} = E/s$ volts per meter. Therefore,

$$\mathcal{E} = \frac{E}{s} = \frac{\mu_0 Hcs}{s} = \mu_0 cH = \frac{\mu_0}{\sqrt{\mu_0\kappa_0}} H = \sqrt{\frac{\mu_0}{\kappa_0}} H \cong 377H \qquad \text{volts/m}$$

which result proves the well-known relation $\mathcal{E}/H \cong 377$ ohms. According to this outcome, we have the Poynting vector as $p = \mathcal{E}^2/377$ watts/m^2 or equal to $377H^2$ watts per square meter. The total power transferred requires the multiplication of p by the cross section of sw square meters, since this is the cross section for the $\mathcal{E}$, H path. The total power then becomes

$$P = swp = \frac{\mathcal{E}^2}{377} sw = \frac{E^2}{377s^2} sw = \frac{1}{377} \frac{w}{s} E^2 = \frac{E^2}{R_r} \qquad \text{watts} \tag{46}$$

for the radiation resistance R_r of the guided energy along and between the conducting ribbons. Hence,

$$R_r = 377 \frac{s}{w} \qquad \text{ohms} \tag{47}$$

in comparison with the value of 377 ohms for the wave impedance of the unguided medium. The conducting ribbons of Fig. 12*a* act like a long coil, of length w and winding area of sl square meters with one turn. The total magnetic flux Φ in webers is the magnetic flux density $B = \mu_0 H$ in webers per square meter multiplied by the cross section of sl square meters, or $\Phi = lsB = ls\mu_0 H$ webers. Since the self-induction in henrys is the total flux Φ per ampere, *i.e.*, $L_t = \Phi/I$, and H amperes per meter $= I/w$, we find for the total self-induction of the ribbon line

$$L_t = \frac{\Phi}{I} = \frac{ls\mu_0 H}{I} = \frac{ls\mu_0(I/w)}{I} = \mu_0 \frac{ls}{w} \qquad \text{henrys} \tag{48}$$

which confirms the result already found, since according to (44) and the derivations after (41), the self-induction per meter length of the ribbon line is $L = (s/w)\mu_0$ henrys for air as a field medium, and for a length l, it is $L_t = lL$ henrys, and as found above [Eq. (48)]. The capacitance of the entire ribbon double line is proportional to the effective area of wl square meters of either long and rectangular condenser layer and inversely proportional to the separation s in meters. The exact formula is readily found by the reasoning already discussed in the last section. Since we have a uniform $\mathcal{E}$ field across and between the conductive layers of wl square meters and discontinuities at respective inner metal surfaces of the ribbons, the electric flux density of $\kappa\mathcal{E}$ coulombs per square meter is stopped at the inner surfaces and causes, in turn, a surface charge of an equal amount, *i.e.*, $\kappa\mathcal{E}$ coulombs per square meter. Hence, across each square meter of condenser plate area, a charge of $\kappa\mathcal{E}$ coulombs is produced. But according to the elementary field theory, the charge of a condenser is all the larger, the larger its capacitance in farads and the larger the voltage E across the plates, or $q = CE$ coulombs. Since $\kappa\mathcal{E}$ coulombs is the charge per square meter, we deal now with a condenser of capacitance C_0 farads with an effective area of 1 sq m and separated by s meters. Therefore, $q = \kappa\mathcal{E}$ coulombs $= C_0 E$, if E is in volts, and

$$C_0 = \frac{\kappa\mathcal{E}}{E} = \frac{\kappa(E/s)}{E} = \frac{\kappa}{s} \qquad \text{farads} \qquad \text{for square-meter condenser plates separated } s \text{ m}$$

The total capacitance C_t of the ribbon line is wl times larger, because the area is wl square meters instead of 1 sq m. Hence,

$$C_t = \kappa_0 \frac{wl}{s} \qquad \text{farads} \tag{49}$$

since for air as the pure dielectric, we have $\kappa \cong \kappa_0$ farads per meter. This outcome, with the elementary theory, also confirms the κ formula of (44), based on the derivations after (41), since the result $C = (w/s)\kappa$ of (44) represents the capacitance of 1 m length of the ribbon line and, for the length of l meters, we have to multiply the value for C simply by l and check the result of (49).

Since the characteristic line impedance, with air as the dielectric, is $Z_0 = \sqrt{L/C} = \sqrt{lL/lC} = \sqrt{L_t/C_t}$ ohms, we find

$$Z_0 = \sqrt{\frac{L_t}{C_t}} = \sqrt{\frac{\mu_0(ls/w)}{\kappa_0(wl/s)}} = \frac{s}{w}\sqrt{\frac{\mu_0}{\kappa_0}} \cong 377\frac{s}{w} = R_r \qquad \text{ohms} \tag{50}$$

Hence, the customary surge impedance, as obtained from the classic transmission-line theory, is equal to the ribbon-guided *radiation resistance*. The field energy that is radiated is "*trapped*" in the space between the ribbons and should *not* be confused with field energy *escape* into the far distant region. It was assumed that the length l is so long that when looking into the ribbon line, we face the classic Z_0 value defined in (50). The result $(s/w)\sqrt{\mu_0/\kappa_0}$ ohms of (50) also confirms the relation for Z_0 given in (44) if we take air as the dielectric in which the conductive ribbons are embedded.

If the width w is not large compared with the spacing s or side guards are not used, the relation $P = psw$ no longer holds, since a nonuniform field will then exist. The relation

$$P = \int dP = \int p\,d(sw) \tag{51}$$

must then be used.

17. Poynting Vector and Radiation from a Conical Antenna. The method just applied for securing trapped radiation energy due to an electromagnetic transverse field can be used for space radiation if we change the double ribbon line to a *single* conical metal surface with a plane metal surface as the return. We have then the conditions shown in Fig. 13. We can also use a double cone (Fig. 14) with a voltage E applied between the apex of each cone. Let us assume again in Fig. 13 that the conical metal surface extends to infinity as well as the metal plane and that we have again quasi-perfect conductivity. An instantaneous impressed voltage of E volts then sends a current $I = I_0\epsilon^{j\omega t}$ amperes into the double line with unlike go and return conductor configurations. We note that the electric $\mathcal{E}$ lines as well as the magnetic H lines are now circles. We see, therefore, that the principal wave is capable of satisfying the boundary conditions along the go and the return

conductors. *Inasmuch as the conducting surfaces are assumed to go to* infinity, only the outgoing principal wave can exist, since no wave train can be reflected back to the source of voltage E. If the conducting surfaces does not go to infinity (or at least far enough to make the arriving energy at the far end negligible), matters are more involved, since the conductors are, so to speak, abruptly stopped and with no current flow at the very ends. We have then also a returning wave train, since reflection takes place at the far ends. The end effect must then also be taken care of, since the outgoing and returning spherical principal waves cannot take care of the nonuniformity at the end, *i.e.*, the end effect. This can *be done only by other types of waves* in addition to the principal

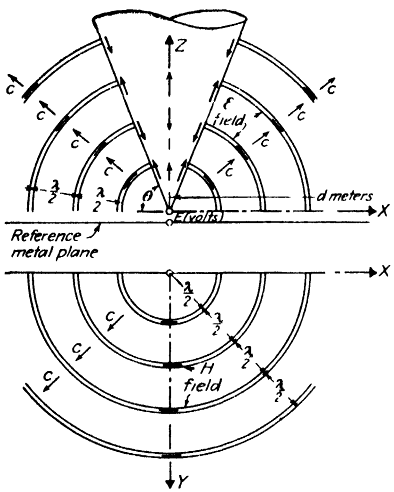


FIG. 13. Circular ℰ lines in meridian planes (such as the ZX plane) and circular H lines in planes parallel with the XY plane for TEM waves (cone infinitely long and reference plane of infinite extension).

waves. These other types of waves have then other field configurations, different propagation constants, etc., from the principal waves.

Treating the case for infinite extensions of the conducting surfaces, at any distance of r meters from the source, we deal with a spherical wave front of r meters radius. The magnetic circulating force is given by the closed-line integral $\oint_l H\,dl$ for which the closed path $l = 2\pi r$ meters. This force is, therefore, equal to $2\pi rH$ amperes at any instant, since r is in meters and the instantaneous magnetic field intensity H is in amperes per meter. The closed-line integral of H represents, therefore, the total current flow I in amperes passing perpendicularly across the area of πr^2 square meters and bounded by the periphery $l = 2\pi r$ meters. By total current, we generally mean the amperian conduction current *and*

the Maxwellian displacement current, since according to (1), page 11, curl $H = J + \partial(\kappa\mathcal{E})/\partial t$ for the conduction current density J in amperes per square meter and the differential for the displacement current density. The integral for $\oint_l H\,dl = I$ amperes expresses the same thing as the curl H relation and takes care of the fact that we deal in Fig. 13 with a finite area πr^2 enclosed by the boundary of $l = 2\pi r$ meters. We have then for the instantaneous magnetic field intensity H, which is tangential to H field circles of radius r meters,

$$H = \frac{I}{2\pi r} \qquad \text{amp/m} \tag{52}$$

where the H circulation $2\pi H = I$ amperes is constant for each circle $2\pi r$ and independent of the magnitude of r, since the current density decreases with r and the circumference increases with r. The instantaneous current I depends on the time function $\epsilon^{j\omega t}$ and the space function $\epsilon^{-j\beta r}$, where the phase constant $\beta = \omega/c = \omega\sqrt{\mu_0\kappa_0}$ circular radians per meter, for $\sqrt{\mu_0\kappa_0} = 10^{-8}/3$ sec/m. The retarded instantaneous current is then

$$I = I_0\epsilon^{j\omega t}\epsilon^{-j\beta r} = I_0\epsilon^{j\omega\left(t-\frac{r}{c}\right)} = \underbrace{I_0\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\text{real component}} + jI_0\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

In the radiation region, the corresponding instantaneous field intensity $\mathcal{E}$, which is tangent to the circular $\mathcal{E}$ lines of force, is

$$\mathcal{E} = \sqrt{\frac{\mu_0}{\kappa_0}}\,H = 120\pi H = 120\pi\frac{I}{2\pi r} = 60\frac{I}{r} \qquad \text{volts/m} \tag{53}$$

The instantaneous Poynting radiation is then

$$p = \text{Re}\,(\mathcal{E}H) = \text{Re}\left(\frac{30}{\pi}\frac{I^2}{r^2}\right) \qquad \text{instantaneous power/m}^2$$

where Re means that the real value of p must be used. The only portion in this expression which is complex is the total current I; and by taking the real value of I, we obtain the correct answer. If we express in Fig. 13 the angle θ in radians, the total area across which p is effective is $r\theta 2\pi r$ square meters, with a total radiated power of

$$P_{\text{avg}} = \underbrace{2\pi r^2\theta}_{\text{area}} p = 2\pi r^2\theta\,\text{Re}\left(\frac{30I^2}{\pi r^2}\right) = 60\theta\,\text{Re}\,(I^2)$$

$$= 60\theta I_0^2\underbrace{\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\text{avg}=\frac{1}{2}}$$

$$= 30\theta I_0^2 = 60\theta I_e^2 = R_r I_e^2 \qquad \text{watts} \tag{54}$$

since the effective current $I_e = I_0/\sqrt{2}$ amperes. The radiation resistance is, therefore,

$$R_r = 60\theta \quad \text{ohms}; \quad \left.\begin{array}{l}\theta \text{ in circular radians } (\theta \text{ deg} = 1.75 \times \\ 10^{-2} \text{ radian})\end{array}\right\} \qquad (55)$$

Hence, for an angle of $\theta = 28.65$ deg, we have $\theta = 28.65 \times 1.75 \times 10^{-2}$ radian, or $\theta = 0.5$ circular radian and a radiation resistance of $R_r = 0.5 \times 60 = 30$ ohms.

From this elementary solution, we note that the principal outgoing wave for the arrangement of Fig. 13 traps the radiated field power carried by the traveling wave in ever-growing and spreading useful radiation space, while in the ribbon line of Fig. 12, the field energy is trapped between the flat and very long conductors and cannot escape.

18. Notes on the Schelkunoff Double-cone Radiators. Figure 14 shows the most useful form of such a radiator, for which the driving voltage is impressed between the apexes. If we have two infinite cones, we may excite TEM waves (principal waves), which travel in case of perfect conductors with the velocity of light only outward from the origin, where the voltage E is applied. The principal wave has then electric lines of force ($\mathcal{E}$ lines) along the meridians of spheres that are concentric with the apex of respective cones. The apexes are then assumed as coinciding even, though they are insulated from each other so that a voltage E can be impressed. On account of the circular symmetry of the waves, the magnetic H lines are circles coaxial with the cones, since the $\mathcal{E}$ lines proceed in axial planes. We have, therefore, respective lines of force as is shown in Fig. 13 for the cone-plane radiator. If we deal with a biconical radiator in unobstructed space of finite length of l meters, according to S. A. Schelkunoff,[1] we can divide the space about the bicones into a region within the spherical boundary S (antenna region), which passes through the end faces of the bicones, and a region external to the spherical S surface. Such a space division is made, since the solution for the external region is not singular on the common axis of the cones. For the region within the S surface, the bicones themselves exclude the common axis, since we have only to satisfy the boundary conditions of respective bicones. The field in the antenna region (inside the S surface) is then the superposition of the principal waves and of complementary waves. For $l = \infty$, we have then only the outgoing principal wave (TEM wave). For l finite, we have an outward-moving principal wave generated by the high-frequency voltage E and an inward-moving principal wave which could be reflected by a conducting sphere. These

[1] *Proc. IRE*, **34**, 23P, January, 1946. *Trans A.I.E.E.*, **57**, 744 (1938). *Proc. IRE*, **29**, 493 (1941). BARROW, CHU, and JANSEN, *Proc IRE*, **27**, 769 (1939)

are then customary transmission-line waves for which, in Fig. 14, we have circular ℰ and circular H lines of force. Inasmuch as in the region beyond the spherical S surface there can be no principal waves (since *no* conductors are present), the principal TEM waves on the inside bound-

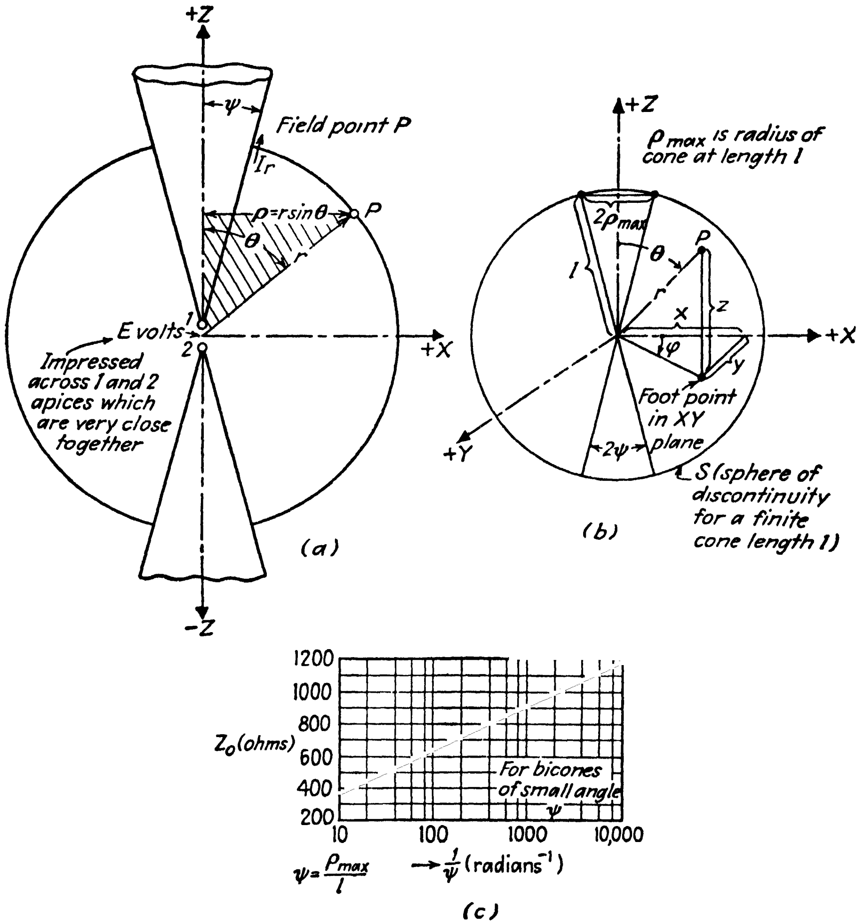


FIG. 14. The Schelkunoff bicone line.

ary of S *cannot* satisfy the fields due to the waves on the outside boundary of surface S. First, there are no principal waves beyond the S surface. Second, on the inside boundary of S are only transverse field components (since TEM waves are present); but on the outside boundary, we have in addition to transverse also a radial field component. This means that we have to add to the principal waves within the S surface also a complementary field with radial ℰ components. Complementary

waves will then satisfy the boundary conditions along the S surface and account for the end effects of the bicones.

As far as the outgoing and the reflected TEM waves represent the well-known principal waves associated with alternating-current energy propagation along two-conductor transmission lines, regardless of whether they are coaxial or parallel wires, we can use as in the preceding sections the expressions of the classic transmission-line theory in connection with Maxwell's equations for TEM waves. We have then

$$\left.\begin{aligned} L &= \frac{\mu}{\pi}\log_\epsilon\left(\cot\frac{\psi}{2}\right) && \text{henrys/m} \\ C &= \frac{\pi\kappa}{\log_\epsilon(\cot\psi/2)} && \text{farads/m} \\ Z_0 &= \sqrt{\frac{L}{C}} = \frac{\sqrt{\mu/\kappa}}{\pi}\log_\epsilon\left(\cot\frac{\psi}{2}\right) && \text{ohms} \end{aligned}\right\} \tag{56}$$

where μ is in henrys per meter; and κ is in farads per meter. For air as the medium about the bicones, we have $\sqrt{\mu/\kappa} = 120\pi = 376.7$ ohms and

$$Z_0 = 120\log_\epsilon\left(\cot\frac{\psi}{2}\right) \quad \text{ohms} \tag{57}$$

These relations are based on $\beta = \omega/v \cong \omega/c$ for air, and the field components

$$\left.\begin{aligned} &\left.\begin{aligned} \mathcal{E}_\theta &= \frac{60I}{r\sin\theta}\epsilon^{-j\beta r} && \text{volts/m} \\ H_\varphi &= \frac{I}{2\pi r\sin\theta}\epsilon^{-j\beta r} && \text{amp/m} \end{aligned}\right\} \text{outgoing} \\ &\left.\begin{aligned} \mathcal{E}_\theta' &= -\frac{60I'}{r\sin\theta}\epsilon^{j\beta r} && \text{volts/m} \\ H_\varphi' &= \frac{I'}{2\pi r\sin\theta}\epsilon^{j\beta r} && \text{amp/m} \end{aligned}\right\} \text{returning (reflected)} \\ &E(r) = \int_\psi^{\pi-\psi} r\mathcal{E}_\theta\, d\theta \qquad \text{transverse volts} \\ &I(r) = 2\pi r H_\varphi \sin\theta = 2\pi r H_\varphi \sin\psi \qquad \text{amp} \end{aligned}\right\} \tag{58}$$

since the transverse $E(r)$ voltage across the conical transmission line at a distance of r meters from the apexes must be the line integral of the field intensity $\mathcal{E}_\theta$. For the corresponding cone current $I(r)$, we have the closed-line integral of the magnetic field intensity H_φ. This is based on the curl relation that can be written in the integral form as $\oint_s H\,ds = I$ amperes and is for the closed path $s = 2\pi r$ a relation $\oint_s H\,ds = 2\pi r H = I$. In our case, we have also the angle θ to take into account, as the following section shows, and find $2\pi r \sin\theta H_\varphi = I(r)$ amperes. The

transverse $E(r)$ voltage and the corresponding cone current $I(r)$ give then the voltages and currents in place of the field intensities $\mathcal{E}_\theta$ and H_φ. Upon integration, the well-known classic transmission-line relations $dE/dr = -j\omega LI$ and $dI/dr = -j\omega CE$ are obtained, since we assume quasi-perfect cone conductivity and a perfect dielectric about the bicones (air as dielectric, for instance).

It must be realized that the expression for the characteristic impedance Z_0 in (56) is based on the L/C ratio, since $Z_0 = \sqrt{L/C}$. Hence, it is *not based on* the ratio of the *electric* and *magnetic field* intensity, but on the ratio of a voltage E and a current I. For the outgoing principal wave train (traveling wave), we have the ratio E/I as the wave impedance Z_0 and, for the returning principal wave (reflected at the open ends of the cones), the ratio E'/I' for Z_0. Hence, $E/I = E'/I'$, where $E = E'$ and $I = I'$ exist only for the case of unity reflection coefficient.

19. Theoretical Details of the Characteristic Impedance Formula of the Double-cone Radiator. We have already found in (42) and (43), for a very long ribbon transmission line, that Z_0 based on the classic surge impedance concept (generally $Z_0 = \sqrt{j\omega L/(G + j\omega C)}$) for perfect conductors yields $Z_0 = k\sqrt{\mu/\kappa_\epsilon}$ ohms, where κ_ϵ farads per meter simplifies to κ farads per meter for a perfect dielectric medium ($\sigma = 0$ mhos/m). We have also generally the propagation constant $\Gamma = j\omega\sqrt{\mu\kappa_\epsilon}$ complex radians per meter. If we compare (42) and (43) with (56) for the biconical transmission line, we shall find that the propagation constant Γ gives the same result as found for the unbounded wave medium (no ribbon conductors or biconical conductors present). But the classic surge impedance concept based on the voltage-to-current ratio, due to an outgoing (traveling) wave disturbance or due to the reflected principle wave, respectively, yields

$$\left.\begin{aligned} Z_0 &= k\sqrt{\frac{\mu}{\kappa}} \quad \text{ohms with } k = \frac{s}{w} \quad \text{for the parallel ribbon line} \\ Z_0 &= k\sqrt{\frac{\mu}{\kappa}} \quad \text{ohms with } k = \frac{\log_\epsilon(\cot\psi/2)}{\pi} \quad \text{for the biconical line} \end{aligned}\right\} \tag{59}$$

in place of unbounded medium impedance of $\sqrt{\mu/\kappa}$ ohms for a perfect dielectric field medium. Hence, as far as the attenuation constant α nepers per meter and the phase constant β circular radians per meter are concerned, ($\Gamma = \alpha + j\beta$ complex radians per meter), it does not matter whether we deal with the classic transmission-line concept or with the unbounded space concept based on $\mathcal{E}$ and H values for Γ. However, for Z_0, we obtain the results of (59). On pp. 403–405, we find a similar result for the parallel wire line (Lecher line), for which $k = \log_\epsilon$ (spacing between centers of wires, divided by the radius of respective wires).

We *must*, therefore, make a distinction between the classic transmission-line impedance Z_0 and the three-dimensional space impedance (unbounded medium impedance). As long as such a distinction is made, the results can be properly applied. In other words, (59) holds for the voltage-to-current ratio of a traveling wave for a double-conductor line, and the free-space impedance $\sqrt{\mu/\kappa}$ ohms holds for the ratio $\mathcal{E}/H$ in the case of

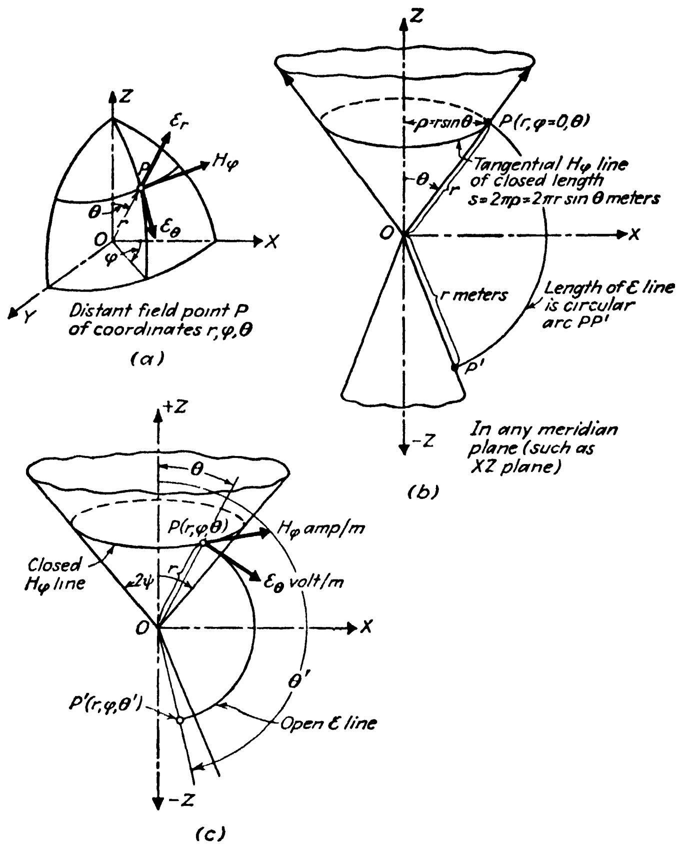


FIG. 15. General case for cones of different spread.

plane TEM waves. Since the phase velocity is equal to ω/β meters per second and $\Gamma = \alpha + j\beta$ complex ohms (generally), we can use the classic transmission-line concept for β and, therefore, also the phase velocity of the unbounded field concept.

Figure 15 shows the field components $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter for the general infinite biconical transmission line. Since we deal with the general case in this illustration, the solid cone

angle is different for the cone along the $+Z$ direction from that along the $-Z$ direction of a spherical coordinate system, or coordinates r, φ, and θ, where r is the distance P in meters from the origin of the XYZ system, φ is the lateral displacement of the field point P with respect to the $+X$ axis, and θ is the off-direction of P with respect to the $+Z$ axis, *i.e.*, the common bicone axis looking along $+Z$.

Since for the principal TEM waves associated with the quasi-perfect conical conductors, only the tangential $\mathcal{E}_\theta$ and H_φ and radial $\mathcal{E}_r$ components can play a part, but not H_θ, $\mathcal{E}_\varphi$, and H_r components, which likewise may constitute TEM waves, we obtain for the curl $\mathcal{E}$ and curl H relations of Eqs. (1), page 11, as is shown in Sec. 31, the expressions

$$\left.\begin{aligned}
&\frac{\partial(\overbrace{\mathcal{E}_\varphi}^{=0}\sin\theta)}{\partial\theta} - \frac{\partial\mathcal{E}_\theta}{\partial\varphi} = -j\omega\mu r\underbrace{H_r}_{=0}\sin\theta \qquad \text{or} \qquad -\frac{\partial\mathcal{E}_\theta}{\partial\varphi} = 0\\
&\frac{\partial(r\mathcal{E}_\theta)}{\partial r} - \frac{\partial\mathcal{E}_r}{\partial\theta} = -j\omega\mu rH_\varphi; \qquad \text{does not simplify}\\
&\frac{\partial\mathcal{E}_r}{\partial\varphi} - \sin\theta\,\frac{\partial(r\overbrace{\mathcal{E}_\varphi}^{=0})}{\partial r} = -j\omega\mu r\underbrace{H_\theta}_{=0}\sin\theta; \qquad \text{reduces to } \frac{\partial\mathcal{E}_r}{\partial\varphi} = 0
\end{aligned}\right\}\ \text{curl } \mathcal{E} \text{ components}$$

$$\left.\begin{aligned}
&\frac{\partial(H_\varphi\sin\theta)}{\partial\theta} - \frac{\partial\overbrace{H_\theta}^{=0}}{\partial\varphi} = (\underbrace{\sigma}_{=0} + j\omega\kappa)r\mathcal{E}_r\sin\theta; \qquad \text{reduces to } \frac{\partial(H_\varphi\sin\theta)}{\partial\theta}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = j\omega\kappa r\mathcal{E}_r\sin\theta\\
&\frac{\partial(r\overbrace{H_\theta}^{=0})}{\partial r} - \frac{\partial\overbrace{H_r}^{=0}}{\partial\theta} = (\underbrace{\sigma}_{=0} + j\omega\kappa)\overbrace{\mathcal{E}_\varphi}^{=0}; \qquad \text{reduces to } 0 = 0\\
&\frac{\partial\overbrace{H_r}^{=0}}{\partial\varphi} - \sin\theta\,\frac{\partial(rH_\varphi)}{\partial r} = (\underbrace{\sigma}_{=0} + j\omega\kappa)r\mathcal{E}_\theta\sin\theta; \qquad \text{reduces to } -\frac{\partial(rH_\varphi)}{\partial r} = j\omega\kappa r\mathcal{E}_\theta
\end{aligned}\right\}\ \text{curl } H \text{ components}$$

For the radiation region, the significant remaining relations are

$$\frac{\partial(r\mathcal{E}_\theta)}{\partial r} = -j\omega\mu(rH_\varphi); \qquad \frac{\partial(rH_\varphi)}{\partial r} = -j\omega\kappa(r\mathcal{E}_\theta) \tag{60}$$

since

$$\frac{\partial(r\mathcal{E}_\theta)}{\partial r} - \underbrace{\frac{\partial\mathcal{E}_r}{\partial\theta}}_{\text{negligible}} = -j\omega\mu rH_\varphi; \qquad \text{reduces to } \frac{\partial(r\mathcal{E}_\theta)}{\partial r} \cong -j\omega\mu rH_\varphi$$

$$\frac{\partial(H_\varphi\sin\theta)}{\partial\theta} = \overbrace{j\omega\kappa r\mathcal{E}_r\sin\theta}^{\text{negligible}}; \qquad \text{reduces to } \frac{\partial(H_\varphi\sin\theta)}{\partial\theta} \cong 0$$

$$-\frac{\partial(rH_\varphi)}{\partial r} = j\omega\kappa r\mathcal{E}_\theta, \qquad \text{remains as it is}$$

The first reduced relation then reads as the corresponding relation in (60), and the relation that remained as it was yields the other relation of (60). Now the argument $r\mathcal{E}_\theta$ to be differentiated with respect to r has the dimension of a voltage, since r is in meters and $\mathcal{E}_\theta$ in volts per meter. For the same reason, the product (rH_φ) has the dimension of a current, since r is in meters and H_φ in amperes per meter. Equations (60) have, therefore, the characteristics of customary transmission-line relations, since $\partial(r\mathcal{E}_\theta)/\partial r$ represents a voltage decrease per unit length (per meter length) and $\partial(rH_\varphi)/\partial r$ the associated current decrease per unit length. A differentiation of the relations (60) yields

$$\frac{\partial^2(r\mathcal{E}_\theta)}{\partial r^2} + j\omega\mu \underbrace{\frac{\partial(rH_\varphi)}{\partial r}}_{=j\omega\mu[-j\omega\kappa(r\mathcal{E}_\theta)]} = 0, \qquad \frac{\partial^2(rH_\varphi)}{\partial r^2} + j\omega\kappa \underbrace{\frac{\partial(r\mathcal{E}_\theta)}{\partial r}}_{=j\omega\kappa[-j\omega\mu(rH_\varphi)]} = 0$$

This means that in either case, we obtain expressions of the form

$$\frac{\partial^2 F}{\partial r^2} + \omega^2\mu\kappa F = 0; \qquad \text{for } F = r\mathcal{E}_\theta \text{ volts or } F = rH_\varphi \quad \text{amp} \tag{61}$$

But $\omega^2\mu\kappa = (\omega/v)^2$, where v stands for the phase velocity $v = 1/\sqrt{\mu\kappa}$ meters per second in the dielectric about the bicones. The ratio ω/v denotes the phase constant β in radians per meter, and the outgoing principal or TEM wave has its outward motion characterized by the phase factor $\epsilon^{j(\omega t-\beta r)}$ On account of the relations

$$\frac{\partial(\mathcal{E}_\theta \sin\theta)}{\partial\theta} \cong 0 \qquad \text{and} \qquad \frac{\partial(H_\varphi \sin\theta)}{\partial\theta} \cong 0$$

the respective $\mathcal{E}_\theta$ and H_φ transverse fields must fall off proportionally to $1/(r \sin\theta)$. Since, in Eqs. (60), we have $\mathcal{E}_\theta = \sqrt{\mu/\kappa}\, H$, we can express these relations either as

$$\frac{\partial(r\sqrt{(\mu/\kappa)}H_\varphi)}{\partial r} + j\omega\mu(rH_\varphi) = 0 \qquad \text{or} \qquad \frac{\partial(rH_\varphi)}{\partial r} + j\omega\sqrt{\mu\kappa}\,(rH_\varphi) = 0$$

$$\frac{\partial(rH_\varphi)}{\partial r} + j\omega\kappa\left(r\sqrt{\frac{\mu}{\kappa}}H_\varphi\right) = 0 \qquad \text{or} \qquad \frac{\partial(rH_\varphi)}{\partial r} + j\omega\sqrt{\mu\kappa}\,(rH_\varphi) = 0$$

or as

$$\frac{\partial(r\mathcal{E}_\theta)}{\partial r} + j\omega\mu\left(r\sqrt{\frac{\kappa}{\mu}}\mathcal{E}_\theta\right) = 0 \qquad \text{or} \qquad \frac{\partial(r\mathcal{E}_\theta)}{\partial r} + j\omega\sqrt{\mu\kappa}\,(r\mathcal{E}_\theta) = 0$$

$$\partial\left(r\sqrt{\frac{\kappa}{\mu}}\mathcal{E}_\theta\right) + j\omega\kappa(r\mathcal{E}_\theta) = 0 \qquad \text{or} \qquad \frac{\partial(r\mathcal{E}_\theta)}{\partial r} + j\omega\sqrt{\mu\kappa}\,(r\mathcal{E}_\theta) = 0$$

But $\omega\sqrt{\mu\kappa}$ is ω divided by the phase velocity, and (60) results in the expressions

$$\frac{\partial F}{\partial r} + j\frac{\omega}{c}F = 0 \qquad \text{for } F = rH_\varphi \text{ or } F = r\mathcal{E}_\theta \tag{62}$$

if air is the dielectric about quasi-perfect bicones. Both (61) and (62) show that the instantaneous field values are

$$H_\varphi = \frac{H_0}{r\sin\theta}\epsilon^{j\left(t-\frac{r}{c}\right)} \quad \text{amp/m}; \qquad \mathcal{E}_\theta = \sqrt{\frac{\mu}{\kappa}}H_\varphi \quad \text{volts/m} \tag{63}$$

From the field values of (63), we can find the instantaneous current $I = I(r)$ across the P circle of Fig. 15*c* and the transverse voltage $E = E(r)$ across the P and P' circles, which are part of the respective cone surfaces. The circle along which H_φ acts is, according to Fig. 15*b*, $2\pi r \sin\theta$ meters in length. Therefore, we have a cone current[1]

$$I(r) = \oint_s H_\varphi\, ds = 2\pi r H_\varphi \sin\theta = 2\pi r \sin\theta \frac{H_0 \epsilon^{j\omega\left(t-\frac{r}{c}\right)}}{r\sin\theta} = 2\pi H_0 \epsilon^{j\omega\left(t-\frac{r}{c}\right)} \quad \text{amp} \quad (64)$$

The transverse $E(r)$ voltage across the P and P' circles of respective conductive conical surfaces is not due to a closed path but to the circular $\mathcal{E}_\theta$ line of force extending from P to P' on respective conical surfaces. Point P refers to an angle $\theta = \theta$, and point P' to an angle $\theta = \theta'$. We obtain the solution for the instantaneous transverse voltage as

$$E(r) = \int_{\theta=\theta}^{\theta=\theta'} r\mathcal{E}_\theta\, d\theta = \int_\theta^{\theta'} r\sqrt{\frac{\mu}{\kappa}}\, H_\varphi\, d\theta = \int_\theta^{\theta'} r\sqrt{\frac{\mu}{\kappa}} \frac{H_0 \epsilon^{j\omega\left(t-\frac{r}{c}\right)}}{r\sin\theta} = H_0\sqrt{\frac{\mu}{\kappa}}\, \epsilon^{j\omega\left(t-\frac{r}{c}\right)} \int_\theta^{\theta'} \frac{d\theta}{\sin\theta} \quad \text{volts}$$

But

$$\int_\theta^{\theta'} \frac{d\theta}{\sin\theta} = \left| -\frac{1}{2}\log_\epsilon \frac{1+\cos\theta}{1-\cos\theta}\right|_\theta^{\theta'} = \left|\log_\epsilon\left[\tan\left(\frac{\theta}{2}\right)\right]\right|_\theta^{\theta'}$$

$$= \log_\epsilon\left(\tan\frac{\theta'}{2}\right) - \log_\epsilon\left(\tan\frac{\theta}{2}\right) = \log_\epsilon \frac{\tan\left(\frac{\theta'}{2}\right)}{\tan\left(\frac{\theta}{2}\right)}$$

$$= \log_\epsilon\left[\tan\left(\frac{\theta'}{2}\right)\cot\left(\frac{\theta}{2}\right)\right]$$

Therefore,

$$E(r) = H_0\sqrt{\frac{\mu}{\kappa}}\, \epsilon^{j\omega\left(t-\frac{r}{c}\right)} \log_\epsilon\left[\tan\left(\frac{\theta'}{2}\right)\cot\left(\frac{\theta}{2}\right)\right] \quad \text{volts} \quad (65)$$

[1] According to Fig. 15, the quantity $r\sin\theta$ represents the radius of the circular conical cross section $\pi r^2 \sin^2\theta$ square meters, a distance r meters away from the apex of the cone. Therefore, $2\pi r \sin\varphi$ is the *tangential* magnetic line of force on the cone, and $2\pi r\sin\varphi \cdot H_\varphi$ must be equal to the total current across the area enclosed by the boundary $2\pi r \sin\varphi$. The total current consists of the amperian conduction current $I(r)$, which flows essentially along the skin of this cross section, and the Maxwellian displacement current, since even vacuum has a dielectric constant of $\kappa_0 = 8.854 \times 10^{-12}$ farad/m. As is shown on p. 16, the displacement current in a good conductor is negligible in comparison with the conduction current and, therefore, the closed-line integral of the tangential H_φ circulation must represent the cone current $I(r)$ amperes.

The characteristic impedance of the biconical transmission line then becomes

$$\left.\begin{aligned} Z_0 &= \frac{E(r)}{I(r)} = \frac{\sqrt{\mu/\kappa}}{2\pi}\log_\epsilon\left[\tan\left(\frac{\theta'}{2}\right)\cot\left(\frac{\theta}{2}\right)\right] \\ &= k\sqrt{\frac{\mu}{\kappa}} \qquad \text{ohms} \\ k &= \frac{\log_\epsilon[\tan(\theta'/2)\cot(\theta/2)]}{2\pi} \end{aligned}\right\} \tag{66}$$

which is k times the characteristic impedance $\sqrt{\mu/\kappa}$ ohms, holding for three-dimensional space and based upon the $\mathcal{E}_\theta/H_\varphi$ ratio. Suppose $\theta = 45$ deg and $\theta' = 25$ deg; then $\tan(\theta'/2) = \tan 12.5 \text{ deg} = 0.2217$ and $\cot(\theta/2) = 1/\tan(\theta/2) = 1/\tan 22.5 \text{ deg} = 1/0.4142$. Therefore, $\tan 12.5 \text{ deg} \cot 22.5 \text{ deg} = 0.2217/0.4142 = 0.538$ and

$$k = \frac{\log_\epsilon(0.538)}{6.28} = \frac{2.303}{6.28}\log_{10}(0.538)$$

$$= \frac{2.303}{6.28}\underbrace{(0.73078 - 1.0)}_{-0.26922} = -0.1005$$

The transmission-line impedance based on the voltage-to-current ratio is then only about 10 per cent of the space impedance based on electric-to-magnetic-field ratio.

If we choose like cones, with an opening of 2ψ degrees, as is indicated for the upper cone of Fig. 15c, we have $\theta = \psi$ degrees and $\theta' = 180$ degrees $- \psi$ degrees, yielding

$$k = \frac{\log_\epsilon\left[\tan\left(\frac{\theta'}{2}\right)\cot\left(\frac{\theta}{2}\right)\right]}{2\pi} = \frac{\log_\epsilon\left[\tan\left(90° - \frac{\psi}{2}\right)\cot\left(\frac{\psi}{2}\right)\right]}{2\pi}$$

$$= \frac{\log_\epsilon\left[\cot^2\left(\frac{\psi}{2}\right)\right]}{2\pi} = \frac{2\log_\epsilon\left[\cot\left(\frac{\psi}{2}\right)\right]}{2\pi} = \frac{\log_\epsilon\left[\cot\left(\frac{\psi}{2}\right)\right]}{\pi}$$

which checks the Z_0 formula given in (56). Therefore, if we have equal bicones as in Fig. 14 with an opening $2\psi = 60$ deg each, we find $\cot \psi/2 = \cot 15 \text{ deg} = 1/0.2679 = 3.72$, and

$$k = \frac{2.303\log_{10}(3.72)}{3.14} = \frac{2.303 \times 0.57054}{3.14} = 0.419$$

The transmission-line characteristic impedance is then about 42 per cent of the characteristic space impedance $\sqrt{\mu/\kappa}$ ohms.

The arrangement of Fig. 15 for cones of unequal opening gives also a

means for the disk-cone radiator, since for $\theta = 90$ deg, the upper cone forms an infinite disk. Suppose we choose $\theta = 90$ deg and $\theta' = 135$ deg; then $\tan(\theta'/2) = \tan 67.5$ deg $= 2.4142$, $\cot(\theta/2) = \cot 45$ deg $= 1$, and

$$k = \frac{2.303 \log_{10}(2.4142)}{6.28} = \frac{2.303 \times 0.38274}{6.28} = 0.14$$

which gives 14 per cent of the space impedance value. It should be realized that the example for $\theta = 45$ deg and $\theta' = 25$ deg yields a bicone

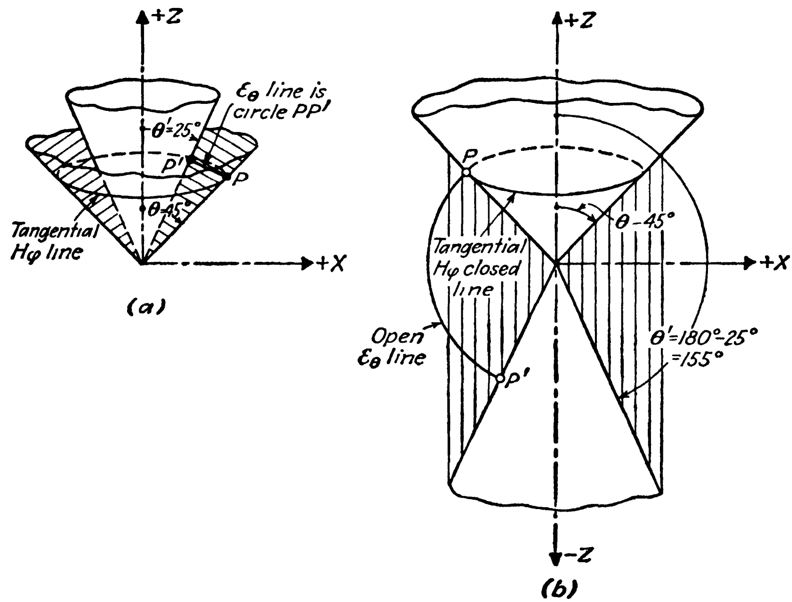


FIG. 16. Biconical double lines for $\theta = 45°$ and $\theta' = 25°$ and $(180 - 25)°$. The apexes are insulated from each other with an impressed h-f voltage across them. The shaded area corresponds to the space for the electromagnetic field energy.

arrangement as in Fig. 16*a*, while Fig. 16*b* gives the complementary case for which $\theta = 45$ deg and $\theta' = 155$ deg. The shaded space marks the space for respective principal waves. For the complementary case, we find $\tan(\theta'/2) = \tan(90 - 12.5)$ deg $= \tan 77.5$ deg $= 4.511$, $\cot(\theta/2) = \cot 22.5$ deg $= 1/0.4142$, and

$$k = \frac{2.303 \log_{10}(4.511/0.4142)}{6.28} = \frac{2.303 \overbrace{\log_{10}(10.94)}^{=1.03902}}{6.28} = 0.384$$

This means that 38.4 per cent of the space impedance acts as characteristic cone line impedance for the complementary case, instead of only 10

per cent. Such a result is to be expected, since there is only a narrow guide space for Fig. 16*a* but quite an open guide space for Fig. 16*b*.

20. Remarks on Formulas for Elementary Radiators. For linear radiators whose diameter is small compared with the length dimension and the operating wavelength, we can relate the distant radiation field to a fixed current distribution along an infinitesimal linear radiator element dl. We have then either the elementary radiator $i\,dl$ of Fig. 17 or the Hertzian doublet of Fig. 18, where two small charges $\pm q$ are separated by a thin connecting link dl. The radiation effect at a distant field point P is then the same, since the current $i = I\epsilon^{j\omega t}$ is constant along dl in either case. In Fig. 17, we deal with an electric *current* moment $i\,dl$ meter · amperes at any instant; and for the doublet in Fig. 18, we deal with the electric *charge* moment $q\,dl$ meter · coulombs, where the time rate of this moment is equal to the moment $i\,dl$, or

$$\frac{d(q\,dl)}{dt} = i\,dl \qquad \text{m} \cdot \text{amp} \tag{67}$$

From Fig. 17*b*, we note that we have the transverse electromagnetic field made up by the electric meridian field component $\mathcal{E}_\theta$ and the normal magnetic field H_φ along a circle around the radiator. Theoretically, we have also a radial electric field component $\mathcal{E}_r$, which according to pages 163, 174, 175, and 186 is vanishingly small in the *distant* radiation region. The $\mathcal{E}_r$ field can by no means be ignored in the near region and the between region (pages 163, and 169-186). Therefore, in the near region, we have also a longitudinal electric field (along the direction of wave propagation). Therefore, generally we have a vertical Z field, which is $|\mathcal{E}_z| = \sqrt{|\mathcal{E}_\theta|^2 + |\mathcal{E}_r|^2}$, where $|\mathcal{E}|$ denotes the absolute amounts in volts per meter. We have, according to Fig. 17*a* and the theory in Secs. 33 and 43, at a field point P, the retarded magnetic vector potentials Λ in volt · seconds per meter with a radial Λ_r and a transverse Λ_θ component, so that $\Lambda_r = \Lambda_z \cos\theta$ and $\Lambda_\theta = \Lambda_z \sin\theta$, while a vector potential Λ_φ cannot exist. The elementary field associations, which play a part in the distant radiation region, are then

$$\left.\begin{aligned} d\mathcal{E}_\theta &= 188.4 \sin\theta \frac{I\,dl}{r \cdot \lambda} \cos(\omega t - \beta r) \\ d\mathcal{E}_\theta &= 0.628 \sin\theta \frac{fI\,dl}{r} \cos(\omega t - \beta r) \end{aligned}\right\} \text{volts/m} \\ dH_\varphi = d\mathcal{E}_\theta \sqrt{\frac{\kappa}{\mu}} = \frac{d\mathcal{E}_\theta}{377} \qquad \text{amp/m} \tag{68}$$

for the free-space wavelength λ and the distance r to the field point P and dl in meters, f in megacycles per second, and I in $i = I\epsilon^{j\omega t}$ in amperes.

It is especially the first expression that is mostly used in applications. For field points P, in the near region of a radiator, the theory shows that the $\mathcal{E}_\theta$ component is complex (real and imaginary terms) with a term varying inversely as the distance r, a term varying inversely with r^2, and a term proportional with $1/r^3$. The magnetic H_φ association is likewise complex, with a term that is proportional to $1/r$ and a term pro-

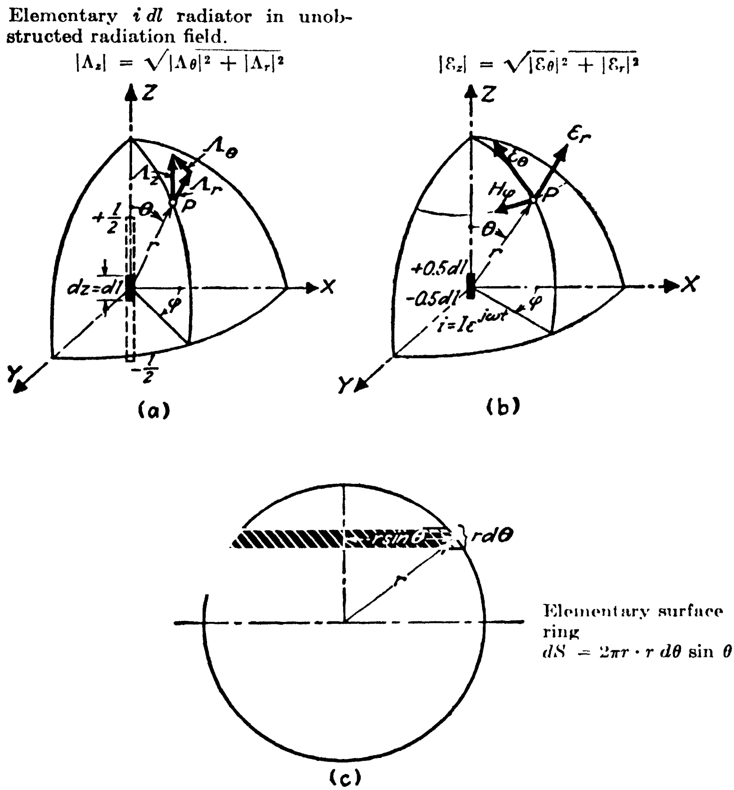


FIG. 17. Components of the retarded magnetic vector potential and of the corresponding electric and magnetic fields at space point $P(r, \varphi, \theta)$. A is in volt-seconds/m, $\mathcal{E}$ in volts/m, H in amp/m, r in m, i in amp.

portional to $1/r^2$, while the radial $\mathcal{E}_r$ field contains a term proportional to $1/r^2$ and a term proportional to $1/r^3$, explaining why this field cannot be significant in the distant radiation region. For the field near a radiator, the terms with the highest inverse distance power, *i.e.*, the $1/r^3$ terms, are most prominent. This means that for distances smaller than 1 m, the $1/r$ terms are the most insignificant terms, with the $1/r^2$ terms more prominent and the $1/r^3$ terms the most prominent field contributors. We have, therefore, in the near region of an antenna

longitudinal and transverse electric fields, in addition to the transverse magnetic field, and a direction finder must point differently in the near region than in the true radiation field. We can then write expressions for the absolute amount $|\mathcal{E}_\theta|$ of the transverse electric field taking all terms into account and for the amount $|\mathcal{E}_{\theta_r}|$ taking only the true radiation field term. We can do the same thing for the magnetic field association H_φ, and find then the ratios

$$R_\epsilon = \frac{|\mathcal{E}_\theta|}{|\mathcal{E}_{\theta_r}|} \qquad \text{and} \qquad R_H = \frac{|H_\varphi|}{|H_{\varphi_r}|}$$

which, if multiplied by 100, express the percentage of the total transverse field in terms of the true radiation field. Since the respective transverse field intensities are complex, we also have a means of finding the phase difference $\delta = \psi_\epsilon - \psi_H$ between the transverse $\mathcal{E}_\theta$ and H_φ fields. The transverse $\mathcal{E}_\theta$ and H_φ fields are then proportional to P_θ and P_φ, for

$$P_\theta = j\omega\left[1 + \frac{c}{j\omega r} - \frac{c^2}{(\omega r)^2}\right]; \quad P_\varphi = \frac{1}{r} + \frac{j\omega}{c}; \quad c = \frac{1}{\sqrt{\mu\kappa}} \cong 3 \times 10^8 \text{ m/sec}$$

Since the ratio $\rho = r/\lambda$, *i.e.*, the distance from the radiator to the field point P in terms of the free-space wavelength, has much to do with the region from which, for all practical purposes, only a true radiation field exists, it is convenient to give the P_θ and P_φ expressions in terms of ρ. Using also the substitution $2\pi/\lambda$ for the phase constant ω/c, we find

$$\left.\begin{aligned} P_\theta &= j\omega\left(1 - \frac{j}{2\pi\rho} - \frac{1}{4\pi^2\rho^2}\right) = j\omega R_\theta \\ P_\varphi &= j\frac{\omega}{c}\left(1 - \frac{j}{2\pi\rho}\right) = j\frac{\omega}{c}R_\varphi \end{aligned}\right\} \tag{69}$$

Therefore, for a distance r equal to the operating wavelength λ in unobstructed space, we find

$$R_\theta = 1 - \frac{j}{6.28} - \frac{1}{6.28^2} = \overbrace{1 - 0.0253}^{0.9747} - j0.1595$$
$$= \sqrt{0.9747^2 + 0.1595^2}\ \underline{/\psi_\epsilon = \tan^{-1}\left(\frac{-0.1595}{0.9747}\right)}$$
$$= \sqrt{0.95062 + 0.02544}\ \underline{\tan^{-1}(-0.164)}$$
$$= 0.988\ \underline{/\psi_\epsilon = -9.3^\circ}$$

$$R_\varphi = 1 - \frac{j}{6.28} = 1 - j0.1595$$
$$= \sqrt{1 + 0.1595^2}\ \underline{/\psi_H = \tan^{-1}\left(\frac{-0.1595}{1.0}\right)}$$
$$= \sqrt{1.02544}\ \underline{/-9.05^\circ} = 1.01\ \underline{/\psi_H = -9.05^\circ}$$

showing that even at a distance as close as 1 wavelength (this is short at VHF's and UHF's since, for instance, even for 100 Mc/sec, we have $\lambda = {}^{300}\!/_{100} = 3$ m), the factors R_θ and R_φ are nearly unity as for the true radiation region, and the phase angle between $\mathcal{E}_\theta$ and H_φ is only $\delta = \psi_\epsilon - \psi_H = -9.3 + 9.05 = -0.25$ time degrees. This is followed up in more detail on pages 86–88, 168–186.

Respective field contributions of formulas (68) refer only to a differential length dl, expressed in meters, of a linear radiator. The total radiated field of a thin linear radiator along the Z direction and l meters in length is then the summed-up effect of all the $i\,dl$ radiator elements along the length l. The amplitude I of $i = I\epsilon^{j\omega t}$ is then fixed only within a particular dl length, since it is generally different for successive line elements dl. This is taken care of by using a current function $I(z)$ instead of the amplitude I in $i = I\epsilon^{j\omega t}$, where $I(z) = IF(z)$ takes the current distribution along l into account as well as its phase The distribution $F(z)$ *must be known*. For an assumed sinusoidal current distribution, with l along the Z axis, we have

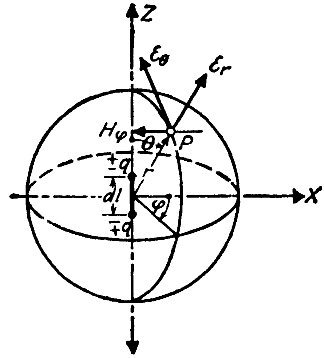


FIG. 18 Hertzian doublet radiator Azimuth φ, elevation $90 - \theta$, and r determine location of distant field point P.

$$F(z) = \sin(\beta l) = \sin\left(\frac{2\pi}{\lambda}\,l\right); \qquad I(z) = I \sin\left(\frac{2\pi}{\lambda}\,l\right)$$

maximum amp (70)

This means that we have an optimum current maximum at a distance of a quarter wavelength from the free end of the radiator, since $2\pi l/\lambda$ for $l = \lambda/4$ yields $\beta l = \pi/2$, corresponding to 90 electrical space degrees, rendering $\sin \beta l = \sin 90\text{ deg} = 1$ and $I(z) = I$ amperes. The total electromagnetic fields at a space point P, along any direction r, φ, θ, as indicated in Fig. 17, then become, on account of $dl = dz$,

$$\left.\begin{aligned}
\mathcal{E}_\theta &= \int_{z=-\frac{l}{2}}^{z=+\frac{l}{2}} F(z)\, d\mathcal{E}_\theta = \int_{-\frac{l}{2}}^{+\frac{l}{2}} 188.4 \sin\theta\, \frac{I(z)\,dz}{r\lambda}\, \epsilon^{j(\omega t - \beta r)} \\
&= 188.4\,\frac{I}{r\lambda}\,\epsilon^{j\omega t} \int_{-\frac{l}{2}}^{+\frac{l}{2}} F(z)\cdot \epsilon^{-j\beta r}\, dz \qquad \text{volts/m} \\
H_\varphi &= \frac{\mathcal{E}_\theta}{Z_0} \quad \text{amp/m}; \qquad Z_0 \sqrt{\frac{\mu}{\kappa}} \cong 377 \text{ ohms for air}
\end{aligned}\right\} \text{for the distant radiation region} \qquad (71)$$

which is followed up in detail on pages 92 and 192–194. It will be noted that the complete retarded phase function $\epsilon^{j(\omega t-\beta r)} = \cos(\omega t - \beta r) + j\sin(\omega t - \beta r)$ is used in (71), instead of only the real value as in (68). *It is the real value that gives* in (68) respective instantaneous time values for the respective elementary fields $d\mathcal{E}_\theta$ and dH_φ. The complete phase function, as used in (71), gives the advantage of a quick separation of the time phase $\epsilon^{j\omega t}$ from the delay $\epsilon^{-j\beta r}$ due to the wave travel through a distance of r meters. It also makes the integration easier. We can also express (68) in the form

$$\left.\begin{aligned} d\mathcal{E}_\theta &= j5 \times 10^5 \mu_0 f^{(\mathrm{Mc})} \frac{\sin\theta}{r} IF(z)\,dz\ \epsilon^{j(\omega t-\beta r)} && \text{volts/m} \\ dH_\varphi &= j1.67 \times 10^{-3} f^{(\mathrm{Mc})} \frac{\sin\theta}{r} IF(z)\,dz \cdot \epsilon^{j(\omega t-\beta r)} && \text{amp/m} \\ \mu_0 &= 1.257 \times 10^{-6}\ \text{henrys/m} \end{aligned}\right\} \quad (72)$$

holding for propagation in unobstructed space. The correctness of these expressions can be checked by the *ratio* (*not differentiation*)

$$Z_0 = \frac{(d\mathcal{E}_\theta)}{(dH_\varphi)} = \frac{j5 \times 10^5 \times 1.257 \times 10^{-6}}{j1.67 \times 10^{-3}} \cong 377\ \text{ohms}$$

The distribution function $F(z)$ is unity for the elementary dz and a very short length Δz.

Instead of the vector potential A (in volt · seconds per meter) and the scalar potential V (in volts), at a distant space point (Fig. 17*a*), we can also use a *polarization* potential. The classic Hertz theory shows that for some practical conditions, an electromagnetic field can be expressed by a *single* vector function, instead of two functions. This function is known as the Hertz vector $\prod$ in meter · volts at a distant field point P, in place of A in Fig. 17*a*. The instantaneous value of this vector is in the *same* direction as the elementary $i\,dz$ radiator and yields at a distant field point P the elementary Hertz contribution

$$\begin{aligned} d\prod &= j11.27 \times 10^{-8} \frac{IF(z)\,dz}{\kappa_0 r f^{(\mathrm{Mc})}} \epsilon^{j(\omega t-\beta r)} \\ &= j1.435 \times 10^{3} \frac{IF(z)\,dz}{r f^{(\mathrm{Mc})}} \epsilon^{j(\omega t-\beta r)} \qquad \text{meter} \cdot \text{volts} \end{aligned} \quad (73)$$

since for atmospheric propagation $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m. As far as an elementary radiator $I\,dz$ is concerned, we have in (73) the distribution $F(z) = 1$; but for an actual radiator length l, this simplification does not hold. The elementary Hertz vector $d\prod$ has the same space orientation as the elementary radiator element dz. Therefore, $d\prod \cdot \cos\theta$ is the Hertz vector effect in a direction θ with the direction of the dz

length. Along the equatorial plane and planes parallel thereto, we have then $d\Pi \cdot \cos 90$ deg $= 0$ m · volts. For a linear radiator, along the vertical Z axis and of finite length of l meters, we find then the optimum value of the Hertz vector as

$$\Pi = j1.435 \times 10^3 \frac{I\epsilon^{j\omega t}}{rf^{(\mathrm{Mc})}} \int_{z=-0.5l}^{z=+0.5l} F(z)\epsilon^{-j\beta r}\,dz \qquad \text{meter} \cdot \text{volts} \tag{74}$$

For a sinusoidal current distribution, as is often approximately true, we have according to (70), the distribution function $F(z) = \sin\left(\frac{2\pi}{\lambda} z\right)$ in the above relation, where I denotes the maximum time value at the current antinode and r is the distance in meters from the radiation center of the linear radiator to the field point P. Since this point is in the *distant* radiation region, the quantity r can be taken out of the integral as far as the inverse distance effect is concerned. This cannot be done for the phase function $\epsilon^{-j\beta r}$, as is brought out in the theoretical part of this text (page 531). For a half-wave radiator, in the unobstructed radiation field, the value of $I(z) = IF(z) = I$ amperes occurs at the mid-point of the length l; and at the open ends ($z = \pm\frac{1}{2}l$ meters), we have no current whatsoever. Such a half-wave current distribution is then satisfied by the distribution function

$$F(z) = \sin\left[180^\circ\left(\frac{z}{l} + 0.5\right)\right] \tag{75}$$

since $\beta l = 2\pi l/\lambda = 2\pi\frac{1}{2}\lambda/\lambda = \pi$ corresponds to 180 deg. If we have a linear radiator of negligible diameter with respect to its length l, which is an integer multiple ($k = 1, 2, 3, 4, 5, \cdots$) of a half wavelength, then in (75) we have to use $k180$ deg in place of 180 deg. This subject matter is dealt with in detail and with many numerical applications on page 214 and in the Appendix.

As shown in Sec. 44, the magnetic field intensity H in amperes per meter can be computed from the curl of the retarded magnetic vector potential A if the result is divided by μ, since $B = \mu H = \mathrm{curl}\ \mathrm{A}$. On account of the relations

$$\frac{\partial \Pi}{\partial t} = v^2 \mathrm{A}, \qquad v = \frac{1}{\sqrt{\mu\kappa}} \qquad \text{m/sec}$$

we find

$$H_\varphi = \frac{\mathrm{curl}\ \mathrm{A}}{\mu} = \frac{\mathrm{curl}\left(\frac{1}{v^2}\frac{\partial\Pi}{\partial t}\right)}{\mu} = \frac{\mu\kappa}{\mu}\,\mathrm{curl}\left(\frac{\partial\Pi}{\partial t}\right) = \kappa\,\mathrm{curl}\left(\frac{\partial\Pi}{\partial t}\right) \qquad \text{amp/m}$$

$$\mathcal{E}_\theta = -\nabla V - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} \qquad \text{volts/m} \tag{76}$$

where ∇V is the gradient of the retarded scalar potential V in volts.

We note, therefore, that in the distant radiation region, we have at a certain field point $P(r, \varphi, \theta)$ the transverse field associations $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter, which can be found from the retarded vector potential Λ volt · seconds per meter by means of $H_\varphi = (\text{curl } \Lambda)/\mu$ and $\mathcal{E}_\theta = H_\varphi \sqrt{\mu/\kappa}$. This retarded magnetic vector potential acts at the space point P in place of the actual radiator, a distance r meters away and such that Λ has components with a resultant effect along the Z direction which is the direction of the linear radiator discussed. The linear radiator has a current moment $I\epsilon^{j\omega t}\,dz$ meter · amperes at the location of the elementary radiator and a retarded current moment $I\epsilon^{j\omega(t-r/c)}\,dz$ at the space point P in effect, with a maximum along the Z direction. At the space point P, we can also use the Hertz vector $\prod$ in meter · volts, which is likewise a maximum along the Z direction at P if the linear radiator is along the Z direction.

21. Notes on TEM, TE, and TM Waves. In the preceding sections, we have mostly dealt with TEM waves, since we were interested in either the principal waves of a two-conductor transmission system with a voltage impressed at the input end and the far ends open or the distant radiation field with *only* a transverse electric and a transverse magnetic field component in the wave front and normal to the direction of wave propagation. In connection with Fig. 14, we found that at the sphere of discontinuity S, which passes through the end faces of a double-cone radiator, we have to make a compromise, since beyond this sphere we have also a radial electric field component besides the end effects of respective cones. We have then principal outgoing and returning or TEM waves within the S sphere plus complementary waves making up for the discontinuity on the S sphere. Figure 19 shows that in a TEM wave the resultant electric $\mathcal{E}$ field and the resultant magnetic H field are normal to each other, forming at a certain space point P a plane wave front (for a small finite area) that moves out with the velocity of propagation of $1/\sqrt{\mu\kappa}$ meter per second in a pure dielectric medium of μ henrys per meter and κ farads per meter. This is, for instance, the case in the true radiation region for which respective field associations are in addition in time phase. This inphase condition would not hold if the medium also had a specific conductivity of σ mhos per meter effective, since then $\mathcal{E}/H$ would yield a complex space impedance $\sqrt{\mu/\kappa_\epsilon}$ ohms with an equivalent dielectric constant $\kappa_\epsilon = \kappa - j\sigma/\omega$ farads per meter and a corresponding phase angle. For the TM wave of Fig. 19, we have the magnetic vector H entirely transverse, for which reason it is called a "transverse magnetic" wave. The resultant electric vector $\mathcal{E}$ is, however, no longer normal to the resultant H field association, so that it can be decomposed into a transverse $\mathcal{E}_z$ component and a longitudinal $\mathcal{E}_x$ component. We have, therefore, also a longitudinal electric field component that is along the direction of wave propagation, for which reason this wave is also called a "longitudinal" $\mathcal{E}$ wave. For the TE wave, we have the resultant

$\mathcal{E}$ vector entirely along the Z direction and the resultant H vector no longer completely transverse, so that we can split it up into a transverse H_y and a longitudinal H_x component. This is the reason why this wave is called a "transverse electric" wave or also a "longitudinal" H wave. All three types of waves of Fig. 19 have, of course, higher modes. What we have to bear in mind is that TE and TM waves have a forward tilted

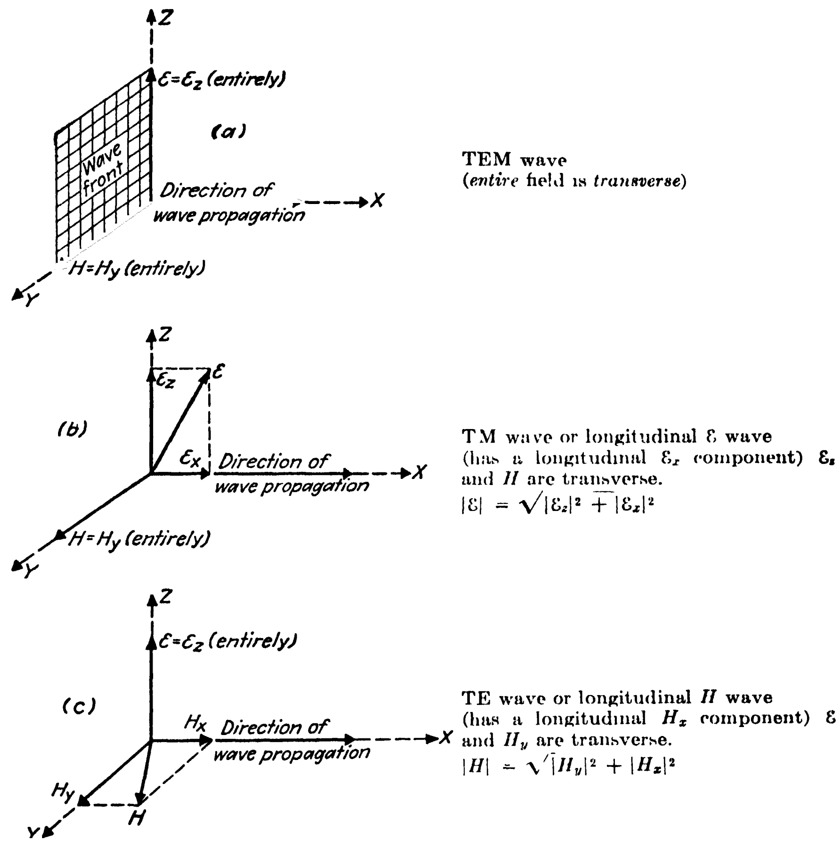


FIG. 19. Field vectors for TEM, TM, and TE waves.

electric $\mathcal{E}$ vector or such a magnetic H vector, respectively, with respect to the customary transverse wave front moving along the X direction. We can also speak of circular magnetic waves for which we have circular H lines of force, just as a circular electric wave is a wave with circular electric lines of force. In a TEM wave both electric and magnetic displacements are transverse to the direction of wave propagation. According to Eqs. (1), page 11, this means that the electric displacement current

density in amperes per square meter as well as the conduction current density in amperes per square meter is transverse to the direction of wave propagation. For the field orientation of Fig. 19, such electric currents would oscillate along the $\pm Z$ directions. It also means that the displacement magnetic current density in volts per square meter is likewise transverse to the direction of wave propagation and along the $\pm Y$ axis in Fig. 19. For the TE wave, we should, for propagation in air, have an electric displacement current density in amperes per square meter that is transverse to the direction of wave propagation. However, the magnetic current density in volts per square meter is partially transverse and partially longitudinal to the direction of wave propagation. For TM waves oriented as in Fig. 19 and propagated in air, the magnetic current density in volts per square meter is transverse to the direction of wave propagation, since it is entirely along the $\pm Y$ directions, but the dielectric displacement current density in amperes per square meter is partially transverse (along the $\pm Z$ directions) and partially longitudinal along the $\pm X$ directions while the wave progresses along the $+X$ direction. This is sometimes also called a "hybrid" TEM wave.

When we deal with customary two-conductor transmission lines, as the parallel-wire line and the concentric line, the *principal waves or TEM waves belong to the conductors.* For the ideal case, for which we have no losses in the dielectric medium and no losses in the go and return conductors, the field of the wave is wholly transverse, with no axial components. Besides, all frequencies are transmitted without attenuation and a fixed phase velocity (3×10^8 m/sec for air as a medium). Dissipation decreases this velocity to some extent But there are also waves that are limited by a lower cutoff frequency below which no power transfer is possible. They are known as "higher order" waves or "complementary" waves. If we have a hollow conducting wave guide, *no principal wave* (*no TEM wave*) can exist. In place of the return conductor, we must have either an axial $\mathcal{E}$ component or an axial H component or both. For an axial $\mathcal{E}$ component in addition to a transverse $\mathcal{E}$ component we close the circuit, as it were, by the dielectric displacement current, just as a closed circuit made up of a coil and a condenser yields a conduction current continued by a displacement current. A principal wave *cannot* exist, and we have only complementary waves. Hollow guides act as attenuators below a certain cutoff frequency but pass waves without attenuation above such a frequency (assuming perfect dielectric within guide and perfect conductivity in guide wall).

Figure 20 shows what may happen in double lines and hollow conductors. Figure 20*a* gives the case for a parallel wire line joining a line with a wider spacing. From the classic line theory, we know that at

junctions BC and $B'C'$, we must have reflection losses. When we deal with fields, we note that the electric $\mathcal{E}$ lines of force are distorted at this junction. The bulged lines of force have then a component perpendicular to respective conductors and also a component that is along the direction of propagation. The magnetic lines of force not shown are circles about the conductors, which are likewise normal to respective conductors.

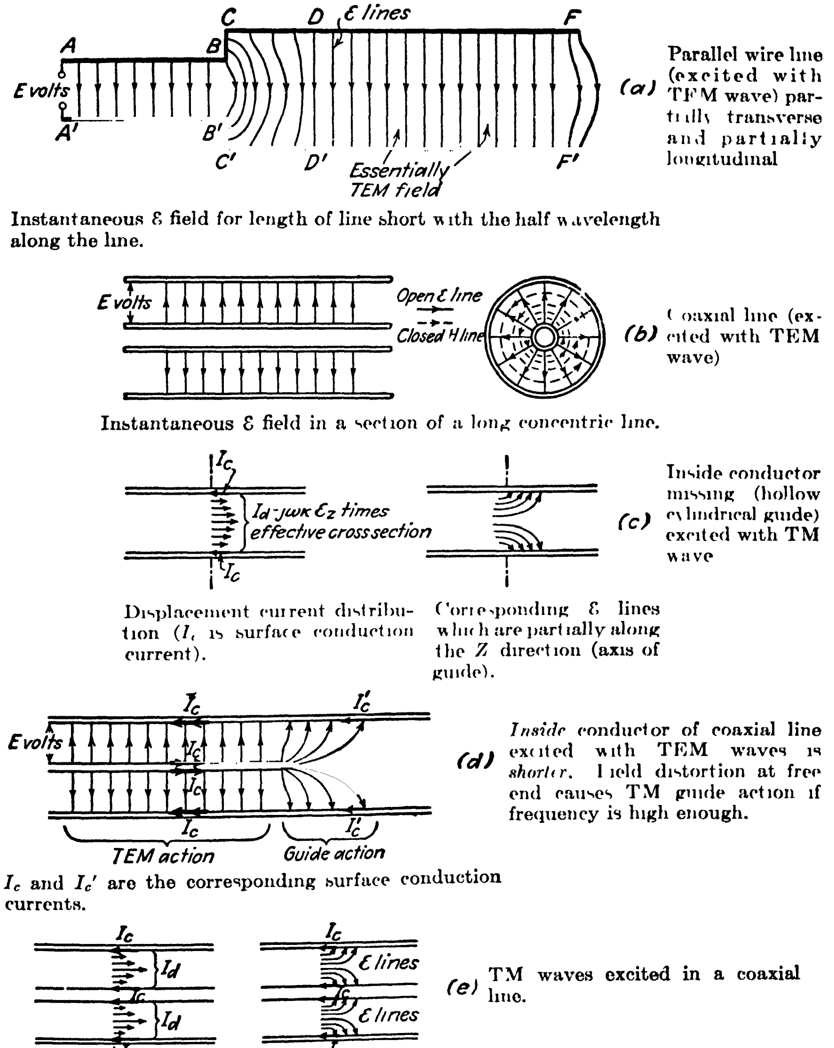


FIG. 20. Fields and their distortions in parallel wire and coaxial lines and fields in a hollow cylindrical tube.

Hence, for portions $ABA'B'$ and $DFD'F'$, we have essentially transverse electric and magnetic fields; but for the BC and $B'C'$ joints, we have transverse and longitudinal $\mathcal{E}$ effects and longitudinal H effects. This illustration shows also a field distortion at the open end of the parallel-wire line. This end effect can be explained by complementary waves and radiation. Figure 20*b* shows the electric lines of force for a coaxial line for the principal wave. Figure 20*c* shows what may happen when the inside conductor of the coaxial line is removed and the frequency is high enough. We may cause TM waves with electric lines of force that have transverse and longitudinal components. For good conductors, the $\mathcal{E}$ lines of force must leave or enter the conductor surfaces under 90 deg and so that right next to the inner conductor surfaces there can be *no* tangential $\mathcal{E}$ field. Figure 20*e* shows a TM wave with the inside conductor not removed. This mode must be properly excited, and the frequency must be high enough so that the dielectric space between the concentric conducting shells acts as a high-pass network to waves. Figure 20*d* shows what happens when the central conductor of a coaxial line extends only through a portion of the outside conductor. We note that field distortion occurs at the free end of the inside conductor. The bulged out $\mathcal{E}$ lines of force terminate normally on the inside surface of the outside cylinder. This will cause electric charges on the inside surfaces and, in turn, cause a conduction current I_c' to flow along the inside surface of the outer cylinder where there is no inside conductor. As the frequency is increased, this charging effect will become more and more pronounced and the end electric field of the inside conductor will penetrate the hollow space more and more and cause an increased displacement current. Both conduction and displacement currents are surrounded by an H field, and both $\mathcal{E}$ and H effects are present in the hollow space beyond the end of the inside conductor. From a certain frequency on and higher, a complementary wave will exist with no attenuation effects. A field distortion will also take place at the end of an open-end concentric line, and electric lines of force will bulge out and charge the outside surface of the outside cylinder at the open end and cause a corresponding conduction current flow.

Any double line (also multiple lines) is capable of guiding complementary waves as long as the operating frequency is chosen sufficiently high to make an unattenuated energy flow of complementary waves possible. The principal wave is the most dominant wave and not limited by a cutoff frequency below which no energy transfer exists. It has a definite propagation constant Γ, which fixes the attenuation constant and the phase constant, and characteristic impedance, Z_0. The phase constant fixes also the phase velocity c', since it is equal to the

angular velocity $\omega = 6.28f$ divided by the phase constant β in $\Gamma = \alpha + j\beta$. The phase constant also fixes the group velocity $c'' = d\omega/d\beta$ meters per second. The phase velocity c' fixes the phase distribution that is the wavelength along the line. The group velocity c'' represents the velocity of propagation of the envelope of a wave that occupies the frequency *band* for which the envelope delay is uniform. It differs from the phase velocity only in a medium in which the phase velocity varies with the frequency or with the direction of wave propagation. As far as the complementary waves are concerned, each particular wave has its own characteristic propagation constant, its own characteristic impedance, and therefore also its own phase velocity. Each particular complementary wave has its characteristic field configuration, as can readily be seen from the various modes along wave guides (Figs. 20, 18, 372 383, and 385- 387). The input voltage and the input current of a double line depend only on the principal wave. Inasmuch as the classic transmission-line theory assumes a simple exponential propagation, depending only on Γ and Z_0, it can represent the actual wave only at a great distance from the physical terminals of the line. The classic theory ignores radiation as well as energy scattering. The theory based on Maxwell's equations requires that the electric $\mathcal{E}$ and magnetic H fields are everywhere continuous and throughout the boundary surfaces. Such a condition can be satisfied only when the complete wave in a region beyond the boundaries is given by an infinite number of waves. We have then for a concentric line along the X direction

$$\left.\begin{aligned} E(x) &= \Big(\underbrace{Z_0 I\epsilon^{-x\Gamma}}_{\text{principal wave}} + \underbrace{\sum_{n=1}^{n=\infty} Z_n I_n\epsilon^{-x\Gamma_n}}_{\text{complementary waves}}\Big)\epsilon^{j\omega t} \qquad \text{volts} \\ I(x) &= \Big(I\epsilon^{-x\Gamma} + \sum_{1}^{\infty} I_n\epsilon^{-x\Gamma_n}\Big)\epsilon^{j\omega t} \qquad \text{amp} \end{aligned}\right\} \tag{77}$$

where also for Γ_n and the characteristic impedance Z_n, we have $n = 1, 2, 3, \ldots$. The propagation constants Γ_n are then of the type found with guides.

22. Magnetic Fields of Displacement Currents and Radiation from Condensers. Equations (1) play a part in this discussion. For a dissipative field medium, areas S_1 and S_2 square meters enclosed by respective finite boundaries of l_1 and l_2 meters, we have

$$\oint_{l_1} \mathcal{E}\,dl = -\frac{d}{dt}\int\!\!\int_{S_1} \mu H\,dS \qquad \text{volts};$$

$$\oint_{l_2} H\,dl = I + \frac{d}{dt}\int\!\!\int_{S_2} \kappa\mathcal{E}\,dS \qquad \text{amp} \tag{78}$$

where $\mathcal{E}$ and H are the electric and magnetic field intensities along the line elements dl forming the respective boundaries of l_1 and l_2 meters. Subscripts 1 and 2 are used, since the closed paths are generally unequal. It is also understood that respective double integrals act normally to respective enclosed surfaces S_1 and S_2, since μH represents the normal magnetic flux density B in webers per square meter (which is also volt · seconds per square meter), and $\kappa\mathcal{E}$ denotes the normal electric flux density D in coulombs per square meter (which is also amperes · seconds per square meter). The negative time rate of the double integral of $\mu H\,dS$ is then the total magnetic current in volts passing normally to surface S_1. The quantity I is the *entire* electric conduction current flowing normally across the surface S_2. The time rate of the double integral of $\kappa\mathcal{E}\,dS$ stands for the *entire* electric displacement current flowing normally across area S_2, both currents being in amperes. We deal usually with harmonic time functions of the form $\epsilon^{j\omega t}$, and find

$$\oint_{l_1} \mathcal{E}\,dl = -j\omega\mu \iint_{S_1} H\,dS \qquad \text{volts;}$$

$$\oint_{l_2} H\,dl = I + \overbrace{j\omega\kappa \iint_{S_2} \mathcal{E}\,dS}^{\text{displacement current } I_d} \qquad \text{amp} \qquad (78a)$$

Let us quickly inspect the orders of magnitude in these relations. If the conduction current I denotes in (78a) the effective value, then $\mathcal{E}$ and H likewise represent effective values. Suppose that we have a linear radiator with outside diameter of $d = 0.5$ in. The conductor may be a copper tube or have a solid copper cross section, since it does not matter at VHF's and UHF's, because the entire conduction current flow occurs essentially along the *surface* of the conductor *next to the field*. If the radiator is oriented in unobstructed radiation space, which is air, according to (78a) we have a conduction current flow of I in the radiator and a displacement current flow in the conductor as well as in the space about the conductor. For copper, we can assume $\kappa = \kappa_0 = 8.854 \times 10^{-12}$ farad/m. Suppose a 100-Mc current of 5 amp exists on a certain portion of the conductor and we have an H circulation around the conductor that is *tangential* with its circumference. Then

$$\oint_{l_2} H\,dl = H\pi d = H\pi \cdot \underbrace{0.5 \times 2.54 \times 10^{-2}}_{\text{meters}} = 0.04H \qquad \text{total amp}$$

By total amperes we mean the conduction current I plus the displacement current I_d vectorially added for the area $S_2 = \pi d^2/4 = 3.14 \times 0.25 \times 2.54^2 \times 10^{-4} = 5.07 \times 10^{-4}$ sq m. According to (78a), the displacement current becomes

$$I_d = j\omega\kappa_0 \iint_{S_2} \mathcal{E}\, dS = j\omega\kappa_0 S_2 \mathcal{E}$$
$$= j2\pi 100 \times 10^6 \times 8.854 \times 10^{-12} \times 5.07 \times 10^{-4}\mathcal{E}$$
$$= j6.28 \times 8.854 \times 5.07 \times 10^{-8}\mathcal{E} = j2.8 \times 10^{-6}\mathcal{E} \quad \text{reactive amp}$$

compared with 5 amp conductive or real amperes. Now let us compute what the order of magnitude of $\mathcal{E}$ must be for certain displacement current values. Let us take a displacement current $I_d = 5$ amp. Then $2.8 \times 10^{-6}\mathcal{E} = 5$. This yields an effective field intensity as high as $\mathcal{E} = 5 \times 10^6/2.8 = 1.78$ million volts/m. This field intensity is difficult to realize practically. First, copper is a very good conductor ($\sigma = 5.8 \times 10^7$ mhos/m) and could not have such an immense electric field intensity acting along a meter length along its surface (the tangential $\mathcal{E}$ field must essentially vanish). Second, according to computations on pages 477 479, the $\mathcal{E}$ field decays rapidly as an electromagnetic field attempts to pass into a conductor. Therefore, we can safely say that for an H circulation tangent to the copper surface, we have the closed-line integral of the magnetic field intensity equal to the *conduction* current flowing on the outer skin of the conductor with a vanishingly small displacement current effect within this skin. Suppose we take an H circulation that forms a circle twice the diameter of the outside diameter of the conductor carrying the conduction current I of 5 amp rms. Then the closed path l_2 must have doubled to a value of $l_2 = 0.08$ m, and the area S_2 quadruples to a value of $S_2 = 2.028 \times 10^{-3}$ sq m. We have then an available dielectric cross section S_2 that is 75 per cent larger than the cross-sectional area defined by the outside diameter of the radiator. This means that the displacement current I_d is also quadrupled and equal to $j1.12 \times 10^{-5}\mathcal{E}$ amperes, if it is assumed that $\mathcal{E}$ acts the same everywhere. This is likewise a most favorable assumption for the magnitude of I_d and is not true in practice, since $\mathcal{E}$ varies as we recede from the conductor. But even for such an assumption, the closed-line integral of the magnetic field vector H still is essentially equal to the conduction current of 5 amp. It would not be correct to use such numerical speculations for H circulations of larger and larger diameters in comparison with the outside diameter of a circular radiator. The reason for this is that first, we have to find the magnitude of $\mathcal{E}$ in an axial direction and, second, we have to take the phase also into consideration at different distances of the conductor surface. If we had, for instance, an H circulation about the radiator along a circular path that is everywhere one-half free-space wavelength from the surface of the radiator, the displacement current across the circular boundary line would be in antiphase with respect to the current flow along the surface of the conductor. The foregoing numerical speculations are meant to show that we have to go to frequencies very much higher than 100 Mc/sec before we can expect the

displacement current to become noticeable in comparison with the conduction current. For the displacement current density in vacuum (practically so in air), we have

$$J_d = \frac{\partial(\kappa_0 \mathcal{E})}{\partial t} = j\omega\kappa_0\mathcal{E} = 6.28 \times 10^6 f^{(\mathrm{Mc})} \times 8.854 \times 10^{-12}\mathcal{E}$$

$$= 5.56 \times 10^{-5} f^{(\mathrm{Mc})}\mathcal{E} \qquad \text{amp/m}^2 \qquad (79)$$

for $\mathcal{E}$ in volts per meter. This means that for 1 amp/m², it would require as much as 180 volts/m field intensity at 100 Mc/sec, 18 volts at 1,000 Mc/sec, and 1.8 volts/m at 10,000 Mc/sec. This would be then a displacement current flow of 1 amp effective across a 10.76-sq-ft area. These are quite high values for the distant radiation region, since for the 100-Mc electromagnetic wave, this would mean an electromagnetic power transfer for each 10.76 sq ft of the wave front of $\mathcal{E}^2/Z_0 = 180^2/377 = 85.5$ watts. Such an abnormal power transfer in the distant radiation region will not happen. For the 1,000-Mc wave, we should have a power transfer by each 10.76 sq ft of the electromagnetic wave front of $18^2/377 = 0.855$ watt; and for the 10,000-Mc wave and a displacement current density of 1 amp per 10.76 sq ft, we have a power transfer of only $1.8^2/377 = 8.85 \times 10^{-3}$ watt, or 8.85 mw, across each 10.76-sq-ft space area normal to the direction of wave propagation. The magnetic field association for the 100-Mc wave would be $\mathcal{E}/377 = 180/377$, or 0.477 amp/m; it would be 0.0477 amp/m for the 1,000-Mc excitation and 4.77 milliamps/m for the 10,000-Mc excitation.

From these examples, we note that at the upper UHF range, we may secure noticeable space currents when electromagnetic waves are present and for electromagnetic field powers that may be possible in the distant radiation field.

The curl $\mathcal{E}$ and curl H relations of Eqs. (1), page 11, give still more information. Let us take air for a perfect dielectric between the condenser plates shown in Fig. 21. For a certain harmonic applied voltage of E volts, then a current of I amperes flows toward and away from respective plates. At a certain instant the direction of I is as indicated, and we have between the plates $\mathcal{E}$ lines of electric force, which cause a displacement current density $J_d = \partial(\kappa_0\mathcal{E})/\partial t$ amperes per square meter. For a time factor of $\epsilon^{j\omega t}$, we have again $J_d = j\omega\kappa_0\mathcal{E}$ amperes per square meter. The electric flux density $D = \kappa_0\mathcal{E}$ coulombs per square meter causes, when stopped at respective condenser plates, $\pm\kappa_0\mathcal{E}$ coulombs charge on each square meter of the respective plates. The total charge on each plate is then at any instant $\pm\kappa_0\mathcal{E} \cdot S$ coulombs, where $S = ab$ is the area in square meters of each plate. Neglecting the edge effects of respective plates, the field intensity $\mathcal{E}$ across the plates is equal to the applied voltage E volts divided by the separation of s meters. Now the charge of a condenser is all the larger the larger its capacitance C

in farads and the more the applied voltage. This means that $\kappa_0 \mathcal{E} S$ must be equal to CE and $\kappa_0 \mathcal{E} S = \kappa_0 ES/s = CE$, or $C = \kappa_0 ab/s$ farads. The displacement current $I_d = SJ_d = j\omega\kappa_0 ab\mathcal{E} = j\omega\kappa_0 abE/s$ amperes is, as it were, the continuation of the conduction current I. It is true that I_d leads the conduction current I by 90 time degrees, since it leads the terminal voltage E of the condenser by 90 deg and E and I are in phase. The conduction current I has H circulations around its flow, just

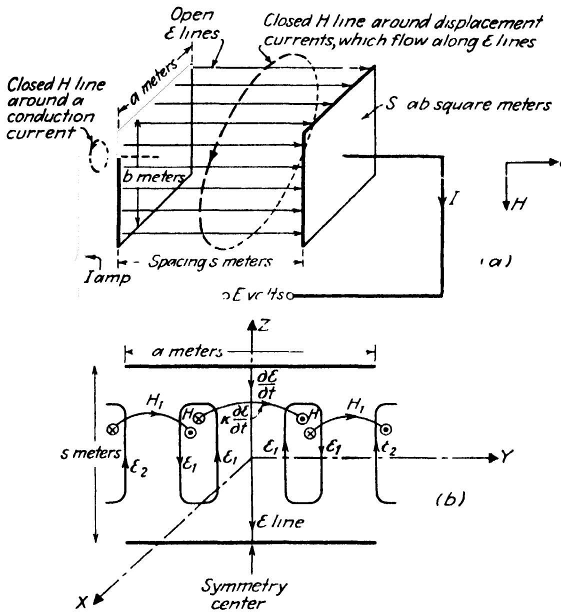


FIG. 21. Shows how magnetic and electric whirling (curling) causes radiation from a condenser.

as the displacement current I_d must have H circulations around its flow. Hence, a transverse magnetic field and a longitudinal electric field exist between the condenser plates as far as the direction normal to the condenser plates is concerned. Nevertheless, the $\mathcal{E}$ and H fields are perpendicular to each other, and there is a radiation possibility normal to the area formed by the $\mathcal{E}$ and H vectors [according to the Poynting theorem and according to the curl $\mathcal{E}$ and curl H relations of Eqs. (1)]. Equations (78) express the same thing in the integral form and show that a variable magnetic flux can cause an electromotive force and a variable electric flux causes a magnetomotive force; the former is in volts, and the latter is in amperes. It is the simplified version of the integral form with the conduction current I of the medium equal to zero that concerns us in the case of a perfect dielectric between the condenser plates shown in Fig. 21. These equations as well as the curl $\mathcal{E}$ and curl H relations,

which are basic in Maxwell's theory, show that a variable $\mathcal{E}$ field may cause a variable H field and vice versa. This may happen in unobstructed electromagnetic field space and may also happen in the dielectric of a condenser as in Fig. 21. With respect to Fig. 21*b*, an increase $\kappa_0 \partial \mathcal{E}/\partial t$ of the electric flux density $\kappa_0 \mathcal{E}$ must occur along the same direction as the direction of the electric $\mathcal{E}$ lines of force between respective condenser plates. According to

$$\oint_{l_2} H\, dl = \frac{d\left(\int\int_{S_2} \kappa_0 \mathcal{E}\, dS\right)}{dt} = j\omega\kappa_0 \int\int_{S_2} \mathcal{E}\, dS$$

for a nondissipative dielectric, an associated magnetic H field is caused by the time variation of the electric flux $\int\int \kappa_0 \mathcal{E}\, dS$. This is an *H circulation around the dielectric displacement current.* But this magnetic H field varies also with time and with it, the magnetic flux $\int\int \mu_0 H\, dS$. According to

$$\oint_{l_1} \mathcal{E}_1\, dl = -\frac{d(\int\int \mu_0 H\, dS)}{dt} = -j\omega\mu_0 \int\int_{S_1} H\, dS$$

the time variation of the magnetic flux $\int\int \mu_0 H\, dS$ causes an increase in the same sense, generating a curling $\mathcal{E}_1$ field which circles around the H lines. Strictly speaking, the $\mathcal{E}_1$ field circles around the magnetic displacement current. The circulating $\mathcal{E}_1$ lines near the symmetry center are in opposition to the original $\mathcal{E}$ lines and therefore weaken, the original $\mathcal{E}$ field near the symmetry center but cause an additive effect toward the outside of the condenser. In exactly the same way, an H_1 field circulation is generated by the time variation of the $\mathcal{E}_1$ field and circles around the $\mathcal{E}_1$ field, etc. Hence, successive looping of H lines around $\mathcal{E}$ lines, of $\mathcal{E}_1$ lines around H lines, and of H_1 lines around $\mathcal{E}_1$ lines, etc., moves electromagnetic field energy very rapidly from the symmetry center toward the outside contour of the condenser. As soon as this contour is reached, space radiation sets in. We have, therefore, the following line integral and corresponding curl actions:

$$\left\{\begin{array}{lll}
(a) & j\omega\kappa_0 \int\int \mathcal{E}\, dS \text{ causes } \oint H\, dl & (\text{is } H \text{ circulation}) \\
(b) & -j\omega\mu_0 \int\int H\, dS \text{ causes } \oint \mathcal{E}_1\, dl & (\text{is } \mathcal{E}_1 \text{ circulation}) \\
(c) & j\omega\kappa_0 \int\int \mathcal{E}_1\, dS \text{ causes } \oint H_1\, dl & (\text{is } H_1 \text{ circulation}) \\
(d) & -j\omega\mu_0 \int\int H_1\, dS \text{ causes } \oint \mathcal{E}_2\, dl & (\text{is } \mathcal{E}_2 \text{ circulation}) \\
(e) & j\omega\kappa_0 \int\int \mathcal{E}_2\, dS \text{ causes } \oint H_2\, dl & (\text{is } H_2 \text{ circulation})
\end{array}\right\}$$

or in the differential form,

$$\left\{\begin{array}{lr} \dfrac{\partial(\kappa_0 \mathcal{E})}{\partial t} \text{ causes curl } H & (a) \\ -\dfrac{\partial(\mu_0 H)}{\partial t} \text{ causes curl } \mathcal{E}_1 & (b) \\ \dfrac{\partial(\kappa_0 \mathcal{E}_1)}{\partial t} \text{ causes curl } H_1 & (c) \\ -\dfrac{\partial(\mu_0 H_1)}{\partial t} \text{ causes curl } \mathcal{E}_2 & (d) \\ \dfrac{\partial(\kappa_0 \mathcal{E}_2)}{\partial t} \text{ causes curl } H_2 & (e) \end{array}\right\} \quad (80)$$

etc., with $\mathcal{E}$, H, $\mathcal{E}_1$, H_1, $\mathcal{E}_2$, H_2, etc., fields moving from the symmetry center of the condenser toward the open boundary between the dielectric of the condenser and the surrounding air outside the condenser. The successive steps in (80) show what is actually meant by the curl $\mathcal{E}$ and curl H relations, which are the Maxwellian main relations. It also brings out what is meant by the integral form of these relations. As mentioned in Sec. 4 in connection with relations (1) and (2), the curl of a field vector F is the F circulation per unit area of an infinitely small circulation loop and so oriented that *maximum* circulation occurs. Hence, we deal with infinitesimally small circulations, since the closed path l of $F\,dl$ encloses an area that is infinitesimally small and the curl F is normal to this elementary area. The differential forms of successive steps (*a*), (*b*), (*c*), (*d*), (*e*), etc., refer, therefore, to differential circulations of H, while the integral forms refer to finite circulations. In practice, the integral forms seem most useful, since they relate, for the general medium, the total enclosed displacement current and the total enclosed conduction current in amperes to the H circulation, for instance.

23. Application of the Ampère-Maxwell Law and the Faraday Induction Law. The curl $\mathcal{E}$ relation of Eqs. (1), page 11, is based on Faraday's induction law, and the curl H relation is the Ampère-Maxwell law. The experimental Faraday law gives the emf around a circuit through which the magnetic flux varies. The Ampère-Maxwell law generalizes the static law that the line integral of the magnetic field vector H around any closed curve enclosing a conduction current I in amperes is numerically equal to the value of I, where H is in amperes per meter and the closed path l is expressed in meters. If I varies with time, according to Maxwell, $\oint_l H\,dl$ is equal to $I + j\omega\kappa \iint_S \mathcal{E}\,dS$, where I is the conduction current and the imaginary component equal to the displacement current in amperes. The conduction current is then generally given by $I = \iint_S \sigma\mathcal{E}\,dS$, where S denotes the area enclosed by the closed boundary l.

According to Eqs. (1) and (2), we can use the differential form or the integral form for a field vector F, where F is either the electric $\mathcal{E}$ or the magnetic H vector, and write

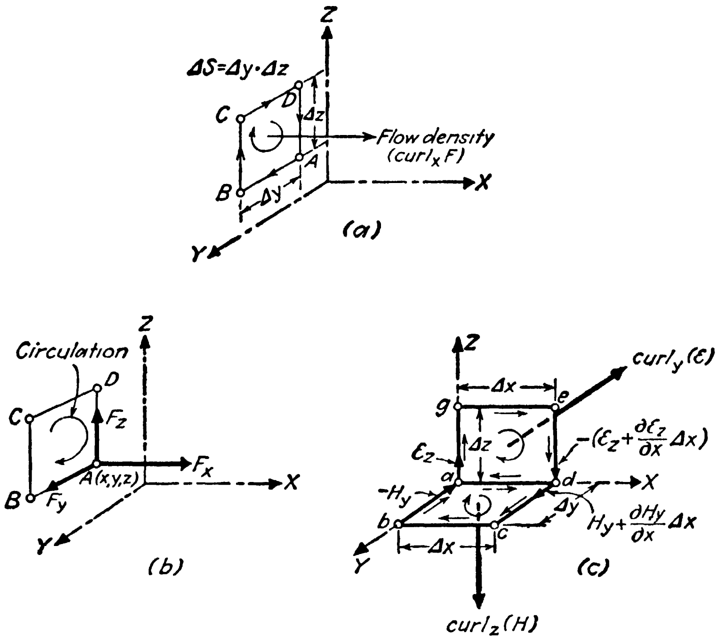
$$\left.\begin{aligned} \text{curl } F &= \text{flow density} \\ \oint_l F \cdot dl &= \text{flow} = \int\int_S \text{curl}_n F \cdot dS \end{aligned}\right\} \tag{81}$$

which means

$$\oint F \cdot dl = \underbrace{(\text{flow density})}_{\text{curl } F} dS \tag{82}$$

if dS is a vanishingly small enclosed area. This relation satisfies the curl definition, since the flow density, called curl F, is uniform over dS. From Eqs. (1), we note that for $F = H$, we have for the flow density a real and an imaginary component in amperes per square meter, since for $\epsilon^{j\omega t}$ variations, curl $H = J + j\omega\kappa\mathcal{E} = k\mathcal{E}$, for $k = \sigma + j\omega\kappa$. For $F = \mathcal{E}$, we have only a negative time rate, or curl $\mathcal{E} = -j\omega\mu H$. Now generally, $\mathcal{E}$ may be oriented arbitrarily in space, so that we have for the absolute values of the XYZ components, $\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2 + \mathcal{E}_z^2}$. In a similar way, we have also $H = \sqrt{H_x^2 + H_y^2 + H_z^2}$ and such vectorial additions, for the respective flow densities Now, in Cartesian coordinates, the infinitesimal *space* element is $dx\,dy\,dz$ measured in cubic meters; and by rotating the XYZ system, we can always place the elementary area dS in any of the three perpendicular planes of the XYZ system. The boundary line of a very small area ΔS is then, for ΔS in the YZ plane as in Fig. 22a and b, given by the closed contour $ABCDA = \Delta y + \Delta z + \Delta y + \Delta z = 2(\Delta y + \Delta z)$. Assuming such a convention that the circulation of the field vector F is like a corkscrew and the flow density is in the direction the screw head moves, then F acts along the closed elementary boundary as is indicated by the arrows. Now F acts like a force, and Δl is a distance; therefore, $F\,\Delta l$ in (81) and (82) is force times distance, which means work. Since F stands for either H or $\mathcal{E}$, the resultant field F has generally components of magnitudes F_x, F_y, and F_z at a space point $A(x, y, z)$. Therefore, for the line element $\Delta l = AB = \Delta y$, we have the portion $F_y\,\Delta y$ of $\oint F\,dl$; along the line element $\Delta l = AD = \Delta z$, we have the portion $-F_z\,\Delta z$ of $\oint F\,dl$, since the circulation arrow points against the F_z component. The F_x component can contribute nothing, since it is normal to the very small area $\Delta S = \Delta y\,\Delta z$ and nowhere along the boundary of ΔS. With respect to the line element $\Delta l = BC = \Delta z$, we have no longer the field value F_z, since BC is separated from F_z by a distance of Δy. Since $\partial F_z/\partial y$ is the change of F_z per unit separation, the quantity $(\partial F_z/\partial y)\,\Delta y$ must be the change for a separation of Δy, and

the field acting along BC must be $F_z + (\partial F_z/\partial y)\,\Delta y$. Therefore, the portion of $\oint F\,dl$ acting along BC must be $[F_z + (\partial F_z/\partial y)\,\Delta y]\,\Delta z$. In exactly the same way, we find for the circulation element $\Delta l = CD = \Delta y$ a value of $-[F_y + (\partial F_y/\partial z)\,\Delta z]$, since this field acts in opposition to



$\Delta y\,\Delta z = \Delta S$ small-area element, which for $\Delta y\,\Delta z \to 0$ becomes $dy\,dz = dS$. F_y acts along $AB = \Delta y$

$F_z + \underbrace{\dfrac{\partial F_z}{\partial y}\,\Delta y}_{\text{change along }\Delta y}$ acts along $BC = \Delta Z$. $-(F_y + \dfrac{\partial F_y}{\partial z}\,\Delta z)$ acts along $CD = \Delta y$. $-F_z$ acts along $DA = \Delta Z$ Positive and negative signs occur, since the direction of the circulation must be taken into account Curl$_x(F)$ is the X component of the curl acting normal to the area $ABCDA$ of area element $dy\,dz$

FIG 22 Details on the curl of a vector F (where F is either the electric $\mathcal{E}$ or magnetic H field vector) and illustrations on the Cartesian components of such a curl.

the polarity $+F_y$, and the portion of $\oint F\,dl$ along CD becomes $-[F_y + (\partial F_y/\partial z)\,\Delta z]\,\Delta y$. Therefore, the *entire* $\oint F\,dl$ circulation around $ABCDA$ becomes

$$\underbrace{\oint_l F\,dl}_{\text{for } ABCDA = l} = \underbrace{F_y\Delta_y}_{\text{for } AB} + \underbrace{\left(F_z + \frac{\partial F_z}{\partial y}\,\Delta y\right)\Delta z}_{\text{for } BC} - \underbrace{\left(F_y + \frac{\partial F_y}{\partial z}\,\Delta z\right)\Delta y}_{\text{for } CD} - \underbrace{F_z\,\Delta z}_{\text{for } DA}$$

or

$$\oint_l F\,dl = F_y \cdot \Delta_y + F_z\,\Delta z + \frac{\partial F_z}{\partial y}\underbrace{\Delta y\,\Delta z}_{\Delta S} - F_y\Delta_y - \frac{\partial F_y}{\partial z}\underbrace{\Delta z\,\Delta y}_{\Delta S} - F_z\,\Delta z$$

$$= \frac{\partial F_z}{\partial y}\Delta S - \frac{\partial F_y}{\partial z}\Delta S$$

But according to (82)

$$\underbrace{\text{curl}_x F}_{\substack{\text{flow density}\\ \text{along X direc-}\\ \text{tion}}} = \frac{\oint F\,dl}{\underbrace{\Delta S \to 0}_{dS}} = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \tag{83}$$

In exactly the same way, we find for $\Delta S = \Delta x\,\Delta z$, which is in the XZ plane, a flow density along the Y direction and for $\Delta S = \Delta x\,\Delta y$, lying in the XY plane, a flow density along the Z direction. These flow densities are known as the "curl F component along the Y direction" and the "curl F component along the Z direction" and read

$$\left.\begin{aligned}
\underbrace{\text{curl}_y F}_{\substack{\text{flow density}\\ \text{along Y direc-}\\ \text{tion}}} &= \frac{\oint F\,dl}{\underbrace{\Delta x\,\Delta z \to 0}_{dx\ dz}} = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\\
\underbrace{\text{curl}_z F}_{\substack{\text{flow density}\\ \text{along Z direc-}\\ \text{tion}}} &= \frac{\oint F\,dl}{\underbrace{\Delta x\,\Delta y \to 0}_{dx\ dy}} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}
\end{aligned}\right\} \tag{84}$$

Such relations are applied throughout this text and compiled in Eqs. (135-139) pages 148-149, also for cylindrical and spherical coordinates. The *three* component relations of (83) and (84) are usually expressed by the *single* relation *curl F = total flow density*. Suppose that F represents the magnetic field vector H in the general medium, with the electromagnetic constants μ, κ, and σ. Then the total flow density consists, according to Eqs. (1), page 11, of a conduction current density $J = \sigma\mathcal{E}$ and a displacement current density $J_d = j\omega\kappa\mathcal{E}$ amperes per square meter. Since in Fig. 22a, the circulation is in the YZ plane, we have

$$\text{Flow density} = \text{curl}_x F = \text{curl}_x H = J_x + J_{d_x}$$
$$= (\sigma + j\omega\kappa)\mathcal{E}_x \qquad \text{amp/m}^2$$

For $F = \mathcal{E}$ volts per meter, we have, according to (1),

$$\text{Flow density} = \text{curl}_x \mathcal{E} = -\frac{\partial(\mu H_x)}{\partial t} = -j\omega\mu H_x \qquad \text{volts/m}^2$$

In Fig. 22c, we have the case of a transverse electromagnetic field with the resultant $\mathcal{E}$ vector along the Z direction and the resultant H vector along the Y direction. We know already that such a field asso-

ciation represents a wave propagation along the X axis. Let us see now how Eq. (82) can be used to prove that such a wave motion is possible. We note that the very small area $\Delta S = \Delta y\,\Delta x$ is associated with the H circulation and that the very small area $\Delta S = \Delta x\,\Delta z$ belongs to the $\mathcal{E}$ circulation. We also note that we have only $\mathcal{E}_z$ and only H_y fields. This means that along the eg, ad, and bc line elements, there can be no work done, since we have no field acting. We find then as above

$$\underbrace{\oint H\,dl}_{abcda=l} = \underbrace{-H_y\,\Delta y}_{\text{for } ab} + \underbrace{0\,\Delta x}_{\substack{\text{for } bc\\(=0)}} + \underbrace{\left(H_y + \frac{\partial H_y}{\partial x}\Delta x\right)\Delta y}_{\text{for } cd} + \underbrace{0\,\Delta x}_{\substack{\text{for } da\\(=0)}}$$

$$= -H_y\,\Delta y + H_y\,\Delta y + \frac{\partial H_y}{\partial x}\underbrace{\Delta x\,\Delta y}_{\Delta S} = + \frac{\partial H_y}{\partial x}\Delta S$$

$$\text{curl}_z\,H = \frac{\oint_l H\,dl}{\underbrace{\Delta x\,\Delta y \to 0}_{dx\,dy}} = + \frac{\partial H_y}{\partial x} = \underbrace{\frac{\partial(\kappa\mathcal{E}_z)}{\partial t}}_{\substack{\text{according to Eqs (1)}\\ \text{for } \sigma = 0 \text{ equal to}\\ \text{curl } H}}$$

Therefore,

$$\frac{\partial H_y}{\partial x} = \kappa\frac{\partial \mathcal{E}_z}{\partial t} \tag{85}$$

For the $\mathcal{E}$ circulation, we find

$$\oint_{l=adega} \mathcal{E}\,dl = \underbrace{0\,\Delta x}_{\substack{\text{for } ad\\(=0)}} - \underbrace{\left(\mathcal{E}_z + \frac{\partial \mathcal{E}_z}{\partial x}\Delta x\right)\Delta z}_{\text{for } de} + \underbrace{0\,\Delta x}_{\substack{\text{for } eg\\(=0)}} + \underbrace{\mathcal{E}_z\,\Delta z}_{\text{for } ga}$$

$$= -\mathcal{E}_z\,\Delta z - \frac{\partial \mathcal{E}_z}{\partial x}\Delta x\,\Delta z + \mathcal{E}_z\,\Delta z = - \frac{\partial \mathcal{E}_z}{\partial x}\Delta x\,\Delta z$$

$$\text{curl}_y\,\mathcal{E} = \frac{\oint_l \mathcal{E}\,dl}{\underbrace{dx\,dz}_{\substack{\text{vanishingly}\\ \text{small}}}} = - \frac{\partial \mathcal{E}_z}{\partial x} = \underbrace{- \frac{\partial(\mu H_y)}{\partial t}}_{\substack{\text{according to Eqs (1)}\\ \text{equal to curl}_y}}$$

and therefore,

$$\frac{\partial \mathcal{E}_z}{\partial x} = \mu\frac{\partial H_y}{\partial t} \tag{86}$$

Differentiating (85) with respect to t and (86) with respect to x, we find

$$\frac{\partial^2 H_y}{\partial x\,\partial t} = \kappa\frac{\partial^2 \mathcal{E}_z}{\partial t^2}; \qquad \frac{\partial^2 \mathcal{E}_z}{\partial x^2} = \mu\frac{\partial^2 H_y}{\partial x\,\partial t}$$

$$\frac{\partial^2 H_y}{\partial x\,\partial t} = \frac{1}{\mu}\frac{\partial^2 \mathcal{E}_z}{\partial x^2} = \kappa\frac{\partial^2 \mathcal{E}_z}{\partial t^2}$$

yielding

$$\frac{\partial^2 \mathcal{E}_z}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \mathcal{E}_z}{\partial t^2} \quad \text{for} \quad v = \frac{1}{\sqrt{\mu \kappa}} \quad \text{m/sec}$$

Differentiating (85) with respect to x and (86) with respect to t gives the same form of equation in terms of H_y, since $\partial^2 \mathcal{E}_z/\partial x \, \partial t$ is eliminated. We have, therefore, the relations

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 F}{\partial t^2} = 0; \quad F = \mathcal{E}_z \quad \text{volts/m}; \quad F = H_y \quad \text{amp/m};$$

$$v = \frac{1}{\sqrt{\mu \kappa}} \quad \text{m/sec} \quad (87)$$

showing that the field function F moves with the velocity v along the X direction and that we have, therefore, a TEM wave motion. This outcome also applies to the illustration of Fig. 21*b*, since, except for the edge effects of the condenser, the original transverse electric $\mathcal{E}$ lines of force are along the $\pm Z$ directions and the magnetic H lines of force circle around the multitudes of elementary displacement currents, flowing likewise normal to the respective condenser plates. Hence, we have an $\mathcal{E}$ field along the $+Z$ directions and H fields not only along the $\pm Y$ directions but also in all directions in the XY plane and planes parallel thereto, *within the space* between the respective condenser plates. Therefore, electromagnetic radiation is possible in all directions around the condenser owing to the transversability of the $\mathcal{E}$ and H fields.

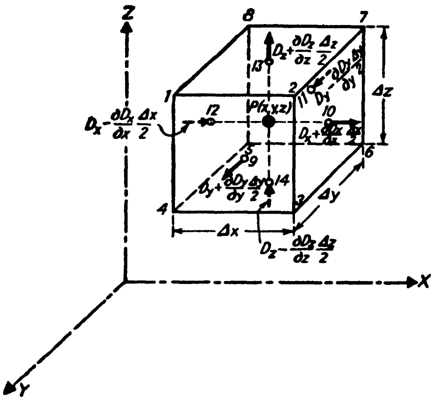
24. Field Energy Relations, Gauss Theorem, and Gauss Law Applied to the Divergence Concept. Here also we deal with differential magnitudes, *i.e.*, with a small volume $\Delta\tau = \Delta x \, \Delta y \, \Delta z$ cubic meters, which in the limit approaches zero value and is known as $d\tau = dx \, dy \, dz$, so that the $\mathcal{E}$ and H values may be taken *constant* for the elementary reference volume (Fig. 23). In energy relations, we *must* deal with $B = \mu H$ webers per square meter and $D = \kappa \mathcal{E}$ coulombs per cubic meter, *i.e.*, with magnetic and electric flux densities, instead of H and $\mathcal{E}$ only, since even for vacuum we still have the values $\mu = \mu_0$ and $\kappa = \kappa_0$ for the electromagnetic medium properties. We have for any field vector F (which may be $\mathcal{E}$ or H, respectively), the *Gauss theorem*

$$\iiint_{\tau = xyz} (\text{div } F) \, d\tau = \iint_S F_n \, dS \quad (88)$$

which transforms a *volume* integral into an equivalent *surface* integral. Corresponding to (81), we have then

$$\text{div } F = \frac{\iint_S F_n \, dS}{\tau \to 0}; \quad \iiint_\tau (\text{div } F) \, d\tau = \iint_S F_n \, dS \quad (89)$$

since the outward flux F over the surface S completely enclosing the volume τ, divided by the volume τ, yields the *average* divergence of F. If the volume τ shrinks to a point ($\tau \to 0$), then the surrounding surface S must also shrink to the same space point ($S \to 0$), and the average divergence simplifies to what is known as "div F." The subscript n expresses only that F acts normal to the surface element dS.



At the symmetry center $P(x, y, z)$, we have D_x, D_y, and D_z components, where $D = \kappa\mathcal{E}$ coulombs/m². Elementary volume: $d\tau = dx\,dy\,dz$. Very small volume $\Delta\tau = \Delta x\,\Delta y\,\Delta z$ which for the limit approaching zero, or $\Delta\tau \to 0$, is $d\tau$.

FIG. 23. Details on the divergence of a vector quantity. The outward flux across the *closed* surface divided by the *enclosed* volume is known as the *average* divergence. The limiting value for which the volume shrinks to a point is known as the divergence of the outward flux at the point.

As is brought out on page 198, div F can also be written as $\nabla \cdot F$, which is a dot product. In Cartesian XYZ coordinates, the operator ∇ stands for $\partial/\partial x + \partial/\partial y + \partial/\partial z$. A simplification occurs for a surface instead of a volume divergence, where then the flux through the boundary of the elementary cells is div $F \cdot d\tau$ and taken only in a definite direction. We deal then only with $dy\,dz$ surfaces of $d\tau$ if we are interested in the field energy escape over one of the $dy\,dz$ surfaces, as is explained in connection with Figs. 4 and 5. The particular surface divergence then reads

$$\operatorname{div}_s F = \frac{\int_l F_n\,dl}{S \to 0} \tag{89a}$$

where S square meters is the area with the outside contour l meters in length. This requires that the surface S shrinks to a point, which is equivalent to saying that l approaches zero value in the limit. A surface divergence would, for a divergence along the X direction, have an operation $\nabla = \partial/\partial y + \partial/\partial z$ if Cartesian coordinates are used. A further simplification occurs for a linear divergence, since then the operator ∇ of div $F = \nabla \cdot F$ is, for instance, simplified to $\nabla = \partial/\partial x$ and div $F = \partial F/\partial x$, which is a gradient.

An application of (89) refers to the classic *Gauss law*, stating that the total electric flux escaping over a sphere of any radius is equal to the net charge of q coulombs within the sphere. As a matter of fact, this

law holds over any closed surface S, since $\iint_S \kappa\mathcal{E}\, dS = q$; and for the charge density of ρ coulombs per cubic meter, we have

$$\iint_S (\kappa\mathcal{E})_n\, dS = \iiint_\tau (\text{div } D)\, d\tau = \iiint_\tau \rho d\tau \tag{90}$$

We have, therefore,

$$\underbrace{\frac{\iint_S \kappa\mathcal{E}\, dS}{\tau \to 0}}_{=\text{div }(\kappa\mathcal{E})} = \underbrace{\frac{\iiint_\tau \rho\, d\tau}{\tau \to 0}}_{=\rho} \tag{90a}$$

expresses that the displaced flux density $\kappa\mathcal{E}$ coulombs/m² streaming out of the infinitesimal volume is equal to the space charge density ρ in coulombs/m³

Since any of the field associations can generally be oriented, we deal with corresponding components of the $\mathcal{E}$ as well of the H field. If we take $D = \kappa\mathcal{E}$ coulombs per square meter as the absolute amount of the resultant electric flux density and $B = \mu H$ webers per square meter as the amount of the resultant magnetic flux density, we have the corresponding Cartesian component relations $D = \sqrt{D_x^2 + D_y^2 + D_z^2}$ and $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$. In Fig. 23, we have a *vanishingly* small volume element $d\tau = dx\, dy\, dz$ at the symmetry center $P(x, y, z)$ of a small-volume element $\Delta\tau = \Delta x\, \Delta y\, \Delta z$ cubic meters. Along the X direction, we have at P the electric flux density $D_x = \kappa\mathcal{E}_x$, as well as the components $D_y = \kappa\mathcal{E}_y$ and $D_z = \kappa\mathcal{E}_z$ coulombs per square meter along the Y and Z directions, respectively. Since the front end face 2-3-6-7 has a $\Delta y\, \Delta z$-square-meter area, the total electric flux across this face in coulombs is $\Delta y \cdot \Delta z$ times larger than the flux density. The flux density over this area is *not* D_x, *but* $D_x + (\partial D_x/\partial x)(\Delta x/2)$, because $\partial D_x/\partial x$ is the increase per unit length and $(\partial D_x/\partial x)(\Delta x/2)$ is the increase for a distance of $\Delta x/2$ from P on the end face 2-3-6-7. This flux density across 2-3-6-7 points *out*ward. In the same way, we find an *in*ward flux density $D_x - (\partial D_x/\partial x)(\Delta x/2)$ coulombs per square meter acting across the rear face 1-4-5-8. Therefore, the surface divergence times the volume $d\tau$ over the rear face 1-4-5-8 and the front face 2-3-6-7 for a volume $\Delta\tau \to 0$, which is $d\tau$, becomes

$$\underbrace{\left(D_x + \frac{\partial D_x}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\substack{\text{total flux toward}\\ \text{the elementary}\\ \text{front face } dy\, dz}} - \underbrace{\left(D_x - \frac{\partial D_x}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\substack{\text{total flux in}\\ \text{coulombs enter-}\\ \text{ing the rear}\\ \text{face } dy\, dz}}$$

$$= D_x\, dy\, dz + \frac{\partial D_x}{\partial x}\underbrace{dx\, dy\, dz}_{d\tau} - D_x\, dy\, dz = \frac{\partial D_x}{\partial x}\, d\tau \qquad \text{coulombs}$$

If we do the same thing for the front and rear faces 1-2-3-4 and 8-7-6-5 of equal differential areas $dx\, dz$ (for $\Delta\tau \to 0$), we obtain the excess total flux of $(\partial D_y/\partial y)\, d\tau$ coulombs. This is then the surface divergence times $d\tau$ along the Y direction. Along the Z direction, we have similarly an excess flux of $(\partial D_z/\partial z)\, d\tau$ coulombs. The net flux leaving the *entire* enclosing surface of the volume element $d\tau$ then yields

$$\frac{\partial D_x}{\partial x}\,d\tau + \frac{\partial D_y}{\partial y}\,d\tau + \frac{\partial D_z}{\partial z}\,d\tau \qquad \text{coulombs}$$

which, according to the divergence F relation of (89), yields the divergence

$$\operatorname{div} D = \operatorname{div}(\kappa\mathcal{E}) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{coulombs/m}^3 \tag{91}$$

since the net flux over the enclosing surface of $d\tau$ must be taken per unit volume (must be divided by $d\tau$). If there are charges of volume density ρ coulombs per cubic meter within a volume, then div $(\kappa\mathcal{E}) = \rho$, according to the Gauss law. For no such charges (no sources) or sinks (no absorbers) within the volume, just as much flux must enter the volume as leaves it. Then div $(\kappa\mathcal{E}) = 0$. This is *always* the case for div (μH), since *always* div $(\mu H) = 0$, because we deal *always with closed lines of magnetic force.* Therefore, just as many H lines must enter a volume element as leave it.

25. Field Energy Relations for Traveling and Stationary Electromagnetic Waves. Let us take $\mathcal{E}_z$ and H_y *resultant* field associations, so that wave propagation in Fig. 23 occurs along the X direction for a progressive wave (which is a traveling wave). Since $\mathcal{E}_z$ is in volts per meter at any instant, the product $\mathcal{E}_z\,dz$ must represent a voltage of dE volts for the separation distance of dz between the upper and the lower surface of the elementary parallelepiped of Fig. 23. Since these surfaces are $dx\,dy$, they must experience an electric flux of $(\kappa\mathcal{E}_z)\,dx\,dy$ coulombs through them, since

$$\int\int_{dx\,dy} (\kappa\mathcal{E}_z)\,dx\,dy = dq \text{ coulombs.}$$

The electric field energy in the volume element $d\tau$ is then

$$P_e = \tfrac{1}{2}dq\,dE = \tfrac{1}{2}\underbrace{\kappa\mathcal{E}_z \cdot dx\,dy}_{dq}\,\mathcal{E}_z\,dz = \tfrac{1}{2}\kappa\mathcal{E}_z^2 \cdot d\tau \qquad \text{joules}$$

yielding an electric energy density of $\tfrac{1}{2}\kappa\mathcal{E}_z^2$ joules per cubic meter. Since the magnetic field intensity H_y is in amperes per meter, the quantity $H_y\,dy$ must be the magnetomotive force in amperes and holds for the separation of the end faces of area $dx\,dz$ experiencing a total magnetic flux $d\Phi = (\mu H_y)\,dx\,dz$ webers, since the magnetic flux density μH_y is in webers per square meter. The magnetic field energy in the volume element $d\tau$ is then

$$P_m = \tfrac{1}{2}\mu H_y\,dx\,dzH_y\,dy = \tfrac{1}{2}\mu H_y^2\,d\tau \qquad \text{joules}$$

and we find a magnetic energy density of $\tfrac{1}{2}\mu H_y^2$ joules per cubic meter and a total electromagnetic energy density of $\tfrac{1}{2}(\mu H_y^2 + \kappa\mathcal{E}_z^2)$ joules per cubic meter at any instant. This result is anticipated from the elementary circuit theory for the instantaneous field energies $\tfrac{1}{2}Li^2$ and $\tfrac{1}{2}Ce^2$ for a magnetic field of a coil and an electric field of a condenser, by replacing L and i by μ and H, and C and e by κ and $\mathcal{E}$. The basic relations

$$\underbrace{H_y\frac{\partial\mathcal{E}_z}{\partial x} = \mu H_y\frac{\partial H_y}{\partial t}}_{\text{is Eq. (86) multiplied by } H_y}; \qquad \underbrace{\mathcal{E}_z\frac{\partial H_y}{\partial x} = \kappa\mathcal{E}_z\frac{\partial\mathcal{E}_z}{\partial t}}_{\text{is Eq. (85) multiplied by } \mathcal{E}_z} \tag{92}$$

for plane waves and $\mathcal{E}_z$ and H_y fields can now be used to derive the field power relation

$$\mathcal{E}_z \cdot H_y = v\underbrace{(\tfrac{1}{2}\mu H_y^2 + \tfrac{1}{2}\kappa\mathcal{E}_y^2)}_{\substack{\text{total energy volume density}\\ \text{in joules/m}^3}} \quad \text{watts/m}^2;$$

$$v = \frac{1}{\sqrt{\mu\kappa}} \quad \text{m/sec} \quad \text{holds for a progressive (traveling) wave} \tag{93}$$

where we deal actually with instantaneous power. But if $\mathcal{E}_z$ and H_y are taken as the effective field values, (93) represents the average power transferred along the X direction across each square-meter area. The details are then as follows:

Adding the two relations of (92), we find

$$H_y\frac{\partial\mathcal{E}_z}{\partial x} + \mathcal{E}_z\frac{\partial H_y}{\partial x} = \underbrace{\mu H_y\frac{\partial H_y}{\partial t} + \kappa\mathcal{E}_z\frac{\partial\mathcal{E}_z}{\partial t}}$$
$$= \frac{1}{2}\mu\frac{\partial H_y^2}{\partial t} + \frac{1}{2}\kappa\frac{\partial\mathcal{E}_z^2}{\partial t} = \frac{\partial\overbrace{(\tfrac{1}{2}\mu H_y^2 + \tfrac{1}{2}\kappa\mathcal{E}_z^2)}^{\text{total field energy density}}}{\partial t} \tag{94}$$

Since for plane waves $\mathcal{E}_z/H_y = \sqrt{\mu/\kappa}$, relations (85) and (86) read

$$\frac{\partial H_y}{\partial x} = \kappa\frac{\partial\mathcal{E}_z}{\partial t} = \kappa\frac{\partial(\sqrt{\mu/\kappa}\,H_y)}{\partial t} = \kappa\sqrt{\frac{\mu}{\kappa}}\frac{\partial H_y}{\partial t} = \kappa\sqrt{\frac{\mu}{\kappa}}\frac{\sqrt{\mu\kappa}}{\sqrt{\mu\kappa}}\frac{\partial H_y}{\partial t}$$
$$= \kappa\sqrt{\frac{\mu}{\mu\kappa^2}}\sqrt{\mu\kappa}\frac{\partial H_y}{\partial t} = \sqrt{\mu\kappa}\frac{\partial H_y}{\partial t} = \frac{1}{v}\frac{\partial H_y}{\partial t}$$
$$\frac{\partial\mathcal{E}_z}{\partial x} = \mu\frac{\partial H_y}{\partial t} = \mu\sqrt{\frac{\kappa}{\mu}}\frac{\partial\mathcal{E}_z}{\partial t} = \mu\sqrt{\frac{\kappa}{\mu}}\frac{1}{v}\frac{1}{\sqrt{\mu\kappa}}\frac{\partial\mathcal{E}_z}{\partial t} = \mu\sqrt{\frac{\kappa}{\mu\mu\kappa}}\frac{1}{v}\frac{\partial\mathcal{E}_z}{\partial t} = \frac{1}{v}\frac{\partial\mathcal{E}_z}{\partial t}$$

rendering the left side of Eq. (94)

$$H_y\frac{\partial\mathcal{E}_z}{\partial x} + \mathcal{E}_z\frac{\partial H_y}{\partial x} = \frac{H_y}{v}\frac{\partial\mathcal{E}_z}{\partial t} + \frac{\mathcal{E}_z}{v}\frac{\partial\mathcal{E}_y}{\partial t}$$
$$= \frac{1}{v}\frac{\partial(\mathcal{E}_z\cdot H_y)}{\partial t}$$

Therefore, (94) then reads

$$\frac{\partial(\mathcal{E}_z\cdot H_y)}{\partial t} = v\frac{\partial(\tfrac{1}{2}\mu H_y^2 + \tfrac{1}{2}\kappa\mathcal{E}_z^2)}{\partial t}$$

which, upon integration with respect to t, yields

$$\mathcal{E}_z\cdot H_y = v\frac{\mu H_y^2 + \kappa\mathcal{E}_z^2}{2}$$

which confirms Eq. (93), since the integration *constant* has nothing to do with *variable* electromagnetic fields.

We deal with harmonic time and space variations with a phase factor $\epsilon^{j(\omega t - \beta x)}$, where $\beta = \omega\sqrt{\mu\kappa}$ denotes the phase constant in space in circular radians per meter. We find then $\mathcal{E}_z = \mathcal{E}_0\epsilon^{j(\omega t - \beta x)}$, and

$$\mathcal{E}_z \cdot H_y = \mathcal{E}_z^2\sqrt{\frac{\kappa}{\mu}} = \mathcal{E}_0^2\sqrt{\frac{\kappa}{\mu}}\underbrace{\cos^2(\omega t - \beta x)}_{\text{time average } \frac{1}{2}} \qquad \text{instantaneous power flow}$$

and

$$p_x = \frac{0.5\mathcal{E}_0^2}{377} = 1.327 \times 10^{-3}\mathcal{E}_0^2 \qquad \text{watts/m}^2 \tag{95}$$

if we take air as the wave medium and $\mathcal{E}_0$ volts per meter is the peak value of the electric field. It is the real value of the retarded time function that we must use, and $\cos(\omega t - \beta x)$ is the *real* component of $\epsilon^{j(\omega t - \beta x)} = \cos(\omega t - \beta x) + j\sin(\omega t - \beta x)$. Since we deal with watts, we must take the *average* power. The only quantity that is variable is the $\cos^2(\omega t - \beta x)$ function with an average value of $\frac{1}{2}$. We also have for $\mathcal{E}_z \cdot H_y$

$$\mathcal{E}_z \cdot H_y = v\frac{\mu H_y^2 + \kappa\mathcal{E}_z^2}{2} = \frac{\mu\frac{\kappa}{\mu}\mathcal{E}_z^2 + \kappa\mathcal{E}_z^2}{2\sqrt{\mu\kappa}} = \frac{\kappa\mathcal{E}_z^2}{\sqrt{\mu\kappa}} = \sqrt{\frac{\kappa}{\mu}}\mathcal{E}_z^2$$

which outcome is used in connection with the derivation of (95). Therefore, the power flow of a progressive space wave with a plane wave front can be computed from (95) if the electric field intensity is known. This field intensity can be measured. The relation (93) shows that the total field energy content $\frac{1}{2}(\mu H_y^2 + \kappa\mathcal{E}_z^2)$ in joules for a cubic-meter radiation space produces all the more watts through 1-sq-m areas normal to the X direction the higher the speed v of wave propagation. This speed is very high for transmission through air (essentially 3×10^8 m/sec).

We know from customary transmission-line theory that for stationary waves, we have a direct (advancing) wave train and a reflected (returning) wave train, where ρ is the ratio of the reflected amplitude $\mathcal{E}_r$ to the direct amplitude $\mathcal{E}_0$ and is known as the "reflection coefficient." We have then

$$\mathcal{E}_z^{(\text{volts/m})} = \underbrace{\mathcal{E}_0\cos(\omega t - \beta x)}_{\text{outgoing wave}} + \underbrace{\mathcal{E}_r\cos(\omega t + \beta x)}_{\text{reflected wave}}$$
$$= \mathcal{E}_0[\cos(\omega t - \beta x) + \rho\cos(\omega t + \beta x)]$$
$$H_y^{(\text{amp/m})} = \underbrace{\sqrt{\frac{\kappa}{\mu}}\mathcal{E}_0\cos(\omega t - \beta x)}_{\text{outgoing wave}} - \underbrace{\sqrt{\frac{\kappa}{\mu}}\rho\mathcal{E}_0\cos(\omega t + \beta x)}_{\text{reflected wave}} \tag{96}$$
$$= \mathcal{E}_0\sqrt{\frac{\kappa}{\mu}}[\cos(\omega t - \beta x) - \rho\cos(\omega t - \beta x)]$$

with the following details:

We have then

$$\cos(\omega t - \beta x) = \cos \omega t \cos \beta x + \sin \omega t \sin \beta x$$
$$\rho \cos(\omega t + \beta x) = \rho \cos \omega t \cos \beta x - \rho \sin \omega t \sin \beta x$$

$$\begin{aligned}
\mathcal{E}_z = \mathcal{E}_0 [\cos(\omega t - \beta x) + \rho \cos(\omega t + \beta x)] \\
&= \mathcal{E}_0 [(1 + \rho) \cos \omega t \cos \beta x + (1 - \rho) \sin \omega t \sin \beta x] \\
&= \mathcal{E}_0 \left[\left(1 + \frac{\mathcal{E}_r}{\mathcal{E}_0}\right) \cos \omega t \cos \beta x + \left(1 - \frac{\mathcal{E}_r}{\mathcal{E}_0}\right) \sin \omega t \sin \beta x\right] \\
&= (\mathcal{E}_0 + \mathcal{E}_r) \cos \omega t \cos \beta x + (\mathcal{E}_0 - \mathcal{E}_r) \sin \omega t \sin \beta x
\end{aligned}$$

yielding a resultant effect, at any instant,

$$\left.\begin{aligned}
\mathcal{E}_z^{(\text{volts/m})} &= \sqrt{(\mathcal{E}_0 + \mathcal{E}_r)^2 \cos^2 \beta x + (\mathcal{E}_0 - \mathcal{E}_r)^2 \sin^2 \beta x} \sin(\omega t + \varphi) \\
\mathcal{E}_z^{(\text{volts/m})} &= \mathcal{E}_0 \sqrt{(1 + \rho)^2 \cos^2 \beta x + (1 - \rho^2) \sin^2 \beta x} \sin(\omega t + \varphi) \\
\varphi &= \tan^{-1}\left(\frac{1 - \rho}{1 + \rho} \tan \beta x\right); \qquad \rho = \frac{\mathcal{E}_r}{\mathcal{E}_0}
\end{aligned}\right\} \tag{97}$$

since

$$\underbrace{(\mathcal{E}_0 + \mathcal{E}_r) \cos \beta x}_{b} \cos \omega t + \underbrace{(\mathcal{E}_0 - \mathcal{E}_r) \sin \beta x}_{a} \sin \omega t = a \sin \omega t + b \cos \omega t$$
$$= A \sin(\omega t + \varphi)$$
$$A \sin(\omega t + \varphi) = \underbrace{A \cos \varphi}_{a} \sin \omega t + \underbrace{A \sin \varphi}_{b} \cos \omega t$$

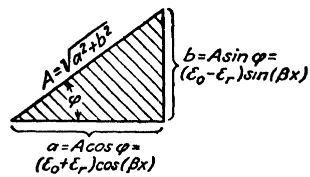
To make this relation hold at any time t, we choose a right triangle as in Fig. 24, with a hypotenuse $A = \sqrt{a^2 + b^2}$ and sides a and b as shown. The angle φ is then given by $\tan \varphi = b/a = [(1 - \rho) \tan \beta x]/(1 + \rho)$. For locations x, for which $\beta x = 2\pi x/\lambda = 0, \pi, 2\pi, 3\pi, \ldots$, we have $\cos^2 \beta x$ equal to unity and $\sin^2 \beta x$ equal to 0, and $\mathcal{E}_z = (\mathcal{E}_0 + \mathcal{E}_r) \sin \omega t$. For locations x, for which the argument $\beta x = \frac{1}{2}\pi, 3\pi/2, 5\pi/2, \ldots$, we have $\cos^2 \beta x = 0$ and $\sin^2 \beta x$ equal to unity, and $\mathcal{E}_z = (\mathcal{E}_0 - \mathcal{E}_r) \times \cos \omega t$, which for ideal wave reflection renders then $\mathcal{E}_z = 0$ volts/m at such places and $\mathcal{E}_z = 2\mathcal{E}_0 \sin \omega t$ for $\beta x = 0, \pi, 2\pi, \ldots$.

For stationary space waves, we have then to take the incident and reflected wave into account and find for the field energy density

$$\begin{aligned}
\frac{1}{2} \mu H_y^2 + \frac{1}{2} \kappa \mathcal{E}_z^2 &= \frac{\mu}{2} \frac{\kappa}{\mu} \mathcal{E}_0^2 [\cos(\omega t - \beta x) - \rho \cos(\omega t + \beta x)]^2 \\
&\quad + \frac{\kappa}{2} \mathcal{E}_0^2 [\cos(\omega t - \beta x) + \rho \cos(\omega t + \beta x)]^2 \\
&= \frac{\kappa}{2} \mathcal{E}_0^2 [\cos^2(\omega t - \beta x) - 2\rho \cos(\omega t - \beta x) \cos(\omega t + \beta x) \\
&\quad + \rho^2 \cos^2(\omega t + \beta x) + \cos^2(\omega t - \beta x) \\
&\quad + 2\rho \cos(\omega t - \beta x) \cos(\omega t + \beta x) + \rho^2 \cos^2(\omega t + \beta x)] \\
&= \kappa \mathcal{E}_0^2 [\underbrace{\cos^2(\omega t - \beta x)}_{\text{forward effect}} + \underbrace{\rho^2 \cos(\omega t + \beta x)}_{\text{reflected effect}}]
\end{aligned}$$

instantaneous joules/m³ (98)

We can now *no longer* multiply this result with the velocity $v = 1/\sqrt{\mu\kappa}$ meters per second in order to obtain the power transfer across square-meter space areas that are normal to the X direction (direction of outgoing wave propagation). We have to deal with an outgoing wave, as well



$$\varphi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left[\frac{\mathcal{E}_0 - \mathcal{E}_r}{\mathcal{E}_0 + \mathcal{E}_r}\tan(\beta x)\right] = \tan^{-1}\left[\frac{1-\rho}{1+\rho}\tan(\beta x)\right]$$

$$\underbrace{(\mathcal{E}_0 + \mathcal{E}_r)\cos(\beta x)}_{=a}\cos(\omega t) + \underbrace{(\mathcal{E}_0 - \mathcal{E}_r)\sin(\beta x)}_{=b}\sin(\omega t) = \sqrt{a^2+b^2}\sin(\omega t + \varphi)$$

FIG. 24. Solution of the equation $(\mathcal{E}_0 + \mathcal{E}_r)\cos(\beta x)\cos(\omega t) + (\mathcal{E}_0 - \mathcal{E}_r)\sin(\beta x)\sin(\omega t)$.

as a reflected wave. But we find such a power transfer from the $\mathcal{E}_z \cdot H_y$ product as found from the $\mathcal{E}_z$ and H_y expressions of Eq. (96), with the solution

$$\mathcal{E}_z \cdot H_y = \mathcal{E}_0^2\sqrt{\frac{\kappa}{\mu}}[\cos(\omega t - \beta x) + \rho\cos(\omega t + \beta x)][\cos(\omega t - \beta x) - \rho\cos(\omega t + \beta x)]$$

$$= \mathcal{E}_0^2\sqrt{\frac{\kappa}{\mu}}[\underbrace{\cos^2(\omega t - \beta x)}_{\substack{\text{has time average}\\ =\frac{1}{2}\text{ value}}} - \rho^2\underbrace{\cos^2(\omega t + \beta x)}_{\text{average }\frac{1}{2}}]$$

instantaneous power/m² (99)

The average power transferred every second across 1 sq m normal to the X direction is then only

$$p_x = \mathcal{E}_0^2\sqrt{\frac{\kappa}{\mu}}\left(\frac{1-\rho^2}{2}\right) = 0.5\,\frac{\mathcal{E}_0^2}{377}(1-\rho^2) = 1.327 \times 10^{-3}(1-\rho^2)\mathcal{E}_0^2$$

watts/m² (100)

Hence, for a reflection factor of 50 per cent, we have $\rho = 0.5$ or $\mathcal{E}_r = \frac{1}{2}\mathcal{E}_0$ and the Poynting vector $p_x = 1.325 \times 10^{-3}(1 - 0.25)\mathcal{E}_0^2 = 0.994 \times 10^{-3}\mathcal{E}_0^2$ watts per square meter, which is 75 per cent of the value for a progressive electromagnetic wave of $\mathcal{E}_0$ peak volts per meter. For a reflection coefficient ρ equal to unity, no power is being transferred. The difference between the available field watts and the radiated watts is then the reactive power due to standing space waves. From (98), we can compute the energy concentration per cubic meter and per second. This will give then the available watts per cubic meter in the electro-

magnetic wave medium. We find then

$$\text{Available field energy} = \frac{1}{2}\mu H_y^2 + \frac{1}{2}\kappa\mathcal{E}_z^2 = \kappa\mathcal{E}_0^2\,\frac{1+\rho^2}{2}$$

$$= \frac{8.854 \times 10^{-12}}{2}(1+\rho^2)\mathcal{E}_0^2$$

$$= 4.427 \times 10^{-12}(1+\rho^2)\mathcal{E}_0^2 \qquad \text{watts/m}^3 \qquad (101)$$

This looks, at first sight, like a negligible amount compared with the result of (100) expressing the watts per square meter transferred, but it is by no means negligible in comparison when we realize that the *speed* of wave propagation enters if we want to use the space power concentration for computing the power flux. Let us take the optimum power transfer. This happens when no reflected wave interferes, *i.e.*, for $\rho = 0$ and $\mathcal{E}_r = 0$ volts/m. For such a case, we have only to move the available field power with the phase velocity $v = 1/\sqrt{\mu\kappa}$ along the $+X$ direction, which happens to be essentially equal to 3×10^8 m/sec for wave propagation in air. We then find the radiation flux in watts through each square-meter area normal to the X direction as $3 \times 10^8 \times 4.427 \times 10^{-12}\mathcal{E}_0^2 = 1.3271 \times 10^{-3}\mathcal{E}_0^2$ watts per square meter, which checks the formula given in Eq. (95), holding for a traveling electromagnetic wave. Therefore, it does not take much energy concentration in an electromagnetic field to produce quite an energy flux in a square meter of the advancing wave front. We have, therefore, also a method of computing the forward-moving wave energy in case of stationary waves for which we have no ideal standing-wave ratio. We know that a portion of the available field energy, given in (101), moves forward with the velocity of $1/\sqrt{\mu\kappa}$ meters per second. It is characterized by the portion marked "forward effect" in (98). A "bucking" field action is due to the "reflected effect" in (98), since we deal with a motion of the electromagnetic field energy with the same velocity, but along the $-X$ direction. Hence, we find

$$\underbrace{\frac{1}{\sqrt{\mu\kappa}}\kappa\mathcal{E}_0^2\cos^2(\omega t - \beta x)}_{v\kappa\mathcal{E}_0^2 \cdot 1/2} - \underbrace{\frac{1}{\sqrt{\mu\kappa}}\rho^2\kappa\mathcal{E}_0^2\cos^2(\omega t + \beta x)}_{v\kappa\rho^2\mathcal{E}_0^2 \cdot 1/2} = \frac{v\kappa\mathcal{E}_0^2}{2}(1-\rho^2)$$

watts/m² for wave interference in space (102)

which is the relation for the *standing* electromagnetic space waves in place of Eq. (93), holding only for *traveling* space waves. We find then

$$\frac{v\kappa\mathcal{E}_0^2}{2}(1-\rho^2) = \frac{\kappa\mathcal{E}_0^2}{2\sqrt{\mu\kappa}}(1-\rho^2)$$

$$= \frac{8.854 \times 10^{-12} \times 3 \times 10^8}{2}(1-\rho^2)\mathcal{E}_0^2 = 1.327 \times 10^{-3}(1-\rho^2)\mathcal{E}_0^2$$

which checks the result of Eq. (100) obtained from the real part of the Poynting vector. It is the power density in watts per square meter in an arriving plane wave front that counts in the case of wave reception.

26. Electromagnetic Field Energy Concentrations in a Dissipative Wave Medium. Such a medium exhibits all three possible electromagnetic properties, namely, μ henrys per meter, κ farads per meter, and σ mhos per meter. We can *no longer* take the available electromagnetic power concentration in watts per cubic meter and move it with the velocity of propagation in order to account for the radiation density in watts per square meter in a progressive wave front. Joulean heat is now produced in the wave medium, owing to the conductivity. Let us take the case where $\mathcal{E} = \mathcal{E}_z$ volts per meter and $H = H_y$ amperes per meter, so that wave propagation occurs along the X direction. We have then the basic field relations

$$\frac{\partial H}{\partial x} = \sigma\mathcal{E} + j\omega\kappa\mathcal{E} \qquad \text{amp/m}^2; \qquad \frac{\partial \mathcal{E}}{\partial x} = j\omega\mu H \qquad \text{volts/m}^2 \quad (103)$$

instead of (85) and (86) for $\sigma\mathcal{E} = 0$, since $\sigma = 0$ in that case. These relations can also be read off from the six Cartesian components on page 148, omitting the terms that vanish, since no field component exists in certain directions. The curl components in Sec. 31 are *most useful* and used as a starting point in such problems.

We have now a complex medium, with a complex characteristic space impedance $\mathcal{E}/H = \sqrt{\mu/\kappa_c}$ in ohms, since $\kappa_c = \kappa - j\sigma/\omega$ farads per meter. We must then take the real value of the $\mathcal{E} \cdot H$ product in order to account for the power flow per unit area and unit time. It is easy to convert a complex expression into a real result. Thus if $\mathcal{E}/H = a + jb$ complex ohms, we have $Z_0 = a + jb$ with a conjugate value $Z^* = a - jb$ and find $Z \cdot Z^* = a^2 + b^2$, which is real. Thus, if we have $\mathcal{E} = \mathcal{E}_0\epsilon^{j\varphi_1} \cos \omega t$ for the instantaneous value in volts per meter, where φ_1 denotes the relative phase of the electric field intensity and $H = H_0\epsilon^{j\varphi} \cos \omega t$ is the time value for the associated magnetic field intensity in amperes per meter, we have for the conjugate value of H the expression $H_0^*\epsilon^{-j\varphi} \cos \omega t$. We find then

$$\mathcal{E} \cdot H^* = \mathcal{E}_0 H_0 \epsilon^{j(\varphi_1 - \varphi)} \cos^2 \omega t = \underbrace{\mathcal{E}_0 H_0 \cos^2 \omega t \cos (\varphi_1 - \varphi)}_{\text{real instantaneous power}} + \underbrace{j\mathcal{E}_0 H_0 \cos^2 \omega t \sin (\varphi_1 - \varphi)}_{\text{reactive}}$$

i.e., a real or power component and an imaginary or field power component. It is the real component that often counts and is written as Re $(\mathcal{E} \cdot H^*)$. By the same procedure we can show that Re $(\mathcal{E} \cdot H^*) =$ Re $(\mathcal{E}^* \cdot H)$, where $\mathcal{E}^* = \mathcal{E}_0\epsilon^{-j\varphi_1} \cos \omega t$. We have then the power transfer along the X axis as

$$p_x = \frac{\text{Re}\,(\mathcal{E} \cdot H^*)}{2} = \frac{\mathcal{E}_0 H_0}{2} \cos (\varphi_1 - \varphi_2) \qquad \text{watts/m}^2 \qquad (104)$$

since the average or constant value of the power counts, and the average value of a

$\cos^2 \omega t$ function is equal to ½. The quantity $\varphi_1 - \varphi_2$ represents then the time phase difference of between the $\mathcal{E}$ and H waves due to the complex medium.

The conjugate expression of the $\partial\mathcal{E}/\partial x$ relation and the unchanged $\partial H/\partial x$ relation of (103) then read for the complex peak values $\mathcal{E}_0$ and H_0

$$\frac{\partial \mathcal{E}_0^*}{\partial x} = -j\omega\mu H_0^*; \qquad \frac{\partial H_0}{\partial x} = \sigma\mathcal{E}_0 + j\omega\kappa\mathcal{E}_0$$

Multiplying the left equation by H_0 and the right equation by the conjugate $\mathcal{E}_0^*$, we have

$$H_0 \frac{\partial \mathcal{E}_0^*}{\partial x} = -j\omega\mu H_0^* \cdot H_0; \qquad \mathcal{E}_0^* \frac{\partial H_0}{\partial x} = \sigma\mathcal{E}_0 \cdot \mathcal{E}_0^* + j\omega\kappa\mathcal{E}_0 \cdot \mathcal{E}_0^*$$

An addition of these two equations yields

$$\underbrace{H_0 \frac{\partial \mathcal{E}_0^*}{\partial x} + \mathcal{E}_0^* \frac{\partial H_0}{\partial x}}_{=\frac{\partial(H_0 \cdot \mathcal{E}_0^*)}{\partial x}} = \sigma\mathcal{E}_0 \cdot \mathcal{E}_0^* + j\omega(\kappa\mathcal{E}_0 \cdot \mathcal{E}_0^* - \mu H_0^* \cdot H_0)$$

$$= A + jB \qquad \text{where } A \text{ and } B \text{ are real} \tag{105}$$

If $\mathcal{E} = \mathcal{E}_0\epsilon^{j\varphi_1} \cos \omega t$ and $\mathcal{E}^* = \mathcal{E}_0\epsilon^{-j\varphi_1} \cos \omega t$ are the instantaneous values, we have for the product

$$\mathcal{E} \cdot \mathcal{E}^* = \mathcal{E}_0^2\epsilon^{j(\varphi_1 - \varphi_1)} \cos^2 \omega t = \mathcal{E}_0^2 \cdot 1 \cdot \cos^2 \omega t = \tfrac{1}{2}\mathcal{E}_0^2 \qquad \text{for the time average}$$

and similarly $H^* \cdot H = \frac{1}{2}H_0^2$. The effective field values are $\mathcal{E}_e = \mathcal{E}_0/\sqrt{2}$ and $H_e = H_0/\sqrt{2}$. But in (105), we deal with the products of conjugate complex peak values for which $\mathcal{E}_0 \cdot \mathcal{E}_0^* = |\mathcal{E}_0^2|$ and $H_0 \cdot H_0^* = |H_0^2|$, *i.e.*, equal to the square of the absolute peak values of respective field associations. Therefore, $\mathcal{E}_0 \cdot \mathcal{E}_0^* = 2\mathcal{E}_e^2$ and $H_0 \cdot H_0^* = 2H_e^2$.

From (105) we find

$$\frac{\partial(\mathcal{E}_0^* \cdot H_0)}{\partial x} = \underbrace{2\sigma\mathcal{E}_e^2}_{2p_h} + j\omega(\underbrace{2\kappa\mathcal{E}_e^2}_{4p_e} - \underbrace{2\mu H_e^2}_{4p_m}) = 2p_h + \overbrace{j4\omega(p_e - p_m)}^{\text{reactive}}$$

$$\text{watts/m}^3 \tag{106}$$

where $p_h = \sigma\mathcal{E}_e^2$ is the heat loss in watts per cubic meter, $p_e = \kappa\mathcal{E}_e^2/2$ is the electric field energy density in joules per cubic meter; and $p_m = \mu H_e^2/2$ is the magnetic field energy density in joules per cubic meter. We have, therefore, a dissipative field power of p_h watts, which is lost in a unit volume of the medium, and useful available reactive watts per cubic meter, since $\omega = 6.28f$, which is the radian frequency per second, has the dimension of 1/sec, and when multiplied by joules per cubic meter, *i.e.*, volt · amp · sec per cubic meter, yields watts per cubic meter.

Formula (106) is the differential form. It is the integral form that is used when dealing with mediums of a finite volume of τ cubic meters, with a surface of S square meters over which field power flows. Integrating $\partial(\mathcal{E}_0^* \cdot H_0)/\partial_x$ over τ yields

$$\frac{1}{2}\int\int\int_\tau \frac{\partial(\mathcal{E}_0^* \cdot H_0)}{\partial_x} \underbrace{dx\,dy\,dz}_{=d\tau} = \underbrace{\frac{1}{2}\int\int_S \mathcal{E}_0^* \cdot \underbrace{H_0\,dy \cdot dz}_{dS}}_{\text{watts/m}^2}$$

$$= \underbrace{\overbrace{P_h}^{\text{heat loss}} + \overbrace{j2\omega(P_e - P_m)}^{\text{radiation watts}}}_{\text{watts for covering heat loss as well as stored electromagnetic power}} \qquad (107)$$

Since the volume integral changes to a surface integral, the surface integral must represent the entire electromagnetic power flow in watts normal to the surface of S square meters, while the right-hand side is the power dissipated plus the reactive field power available in the finite volume of τ cu m. This is true only if there are no electromagnetic sources within this volume. The medium dissipation P_h in the form of heat loss may be taken as a sink in the field. Therefore, the power leaving the end face of the finite volume τ must be diminished by the amount of P_h watts.

The above solution applies to wave propagation of a plane electromagnetic wave along the X direction, since the perpendicular field vectors are $\mathcal{E} = \mathcal{E}_z$ and $H = H_y$. A more general solution is based on the curl $\mathcal{E}$ and curl H relations given in Eqs. (1) on page 11. They read

$$\left.\begin{aligned} &\text{curl } \underset{\cdot}{H} = \underset{\cdot}{J} + \frac{\partial D}{\partial t}; \qquad \text{curl } \underset{\cdot}{\mathcal{E}} = -\frac{\partial B}{\partial t} \\ &\text{curl } \underset{\cdot}{H} = \underset{\cdot}{J} + j\omega \underset{\cdot}{D}; \qquad \text{curl } \underset{\cdot}{\mathcal{E}} = -j\omega B \qquad \text{for harmonic fields} \\ &\underset{\cdot}{J} = \sigma\underset{\cdot}{\mathcal{E}}; \qquad \underset{\cdot}{B} = \mu \underset{\cdot}{H}; \qquad \underset{\cdot}{D} = \kappa\underset{\cdot}{\mathcal{E}} \end{aligned}\right\} \qquad (108)$$

where the dot below means that we deal with a vector. This dot is *often omitted* in this text for *simplicity's* sake, but understood. Applying the well-known vector identity

$$\text{div } (\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H}) \equiv \underset{\cdot}{H} \cdot \text{curl } \underset{\cdot}{\mathcal{E}} - \underset{\cdot}{\mathcal{E}} \cdot \text{curl } \underset{\cdot}{H} \qquad (109)$$

as given on pages 202–205, and often also written in the form

$$\nabla \cdot (\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H}) = \underset{\cdot}{H} \cdot (\nabla \times \underset{\cdot}{\mathcal{E}}) - \underset{\cdot}{\mathcal{E}} \cdot (\nabla \times \underset{\cdot}{H})$$

since, according to pages 22–23, the divergence of a vector is a *dot* product and the curl of a vector is a *cross* product, and the curl of a vector is another vector. Multiplying again the curl $\underset{\cdot}{\mathcal{E}}$ relation by the conjugate $\underset{\cdot}{H}^*$ of the complex vector $\underset{\cdot}{H}$ of a general medium and the conjugate curl $\underset{\cdot}{H}^*$ relation by $\underset{\cdot}{\mathcal{E}}$, we have

$$\underset{\cdot}{H}^* \cdot \text{curl } \underset{\cdot}{\mathcal{E}} = -j\omega \underset{\cdot}{B} \cdot \underset{\cdot}{H}^*; \qquad \underset{\cdot}{\mathcal{E}} \cdot \text{curl } \underset{\cdot}{H}^* = \underset{\cdot}{\mathcal{E}} \cdot \underset{\cdot}{J}^* - j\omega \underset{\cdot}{D}^* \cdot \underset{\cdot}{\mathcal{E}}$$

where the multiplication is a customary scalar multiplication.

Subtracting the right-hand equation from the left-hand equation yields

$$\underbrace{\underline{H}^* \text{ curl } \mathcal{E} - \mathcal{E} \text{ curl } \underline{H}^*}_{=\text{div }(\mathcal{E}\times\underline{H}^*)} = j\omega \underline{D}^* \cdot \underline{\mathcal{E}} - j\omega \underline{B} \cdot \underline{H}^* - \underline{\mathcal{E}} \cdot \underline{J}^*$$
$$= j\omega\kappa\underline{\mathcal{E}}^* \cdot \underline{\mathcal{E}} - j\omega\mu\underline{H} \cdot \underline{H}^* - \underline{\mathcal{E}} \cdot \sigma\underline{\mathcal{E}}^*$$

$$\text{div } \{(\mathcal{E}_0 \cos \omega t) \times [H_0 \cos (\omega t - \varphi)]\} = j\omega\kappa\mathcal{E}_0^2 \cos^2 \omega t - j\omega\mu H_0^2 \cos^2 (\omega t - \varphi) - \sigma\mathcal{E}_0^2 \cos \omega t \qquad \text{as time functions}$$

since, in a dissipative medium, a relative time phase φ exists between the electric $\underline{\mathcal{E}}$ and magnetic field $\underline{H}$ vectors. The divergence of vectors refers no longer to a vector, even though the "cross" product $\underline{\mathcal{E}} \times \underline{H}^*$ is a product of vectors $\mathcal{E}$ and $\underline{H}^*$. The divergence of $\mathcal{E} \times \underline{H}^*$ represents a power density in watts per unit volume for which the volume approaches zero value in the limit. This power density in watts per unit volume can also be transformed (by means of the Gauss theorem) into a surface density of the power. Hence, in each case we deal only with a scalar amount of something that is the power in watts. For the power in watts, we mean the energy per second, since energy is in joules or volt · ampere · seconds. The average time value of the above relation is what counts. The only time variables are the $\cos^2$ functions with an average value of $\frac{1}{2}$. We find then the average divergence[1]

$$\text{div}_{\text{avg}} (\underline{\mathcal{E}} \times \underline{H}^*) = -\sigma \frac{\mathcal{E}_0^2}{2} + j\omega \left(\frac{\kappa\mathcal{E}_0^2}{2} - \frac{\mu H_0^2}{2}\right)$$
$$= -\sigma\mathcal{E}_e^2 + j\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2) = -\sigma\mathcal{E}_e^2 + j2\omega\left(\frac{\kappa\mathcal{E}_e^2 - \mu H_e^2}{2}\right)$$
$$= \underbrace{-\sigma\mathcal{E}_e^2 + j2\omega\left(\frac{\kappa\mathcal{E}_e^2 - \mu\frac{\kappa_c}{\mu}\mathcal{E}_e^2}{2}\right)}_{\text{Note! watts/m}^3\text{ for infinitesimal } d\tau \text{ volume which is } \tau \to 0} = \underbrace{-\sigma\mathcal{E}_e^2}_{\text{heat loss}} + j2\omega\underbrace{\left[\frac{(\kappa - \kappa_c)\mathcal{E}_e^2}{2}\right]}_{\substack{\text{mean value of } \textit{difference} \text{ of electric and}\\ \text{magnetic energy volume densities}}}$$

watts/m³ (110)

For a finite volume τ cubic meters completely enclosed by S square meters, we find the integral solution for the right-hand side of the above equation:

[1] The electric energy density is

$$\int_0^D \underline{\mathcal{E}}\, d\underline{D} = \int_0^{\kappa\mathcal{E}} \underline{\mathcal{E}}\, d(\kappa\underline{\mathcal{E}}) = \kappa \int_0^{\mathcal{E}} \underline{\mathcal{E}}\, d\underline{\mathcal{E}} = \kappa\underline{\mathcal{E}}\underline{\mathcal{E}} = \kappa\mathcal{E}_0^2 \qquad \text{joules/m}^3$$

The magnetic energy density is

$$\int_0^B \underline{H}\, d\underline{B} = \int_0^{\mu H} \underline{H}\, d(\mu\underline{H}) = \mu H_0 \qquad \text{joules/m}^3$$

We have then the form $F_0^2 = \frac{1}{2}F_e$ for $F = \mathcal{E}$ or H, respectively, and F_e the respective effective field value. Hence the electromagnetic *energy* density has an average value

$$\kappa\mathcal{E}_0^2 \underbrace{\cos^2 \omega t}_{\frac{1}{2}} + \mu H_0^2 \underbrace{\cos^2 (\omega t - \varphi)}_{\frac{1}{2}} = \frac{\kappa\mathcal{E}_0^2 + \mu H_0^2}{2} = \kappa\mathcal{E}_e^2 + \mu H_e^2 \qquad \text{joules/m}^3$$

The corresponding *power* density then is

$$\frac{\partial(\kappa\mathcal{E}_e^2 + \mu H_e^2)}{\partial t} = -2\omega(\kappa\mathcal{E}_e^2 + \mu H_e^2) \qquad \text{watts/m}^3$$

$$\iiint_\tau \operatorname{div}\,(\dot{\mathcal{E}} \times \dot{H}^*)\,d\tau = \iint_S (\dot{\mathcal{E}} \times \dot{H}^*)_n\,dS$$ application of the Gauss theorem, Eq. (88)

reducing the volume integral to an equivalent surface integral. The complete expression then reads

$$\underbrace{\tfrac{1}{2}\iint_S (\dot{\mathcal{E}} \times \dot{H}^*)_n \cdot dS}_{\substack{\text{is complex and the}\\ \text{real part in watts}\\ \text{toward } S \text{ sq m}\\ \text{supplies the}\\ \text{field power}\\ \text{dissipation}}} = \underbrace{\tfrac{1}{2}\iiint_\tau [j\omega(\kappa\dot{\mathcal{E}}^* \cdot \dot{\mathcal{E}} - \mu\dot{H} \cdot \dot{H}^*) - \overbrace{\sigma\dot{\mathcal{E}}^* \cdot \dot{\mathcal{E}}}^{\text{dissipated}}]\,d\tau}_{\substack{\text{Complex watts flowing in } \tau \text{ cu m}\\ \text{toward the enclosing surface of } S \text{ sq m}}}$$

$$\text{since } \underbrace{\iiint_\tau -\frac{\sigma\dot{\mathcal{E}}^* \cdot \dot{\mathcal{E}}}{2}\,d\tau}_{} = \iiint_\tau -\sigma\mathcal{E}_e^2\,d\tau$$

This relation can also be written as

$$\left.\begin{aligned} &\underbrace{\tfrac{1}{2}\iint_S (\dot{\mathcal{E}} \times \dot{H}^*)_n\,dS}_{\substack{\text{complex power flow}\\ \text{in watts toward}\\ \text{the enclosing surface}\\ \text{of } S \text{ sq m}}} \\ &= \underbrace{-\sigma\iiint_\tau \mathcal{E}_e^2\,d\tau}_{\substack{\text{watts lost}\\ \text{in } \tau \text{ cu m}}} + \underbrace{j\omega\Big(\kappa\overbrace{\iiint_\tau \mathcal{E}_e^2\,d\tau}^{} - \mu\underbrace{\iiint_\tau H_e^2\,d\tau}_{\iiint \kappa_e\mathcal{E}_e^2\,d\tau}\Big)}_{2\omega \text{ times the difference of the mean values } \iiint_\tau \frac{\kappa\mathcal{E}_e^2}{2}\,d\tau \text{ and } \iiint_\tau \frac{\mu H_e^2}{2}\,d\tau} \end{aligned}\right\} \quad (111)$$

From the foregoing solution, we note that the complex power flow normal to the surface elements dS consists of a reactive term (j term) and a dissipative term. The expression shows also that Re $\frac{1}{2}(\dot{\mathcal{E}} \times \dot{H}^*)$ is the average power in watts across 1-sq-m surface. In these relations, the product $\dot{\mathcal{E}}^* \cdot \dot{\mathcal{E}}$ has an absolute amount of $\mathcal{E}_0^2$ volts per meter, which is the peak value of the electric field intensity. In the same way, $\dot{H} \cdot \dot{H}^*$ is H_0^2 amperes per meter. Therefore, the real component of the complex power flow over the surrounding surface S must be equal to the dissipated electromagnetic field watts.

The corresponding differential relation can also be found directly from (109) and (108), since

$$\begin{aligned} \operatorname{div}\,(\mathcal{E} \times H) &= H \cdot \operatorname{curl} \mathcal{E} - \mathcal{E} \cdot \operatorname{curl} H \\ &= H \cdot \left(-\frac{\partial B}{\partial t}\right) - \mathcal{E} \cdot \left(J + \frac{\partial D}{\partial t}\right) \\ &= -\frac{1}{2}\frac{\partial(\mu H^2)}{\partial t} - \mathcal{E}J - \frac{1}{2}\frac{\partial(\kappa\mathcal{E}^2)}{\partial t} \end{aligned}$$

since

$$\frac{1}{2}\frac{\partial(\mu H^2)}{\partial t} = \frac{1}{2}\frac{\partial(H \cdot B)}{\partial t} \qquad \text{and} \qquad \frac{1}{2}\frac{\partial(\mu H^2)}{\partial t} = H\frac{\partial B}{\partial t}$$

All field values hold now at any instant, and we find

$$\underbrace{\operatorname{div}(\mathcal{E} \times H)}_{\substack{\text{divergence of}\\ \text{field energy over}\\ \text{enclosing surface}}} + \underbrace{\frac{\partial(\frac{1}{2}\mu H^2 + \frac{1}{2}\kappa\mathcal{E}^2)}{\partial t}}_{\substack{\text{rate at which}\\ \text{electromagnetic}\\ \text{field power increases}}} = \underbrace{-\mathcal{E}J}_{\substack{\text{watt loss}\\ \text{in watts/m}^3}} \tag{112}$$

With respect to the complex Poynting vector $p = \frac{1}{2}\mathcal{E} \times H^*$ in relations (111) and (110), we find

$$\begin{aligned}\operatorname{div} p &= \frac{1}{2}\operatorname{div}(\mathcal{E} \times H^*) = -\frac{1}{2}\sigma\mathcal{E}^* \cdot \mathcal{E} + j\frac{\omega}{2}(\kappa\mathcal{E}^* \cdot \mathcal{E} - \mu H^* \cdot H)\\ &= -\sigma\frac{\mathcal{E}_0^2}{2} + j\omega\left(\frac{1}{2}\kappa\mathcal{E}_0^2 - \frac{1}{2}\mu H_0^2\right)\\ &= \underbrace{-\sigma\mathcal{E}_e^2}_{\text{real}} + \underbrace{j\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2)}_{\text{imaginary}} \qquad \text{watts/m}^3\end{aligned} \tag{113}$$

where $\mathcal{E}_0$ and H_0 represent the peak field values and $\mathcal{E}_e$ and H_e the effective values in volts per meter and amperes per meter, respectively. The real portion Re (div p) $= -\sigma\mathcal{E}_e^2 = \mathcal{E}_e \cdot J_e$ watts per cubic meter must be the power flowing that supplies the joulean heat loss per unit volume of the dissipative wave medium, while the divergence of the imaginary portion, *i.e.*, Im (div p) $= \omega(\kappa\mathcal{E}_e^2 - \mu H_e^2)$ is the reactive electromagnetic field power per unit volume. Since $\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2) = 2\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2)/2$, the reactive total electromagnetic field watts per unit volume is $1.256 \times 10^7 f$ times the difference of the average values of the electric and magnetic field energies in joules per unit volume, where f is in megacycles per second. According to (111), we have for a finite volume of τ cubic meters completely enclosed by S square meters

$$\underbrace{\iiint_\tau \operatorname{div} p\, d\tau = \iint_S p_n\, dS}_{\substack{\text{Gauss theorem transformation of a}\\ \text{volume into a surface integral, } p_n \text{ means}\\ p = \mathcal{E} \times H \text{ is } normal \text{ to each } dS\\ \text{element.}}}$$

$$= \underbrace{\operatorname{Re}\iint_S p_n\, dS}_{-\iint_S \sigma\mathcal{E}_e^2 \cdot dS} + \underbrace{\operatorname{Im}\iint_S p_n\, dS}_{1.256 \times 10^7 f(\text{Mc})\iint_S (\bar{p}_e - \bar{p}_m)\, dS} \qquad \text{watts toward } S\text{-sq-m surface} \tag{114}$$

$\bar{p}_e = \frac{1}{2}\kappa\mathcal{E}_e^2$ joules/m³
$\bar{p}_m = \frac{1}{2}\mu H_e^2$ joules/m³

Now let us see what these relations actually mean. First, we note that

the ratio of the real value $\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2)$ of the imaginary part of the divergence of p relation of Eq. (113) to the real part yields

$$\frac{\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2)}{-\sigma\mathcal{E}_e^2} = -\frac{\omega\left(\kappa\mathcal{E}_e^2 - \mu\frac{\kappa_e}{\mu}\mathcal{E}_e^2\right)}{\sigma\mathcal{E}_e^2} = \frac{\omega(\kappa_e - \kappa)}{\sigma} = Q_e - Q \quad (115)$$

since the space impedance is no longer $Z_0 = \sqrt{\mu/\kappa}$ pure resistive ohms but is $Z_0 = \sqrt{\mu/\kappa_e}$ complex ohms, with $\kappa_e = \kappa - j\sigma/\omega$ complex farads per meter. Hence, the above-mentioned ratio is the difference between the equivalent figure of merit $Q_e = \omega\kappa_e/\sigma$ of the general medium that is dissipative and the figure of merit $Q = \omega\kappa/\sigma$. We have then

$$\frac{\text{Im (div } p)}{\text{Re (div } p)} = j(Q_e - Q) = j\frac{\omega(\kappa_e - \kappa)}{\sigma} = j\frac{\omega}{\sigma}\left(\kappa - j\frac{\sigma}{\omega} - \kappa\right) = 1 \quad (116)$$

This simple result applies to the elementary divergence, *i.e.*, the average divergence as a volume τ contracts to a point, which is the same as saying to the divergence across the surrounding infinitesimal surface dS. It is the integral form of (111) that applies to finite volumes of τ cubic meters completely enclosed by S square meters in the mks system of units. For such a condition, the value of the electric as well as the value of the magnetic field vectors changes in the volume, since energy is being used to cover the joulean heat losses of the medium. If we have then the case of a plane wave front, for instance, an $\mathcal{E}_z$, H_y wave front, moving along the X direction, then the electromagnetic energy can enter only over the rear face of the finite volume of τ cubic meters and leave it only over the front face of the volume, since the wave energy travels in planes (for areas S that are not too large) formed by the $\mathcal{E}_z$ and H_y vectors, along a direction X normal to such planes. Hence, according to Eq. (111), only the reactive amount

$$j\frac{\omega}{2}\iiint_\tau (\kappa\dot{\mathcal{E}}^* \cdot \dot{\mathcal{E}} - \mu\dot{H} \cdot \dot{H}^*)\, d\tau = j\omega\iiint_\tau (\kappa\mathcal{E}_e^2 - \mu H_e^2)\, d\tau \qquad \text{watts}$$

minus the real amount of

$$\iiint_\tau \sigma\mathcal{E}_e^2 \cdot d\tau \qquad \text{watts}$$

can ever reach the front surface and escape there, since the latter amount is turned into heat in the volume of τ cubic meters. Along the direction X of wave propagation inside the volume, we have, therefore, a medium attenuation of α nepers per meter, which is in the X direction for the above-mentioned $\mathcal{E}$ and H polarizations along the $\pm Z$ and $\pm Y$ directions, respectively. Therefore, the effective values $\mathcal{E}_e$ and H_e can no longer be fixed along the path between the parallel rear and front faces, which the plane wave front covers during its travel between these two surfaces of

the volume τ. This is the reason why we must deal with the instantaneous value of the rate at which energy traverses a unit surface, normal to its flow in an electromagnetic field, which is given by the Poynting vector $p = \mathcal{E} \times H$ with an average value of $p = \frac{1}{2}\mathcal{E} \times H^*$ for a complex medium (also σ mhos per meter effective). We have then for the instantaneous electric and magnetic field polarizations of a plane $\mathcal{E}_z$, H_y wave along the X direction

$$\left.\begin{aligned} \mathcal{E}_z &= \mathcal{E}_0 \epsilon^{-\alpha x} \epsilon^{j(\omega t - \beta x)} \qquad \text{volts/m} \\ H_y &= \sqrt{\frac{\kappa_e}{\mu}}\, \mathcal{E}_0 \epsilon^{-\alpha x} \epsilon^{j(\omega t - \beta x + \varphi)} \qquad \text{amp/m} \\ \sqrt{\kappa_e} &= \sqrt{\left(\kappa - j\frac{\sigma}{\omega}\right)} \underline{/2\varphi} = \underbrace{\left|\sqrt{\sqrt{\kappa^2 + \frac{\sigma^2}{\omega^2}}}\right|}_{\text{magnitude}} \underline{/\varphi} = 0.5 \tan^{-1}\left(-\frac{\sigma}{\omega\kappa}\right) \end{aligned}\right\} \quad (117)$$

We have then for the electric $\mathcal{E}_z$ and the conjugate magnetic H_y^* vectors

$$\mathcal{E}_z = \mathcal{E}_0 \epsilon^{-\alpha x + j(\omega t - \beta x)}; \qquad H_y^* = \sqrt{\frac{\kappa + j\frac{\sigma}{\omega}}{\mu}}\, \mathcal{E}_0 \epsilon^{-\alpha x - j(\omega t - \beta x + \varphi)}$$

where $-\alpha x$ is the neper attenuation for a wave travel along x meters of the general medium, while $\beta x = 2\pi x/\lambda = \omega x/v$ is the time phase delay for a wave travel of x meters, and φ represents the phase angle between the electric and magnetic field vectors due to the complex medium, *i.e.*, due to the phase angle of $Z_0 = \sqrt{\mu/\kappa_e}$ complex ohms. We have, therefore, for the complex Poynting vector,

$$p = \frac{1}{2}\mathcal{E}_z \cdot H_y = \mathcal{E}_0^2 \sqrt{\frac{\kappa + j\frac{\sigma}{\omega}}{\mu}}\, \epsilon^{\overbrace{\alpha x + j(\omega t - \beta x) - \alpha x - j(\omega t - \beta x + \varphi)}^{= -2\alpha x - j\varphi}}$$

or

$$\left.\begin{aligned} p &= \frac{1}{2}\mathcal{E}_0^2 \sqrt{\frac{\sqrt{\kappa^2 + (\sigma/\omega)^2}}{\mu}}\, \epsilon^{-2\alpha x - j\varphi} \\ &\qquad = 0.5\sqrt{\frac{\sqrt{\kappa^2 + (\sigma/\omega)^2}}{\mu}}\, \mathcal{E}_0^2 \epsilon^{-2\alpha x}(\cos\varphi - j\sin\varphi) \\ &= \underbrace{A\cos\varphi}_{\text{Re}(p)} - \underbrace{jA\sin\varphi}_{\text{Im}(p)} \qquad \text{watts/m}^2 \\ A &= 0.5\sqrt{\frac{\sqrt{\kappa^2 + (\sigma/\omega)^2}}{\mu}}\, \mathcal{E}_0^2 \epsilon^{-2\alpha x} \qquad \text{watts/m}^2 \\ \text{div}\, p &= \frac{\partial p}{\partial x} = -2\alpha A\cos\varphi + j2\alpha A\sin\varphi \qquad \text{watts/m}^3 \end{aligned}\right\} \quad (118)$$

Since A cos φ denotes the real part of the complex Poynting vector and A sin φ the imaginary part of it, we can realize why (116) gives unity as the answer. The result of (118) also takes into account the relative time phase between the electric and magnetic field intensity. The real part A cos φ denotes, therefore, the energy in joules per second streaming perpendicularly across the average unit area in the ZY plane and accounts for the average power needed for covering the heat losses due to the conductivity of the general medium. The imaginary portion jA sin φ represents the reactive electromagnetic energy flux per second across unit area. As is shown below, we can also express in (117) the value of $\sqrt{\kappa_e/\mu}$ by $\sqrt{\alpha^2 + \beta^2}/(\mu\omega)$ and φ can also be computed from $\tan^{-1}(\alpha/\beta)$ with $\sin\varphi = \alpha/\sqrt{\alpha^2 + \beta^2}$ and $\cos\varphi = \beta/\sqrt{\alpha^2 + \beta^2}$, leading to the complex Poynting vector

$$p = \underbrace{\frac{\mathcal{E}_0^2}{2\mu\omega}\sqrt{\alpha^2 + \beta^2}}_{=A}\,\epsilon^{-2\alpha x}\cdot\epsilon^{-j\varphi} = A\cos\varphi - jA\sin\varphi$$

$$= A\frac{\beta}{\sqrt{\alpha^2+\beta^2}} - jA\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$$

$$= \underbrace{\underbrace{7.96\times 10^{-8}\frac{\beta\mathcal{E}_0^2}{\mu f^{(\mathrm{Mc})}}\epsilon^{-2\alpha x}}_{\text{real part, supplies space dissipation}} - \underbrace{j7.96\times 10^{-8}\frac{\alpha\mathcal{E}_0^2}{\mu f^{(\mathrm{Mc})}}\epsilon^{-2\alpha x}}_{\text{imaginary part, supplies reactive radiation stream}}}_{\text{watts/m}^2} \qquad (118a)$$

These formulas then entail the knowledge of the attenuation constant α in nepers per meter and the phase constant β in circular radians per meter of the general medium. They are given in Eqs. (30) on page 53 in a most convenient form, based on the merit $Q = \omega\kappa/\sigma$ of the dissipative medium. These formulas are also expressed in terms of square-root factors for which one term of a sum is unity. This gives then a ready means of seeing at a glance how much the conductivity affects the result in comparison with a pure dielectric wave medium. We have then the following details:

For $\mathcal{E}_z$ and H_y plane waves, we have of the six equations for the curl $\mathcal{E}$ and curl H components on page 148, only the remaining relations

$$-\frac{\partial\mathcal{E}_z}{\partial x} = -j\omega\mu H_y \qquad \text{and} \qquad -\frac{\partial H_y}{\partial x} = (\sigma + j\omega\kappa)\mathcal{E}_z = Y\mathcal{E}_z$$

The first relation reads also

$$\frac{\partial\mathcal{E}_z}{\partial x} = j\omega\mu H_y = ZH_y$$

This leads to the propagation constant Γ and three-dimensional wave impedance Z_0, according to transmission-line practice, to

$$\Gamma = \sqrt{ZY} = \sqrt{j\omega\mu(\sigma + j\omega\kappa)} = j\omega\sqrt{\mu\left(\frac{\sigma}{j\omega} + \kappa\right)} = j\omega\sqrt{\mu\left(\kappa - \frac{j\sigma}{\omega}\right)} = j\omega\sqrt{\mu\kappa_e}$$

$$= \alpha + j\beta; \qquad Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\kappa}} = \sqrt{\frac{\mu}{\kappa_e}} = R_0 + jX_0$$

From the β expression of Eq. (30) on page 53, we find

$$\beta = \omega\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}} = \omega\sqrt{\frac{\mu\kappa}{2}(1+q)} = \omega\sqrt{\frac{\mu\kappa}{2}\left(1 + \sqrt{1 + \frac{1}{Q^2}}\right)}$$

$$= \omega\sqrt{\frac{\mu\kappa}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\kappa^2}} + 1\right)} \qquad \text{circular radians/m}$$

From the α expression of Eq. (30), we find

$$\alpha = \underbrace{\frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}}_{a}\sqrt{\frac{2}{1+q}} = a\sqrt{\frac{2(q-1)}{(q+1)(q-1)}} = \omega a\sqrt{\frac{2(q-1)}{\omega^2(q^2-1)}}$$

$$= \omega a\sqrt{\frac{2(q-1)}{\omega^2\left(1 + \frac{1}{Q^2} - 1\right)}} = \omega a\sqrt{\frac{2Q^2(q-1)}{\omega^2}} = \omega\sqrt{\frac{2a^2Q^2(q-1)}{\omega^2}}$$

$$= \omega\sqrt{\frac{2\frac{\sigma^2}{4}\frac{\mu}{\kappa}\frac{\omega^2\kappa^2}{\sigma^2}(q-1)}{\omega^2}} = \omega\sqrt{\frac{\mu\kappa}{2}(q-1)} = \omega\sqrt{\frac{\mu\kappa}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\kappa^2}} - 1\right)}$$

nepers/m

Therefore, for $\rho = \sqrt{1 + (\sigma^2/\omega^2\kappa^2)}$

$$\alpha^2 + \beta^2 = \omega^2\frac{\mu\kappa}{2}(\rho - 1) + \omega^2\frac{\mu\kappa}{2}(\rho + 1) = \omega^2\mu\kappa\rho = \omega^2\mu\kappa\sqrt{1 + \frac{\sigma^2}{\omega^2\kappa^2}}$$

$$= \omega^2\mu\kappa\sqrt{\frac{\kappa^2 + \frac{\sigma^2}{\omega^2}}{\kappa^2}} = \omega^2\mu\sqrt{\kappa^2 + \frac{\sigma^2}{\omega^2}}$$

$$\frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega} = \frac{\sqrt{\mu\omega^2}\sqrt{\kappa^2 + (\sigma^2/\omega^2)}}{\mu\omega}$$

must be equal to $\sqrt{\kappa_e/\mu}$
But

$$\sqrt{\frac{\kappa_e}{\mu}} = \sqrt{\frac{\mu\omega^2}{\mu\omega^2}\frac{\kappa_e}{\mu}} = \frac{\sqrt{\mu\omega^2\kappa_e}}{\mu\omega} = \frac{\sqrt{\mu\omega^2\left(\kappa - j\frac{\sigma}{\omega}\right)}}{\mu\omega} = \frac{\sqrt{\mu\omega^2\sqrt{\kappa^2 + \frac{\sigma^2}{\omega^2}}}}{\mu\omega}$$

which checks and confirms the relations leading to Eq. (118*a*). We still have to prove the relation for the relative phase φ between the electric and magnetic field vectors in terms of α and β, as applied in (118*a*). This phase angle is due to the complex

space impedance $Z_0 = \sqrt{Z/Y}$; and in comparison with the propagation constant $\Gamma = \sqrt{YZ}$, we find for

$$\frac{\partial H_y}{\partial x} = (\sigma + j\omega\kappa)\mathcal{E}_z = Y\mathcal{E}_z \qquad \text{and} \qquad \frac{\partial \mathcal{E}_z}{\partial x} = j\omega\mu H_y = ZH_y$$

the derivations

$$\Gamma = \sqrt{YZ} = \sqrt{j\omega\mu(\sigma + j\omega\kappa)} = \sqrt{j\omega\mu\sigma - \omega^2\mu\kappa} = j\sqrt{\omega^2\mu\kappa - j\omega\mu\sigma} = \alpha + j\beta$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\kappa}} = \sqrt{\frac{\mu\omega}{\mu\omega}\cdot\frac{\omega\mu}{\frac{\sigma}{j} + \omega\kappa}} = \frac{\mu\omega}{\sqrt{\omega^2\mu\kappa - j\omega\mu\sigma}}$$

$$= \frac{\mu\omega}{\frac{\alpha}{j} + \beta} = \frac{\mu\omega}{\beta - j\alpha} = \frac{\mu\omega}{\alpha^2 + \beta^2}(\beta + j\alpha)$$

$$= \frac{\mu\omega}{\alpha^2 + \beta^2}\sqrt{\beta^2 + \alpha^2}\ \Big/ \varphi = \tan^{-1}\frac{\alpha}{\beta} = \frac{\mu\omega}{\sqrt{\alpha^2 + \beta^2}}\epsilon^{j\varphi}$$

But we are interested in the expression for H_y, of Eq. (117), in $\sqrt{\kappa_e/\mu} = 1/Z_0$, and have

$$\frac{1}{Z_0} = \sqrt{\frac{\kappa_e}{\mu}} = \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega}\epsilon^{-j\varphi} = \frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega}\cos\varphi - j\frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega}\sin\varphi$$

Equations (118a) give the complex Poynting vector. The divergence of the real part expresses the power in watts per unit volume. The imaginary part of p refers to reactive power. We find

$$\operatorname{div} p = \nabla\cdot p = \operatorname{grad}_x p = \frac{\partial p}{\partial x} = \frac{7.96 \times 10^{-8}\mathcal{E}_0^2}{\mu f^{(\mathrm{Mc})}}\left[\beta\frac{\partial(\epsilon^{-2\alpha x})}{\partial x} - j\alpha\frac{\partial(\epsilon^{-2\alpha x})}{\partial x}\right]$$

$$= \frac{7.96 \times 10^{-8}\mathcal{E}_0^2}{\mu f^{(\mathrm{Mc})}}(-2\alpha\beta\epsilon^{-2\alpha x} + j2\alpha^2\epsilon^{-2\alpha x})$$

$$= -1.58 \times 10^{-7}\frac{\alpha\beta\mathcal{E}_0^2\epsilon^{-2\alpha x}}{\mu f^{(\mathrm{Mc})}} + j1.58 \times 10^{-7}\frac{\alpha^2\mathcal{E}_0^2\epsilon^{-2\alpha x}}{\mu f^{(\mathrm{Mc})}} \qquad \text{watts/m}^3$$

But

$$\alpha\beta = \underbrace{\frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{1+q}}}_{\alpha}\,\omega\,\underbrace{\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}}}_{\beta} = \frac{\sigma\omega}{2}\sqrt{\frac{\mu\mu\kappa}{\kappa}}$$

$$= \frac{\mu\sigma\omega}{2} = 3.14 \times 10^6 f^{(\mathrm{Mc})}\mu\sigma$$

$$\alpha^2 = \frac{\sigma^2}{4}\frac{\mu}{\kappa}\frac{2}{1+q} = \frac{\mu\sigma^2}{2\kappa}\frac{1}{1 + \sqrt{1 + \frac{1}{Q^2}}} = \frac{\mu\sigma^2}{2\kappa}\frac{1}{1 + \frac{\sqrt{1+Q^2}}{Q}}$$

$$= \frac{\mu\sigma^2}{2\kappa}\frac{Q}{Q + \sqrt{1+Q^2}}$$

$$= \frac{\mu\sigma\omega}{2(Q + \sqrt{1+Q^2})} = \frac{3.14 \times 10^6 f^{(\mathrm{Mc})}\mu\sigma}{Q + \sqrt{1+Q^2}}$$

Therefore,

$$\operatorname{div} p = -1.58 \times 10^{-7} \frac{3.14 \times 10^6 f^{(\mathrm{Mc})} \mu\sigma}{\mu f^{(\mathrm{Mc})}} \mathcal{E}_0^2 \epsilon^{-2\alpha x} + j1.58 \times 10^{-7} \frac{3.14 \times 10^6 f^{(\mathrm{Mc})} \mu\sigma}{(Q + \sqrt{1 + Q^2}) \mu f^{(\mathrm{Mc})}} \mathcal{E}_0^2 \epsilon^{-2\alpha x}$$

or

$$\operatorname{div} p = \underbrace{-\frac{1}{2}\sigma \mathcal{E}_0^2 \epsilon^{-2\alpha x}}_{\substack{-\sigma\mathcal{E}_0^2\epsilon^{-2\alpha x} \\ \text{watt loss in dissipative} \\ \text{medium of unit volume}}} + \underbrace{j\frac{1}{2}\frac{\sigma}{Q + \sqrt{1 + Q^2}} \mathcal{E}_0^2 \epsilon^{-2\alpha x}}_{\substack{j\frac{\sigma\mathcal{E}_0^2\epsilon^{-2\alpha x}}{Q+\sqrt{1+Q^2}} \\ \text{wattless (reactive)}}} \quad \text{watts/m}^3 \qquad (119)$$

27. Numerical Examples of the Formulas for the Complex Poynting Vector and the Divergence Expression of a Dissipative Electromagnetic Medium. Let us apply the formulas of Secs. 25 and 26 to a medium that represents dry rocky ground of electromagnetic mean values of $\sigma = 10^{-5}$ mho/m, $\kappa = 8\kappa_0 = 8 \times 8.854 \times 10^{-12} = 7.0832 \times 10^{-11}$ farad/m, and $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m. We find then at 100 Mc/sec, a figure of merit of $Q = \omega\kappa/\sigma = 6.28 \times 100 \times 10^6 \times 7.083 \times 10^{-11} \times 10^5 = 44.6$. We have then in the formulas of Eq. (30), page 53,

$$q = \sqrt{1 + \left(\frac{1}{Q^2}\right)} = \sqrt{1 + \frac{1}{1{,}989}} \cong 1$$

and an attenuation constant, phase constant, and relative time phase of

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{1+q}} \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}} = 5 \times 10^{-1} \times 10^{-5}\sqrt{\frac{1.257 \times 10^{-6}}{7.083 \times 10^{-11}}}$$

$$= 5 \times 10^{-6} \times 1.33 \times 10^2 = 6.65 \times 10^{-4} \quad \text{nepers/m}$$

$$\beta = \omega\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}} \cong \omega\sqrt{\mu\kappa}$$

$$= 6.28 \times 10^8 \sqrt{1.257 \times 10^{-6} \times 7.083 \times 10^{-11}}$$

$$= 6.28 \times 10^8 \times 9.43 \times 10^{-9} = 5.92 \text{ radians/m}$$

$$\varphi = \tan^{-1}\left(\frac{\alpha}{\beta}\right) = \tan^{-1}\left(\frac{6.65 \times 10^{-4}}{5.92}\right) = \tan^{-1}(0.000112)$$

$$= 0.000112 \text{ circular radians} = 0.000112 \times 57.3 = 0.00642°$$

This relative phase between $\mathcal{E}_z$ and H_y field vectors can also be checked by means of the formula given in (117), and we find

$$\varphi = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma}{\omega\kappa}\right) = \frac{1}{2}\tan^{-1}\left(-\frac{10^{-5}}{6.28 \times 10^8 \times 7.083 \times 10^{-11}}\right)$$

$$= -\frac{1}{2}\tan^{-1}(0.000224) = -\frac{1}{2}0.000224 = -0.000112 \text{ circular radians}$$

$$= -0.00642°$$

We have here a minus sign, since in (117), this is the angle by which the H_y field lags behind the $\mathcal{E}_z$ field. This is also the case for the first calculation, since we find under details that

$$\frac{1}{Z_0} = \sqrt{\frac{\kappa_c}{\mu}} = \left(\frac{\sqrt{\alpha^2 + \beta^2}}{\mu\omega}\right)\epsilon^{-j\varphi}$$

i.e., a negative φ angle. It really does not matter in this particular application, since the real cos φ term is positive for *negative* as well as positive values of φ. We note that this angle is very small, so that cos φ becomes almost unity value and $\sin\varphi = \tan\varphi = \varphi$ circular radians; *i.e.*, $\sin\varphi = 0.000112$, which is surely a negligible quantity in comparison with cos φ essentially of unity value. Suppose that we have an effective electric field of 50 mv/m at a place that corresponds to a location $x = 0$ m. Then after a wave propagation over $x =$ 1-m distance, we have an effective electric field of

$$\mathcal{E}_e = 50 \times 10^{-3}\epsilon^{-\alpha x} = 50 \times 10^{-3}\epsilon^{\,6.65\times10^{-4}} = 50 \times 10^{-3}\epsilon^{-0.000665} \qquad \text{volts/m}$$

Even a negative exponent as large as 0.01 would still give $\epsilon^{-0.01} = 0.99$, which is essentially unity. Therefore, we can put $\epsilon^{-2\alpha}$ equal to unity. Therefore, we have, according to (119), the watt loss per unit volume equal to $\sigma\mathcal{E}_e^2 = 10^{-5} \times 50^2 \times 10^{-6} = 2.5 \times 10^{-8}$ watts/m³, *i.e.*, only 0.025 μw/m³. If we had had, even an effective field value as high as 1 volt/m, we still should have a loss of only 10^{-5} watt/m³, which is only 10 μw/m³. Hence, for a frequency of 100 Mc/sec and higher, these computations hold. Let us compare now this loss with the useful radiation power. Since $H_e = \sqrt{\kappa_c/\mu}\,\mathcal{E}_e = \sqrt{\kappa_c/\mu}$ amperes per meter for 1 volt/m effective field voltage, we find

$$H_e = \sqrt{\frac{\kappa - j\dfrac{\sigma}{\omega}}{\mu}} = \sqrt{\frac{7.083 \times 10^{-11} - j\dfrac{10^{-5}}{6.28 \times 10^{8}}}{1.257 \times 10^{-6}}}$$

$$= \sqrt{\frac{7.083 \times 10^{-11} - j0.00159 \times 10^{-11}}{1.257 \times 10^{-6}}}$$

$$\cong \sqrt{\frac{7.083 \times 10^{-11}}{1.257 \times 10^{-6}}} = 7.48 \times 10^{-3} \text{ amp/m}$$

This means that the characteristic space impedance $\mathcal{E}_e/H_e$ is now equal only to $1/(7.48 \times 10^{-3}) = 134$ ohms instead of 377 ohms in case of air. From the very small phase angle of 0.00642 deg by which H_e lags $\mathcal{E}_e$, we know that cos 0.00642 deg $\cong 1$ and $\mathcal{E}_e \cdot H_e = 1 \times 7.48 \times 10^{-3} = 7.48 \times 10^{-3}$ watt/m² are in the plane wave front that moves with a velocity $v = \omega/\beta = 6.28 \times 10^8/5.92 = 1.06 \times 10^8$ m/sec as against 3×10^8 m/sec in air. According to (93), we find that $\mathcal{E}_e H_e$ divided by the velocity of

propagation yields the electromagnetic field energy density; *i.e.*, $7.48 \times 10^{-3}/(1.06 \times 10^8) = 7.05 \times 10^{-11}$ joule/m³. According to (118*a*), we have the complex Poynting vector p; and since $\mathcal{E}_0 = \sqrt{2}\,\mathcal{E}_e$,

$$p = 7.96 \times 10^{-8} \frac{2\mathcal{E}_e^2 \epsilon^{-2\alpha x}}{\mu f^{(\text{Mc})}} (\beta - j\alpha) = 1.592 \times 10^{-7} \frac{\mathcal{E}_e^2 \epsilon^{-2\alpha x}}{\mu f^{(\text{Mc})}} (\beta - j\alpha)$$

$$= 1.592 \times 10^{-7} \frac{\epsilon^{-2\alpha x}}{1.257 \times 10^{-6} \times 10^2} (\beta - j\alpha)$$

$$= 1.26 \times 10^{-3} \epsilon^{-2\times 6.65\times 10^{-4}}(5.92 - j6.65 \times 10^{-4})$$

$$= \underbrace{7.48 \times 10^{-3} \epsilon^{-1.33\times 10^{-3}x}}_{\substack{\text{dissipative wave-front watt}\\ \text{density (supplies space}\\ \text{dissipation)}}} - \underbrace{j8.38 \times 10^{-7} \epsilon^{-1.33\times 10^{-3}x}}_{\substack{\text{reactive wave-front}\\ \text{watt density}}} \quad \text{watts/m}^2$$

When we take again the ratio p/v, where $v = 1.06 \times 10^8$ m/sec is the velocity with which the wave front moves along the X direction, we find a dimension watts per square meter divided by meters per second, which yields joules per cubic meter. Therefore, we can account for the field energy in volt · ampere · seconds when the wave front moves through $x = 1$ m along a direction normal to its plane. This means that the decay factor becomes $\epsilon^{-2\alpha x} = \epsilon^{-2\alpha} = \epsilon^{-1.33\times 10^{-3}} \cong 0.999$, which is essentially unity. This yields for the real value of p

$$\text{Re}\left(\frac{p}{v}\right) = \text{Re}\left(\frac{7.48 \times 10^{-3}}{1.06 \times 10^8}\right) = 7.05 \times 10^{-11} \text{ joule/m}^3$$

which checks the result obtained above.

Since for the electromagnetic properties σ, κ, and μ of dry rocky earth crust, we find for 1-m propagation the decay factors $\epsilon^{-\alpha}$ for the respective *fields* and $\epsilon^{-2\alpha}$ for the respective *field energies* essentially equal to unity, we can compute the joulean heat loss per cubic meter directly by means of the divergence formula (119). We have already evaluated the real value $\text{Re}(\text{div } p) = \sigma\mathcal{E}_e^2\epsilon^{-2\alpha} = \sigma\mathcal{E}_e^2 = 10^{-5}$ watt/m³ for $\mathcal{E}_e = 1$ volt/m. Now the differential form of (112) for a finite volume τ completely bounded by a surface S, reads[1]

$$\underbrace{\iint_S (\mathcal{E} \times H)_n \, dS + \iiint_\tau \sigma\mathcal{E} \cdot \underset{J\ \ \mathcal{E}}{\mathcal{E}} \cdot d\tau}_{\substack{\text{must be the loss of the stored electromagnetic}\\ \text{field energy}}} = \underbrace{- \iiint_\tau \left[\mathcal{E} \cdot \frac{\partial(\kappa\mathcal{E})}{\partial t} + H \frac{\partial(\mu H)}{\partial t}\right] d\tau}_{\substack{\text{rate of decrease of electromagnetic}\\ \text{field energy stored in } \tau \text{ cu m}}}$$

The term $\iiint_\tau \mathcal{E}J\,d\tau$ is in (volt/m) · (amp/m²) · (m³), *i.e.*, watts dissipated in τ cubic meters. Just as in customary circuit theory, we have a source with an impressed electric field $\mathcal{E}_i$ so that the current density $J = \sigma\mathcal{E} + \sigma\mathcal{E}_i$, which for an

[1] For a dissipative wave medium, the em field power entering a certain medium volume τ must be partially used up in τ for supplying heat losses, and the power left in the field of τ will escape over the enclosing surface S of τ into the adjacent volume region as available forward power along the direction of wave propagation. For other details consult also pp. 201–204.

ideal conductivity σ mhos per meter would render $\mathcal{E}_i + \mathcal{E} = J/\infty = 0$, which means that $\mathcal{E}$ would be equal and opposite to the impressed field $\mathcal{E}_i$. Such a condition cannot exist and would not be useful in wave propagation; but applied to the joulean heat loss integral, it reads

$$\iiint_\tau \mathcal{E} \cdot J \, d\tau = \iiint_\tau \left(\frac{J}{\sigma} - \mathcal{E}_i\right) J \, d\tau = \underbrace{\iiint_\tau \frac{J^2}{\sigma} d\tau}_{\substack{\iiint \frac{J_0^2}{\sigma} \cos^2 \omega t \, d\tau \\ = \iiint_\tau \frac{J_e^2}{\sigma} d\tau \\ \text{(time average)}}} - \underbrace{\iiint_\tau \mathcal{E}_i \cdot J \, d\tau}_{\substack{\text{work done by } \mathcal{E}_i \\ \text{against the heat loss} \\ \iiint \frac{J_e^2}{\sigma} d\tau}}$$

We have then

$$\left.\begin{aligned} &\iint_S (\mathcal{E} \times H)_n \cdot dS + \iiint_\tau \frac{J^2}{\sigma} d\tau + \frac{\partial\left[\iiint_\tau \frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)\, d\tau\right]}{\partial t} \\ &\qquad = \overbrace{\iiint_\tau \mathcal{E}_i J \, d\tau}^{\text{impressed}} \qquad \text{integral form} \\ &\operatorname{div}(\mathcal{E} \times H) + \frac{J^2}{\sigma} + \frac{\partial[\frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)]}{\partial t} = \overbrace{\mathcal{E}_i \cdot J}^{\text{impressed}} \qquad \text{differential form} \end{aligned}\right\} \quad (120)$$

The expressions to the right are complex, with the real portion expressing the average power used by the impressed electromagnetic field to support the progressive electromagnetic wave (keeping it in a state of oscillation). A portion of the impressed field power supplies the joulean heat loss in the wave medium (second term on the left side). The remaining field energy leaves the volume τ normal to the dS boundary surfaces. Since the relations of (120) are equations of continuity, the $\mathcal{E}_i \cdot J$ term represents then the rate of the work done by the impressed field forces for sustaining the impressed current. This will cause electromagnetic field energy, which is partly used up as joulean heat loss (the J^2/σ term) and appears partly as the time rate of the electromagnetic field energy ($\kappa\mathcal{E}^2/2$ and $\mu H^2/2$ terms) and partly as the time rate of field energy flow across the surrounding surface S. This assumes that the electromagnetic field energies are distributed throughout the field region.[1] Applying the

[1] The definition of the *classical* Poynting vector $p = \mathcal{E} \times H$ watts/m² represents the field energy flow per second across one-square-meter area normal to vector $\mathcal{E} \times H$. This yields the classical differential relation

$$\operatorname{div}(\mathcal{E} \times H) + \frac{J^2}{\sigma} + \partial[0.5(\kappa\mathcal{E}^2 + \mu H^2)]/\partial t = \mathcal{E}_i \cdot J$$

for the total instantaneous electromagnetic field energy $w = w_e + w_m$ with respective space densities

$$w_e = \int_0^D \mathcal{E} d(\kappa\mathcal{E}) \qquad \text{and} \qquad w_m = \int_0^B H d(\mu H)$$

and the corresponding time averages $\bar{w}$, $\bar{w}_e$ and $\bar{w}_m$. If p denotes now the *complex*

relations of (120) to harmonic fields and using the results of (110), we find

$$\left.\begin{aligned} \tfrac{1}{2}\iiint_\tau \mathcal{E}_i \cdot J^*\, d\tau &= -\tfrac{1}{2}\iint_S (\mathcal{E} \times H^*)_n \cdot dS \\ &\quad - \tfrac{1}{2}\iiint_\tau \sigma\mathcal{E}^* \cdot \mathcal{E}\, d\tau + \tfrac{1}{2}j\omega \iiint_\tau (\kappa\mathcal{E}^* \cdot \mathcal{E} - \mu H \cdot H^*)\, d\tau \\ &= -\tfrac{1}{2}\iint_S (\mathcal{E} \times H^*)_n\, dS - \sigma \iiint \mathcal{E}_e^2\, d\tau \\ &\quad + \underbrace{j\omega \iiint (\kappa\mathcal{E}_e^2 - \mu H_e^2)\, d\tau}_{j2\omega \iiint_\tau \left(\frac{\kappa - \kappa_e}{2}\mathcal{E}_e^2\right) d\tau} \\ \kappa_e &= \kappa - j\frac{\sigma}{\omega} \qquad \text{farads/m} \end{aligned}\right\} \qquad (121)$$

where the triple integrals refer to watts in τ-cubic-meter field space and the double integral is the complex power flow across the boundary surface S enclosing the volume τ. The integral to the left represents again the complex impressed power, the real portion of which is the average power caused by the impressed field force to keep the advancing electromagnetic wave in a state of oscillation. We note that the created electromagnetic field supplies the medium heat dissipation ($\sigma\mathcal{E}_e^2$ term) and the Poynting radiation (of complex $\mathcal{E} \times H^*$ term) across the surrounding surface of S square meters. The $(\kappa - \kappa_e)\mathcal{E}_e^2/2$ term, appears as 2ω times the difference between the mean electric and magnetic power that is stored in τ cubic meters within the S boundary. This quantity, therefore, becomes small when the imaginary portion $j\sigma/\omega$ happens to be small compared with the actual dielectric constant κ of the medium. In our numerical example, this is the case, since

$$\kappa - \kappa_e = \kappa - \kappa + j\frac{\sigma}{\omega} = j\frac{10^{-5}}{6.28 \times 10^8} = j1.592 \times 10^{-14} \qquad \text{farad/m}$$

For $\tau = 1$ cu m, we find then for $\mathcal{E}_e = 1$ volt/m,

$$\begin{aligned} j2\omega \iiint_\tau \frac{(\kappa - \kappa_e)}{2}\mathcal{E}_e^2\, d\tau &= -1.592 \times 10^{-14}\omega \\ &= -1.592 \times 10^{-14} \times 6.28 \times 10^8 \\ &= -10^{-5}\ \text{watt/m}^3 \end{aligned}$$

Poynting vector, we have $p = 0.5\mathcal{E} \times H^*$, where the *real* part of p yields the time average $\bar{p}$. If $\mathcal{E}_i$ is not impressed, we have

$$w = \frac{J^2}{\sigma} - \mathcal{E}_i J = \frac{J^2}{\sigma}$$

$$\begin{aligned} \operatorname{div} p &= -0.5\sigma\mathcal{E}\mathcal{E}^* - j0.5\omega(\mu H H^* - \kappa\mathcal{E}\mathcal{E}^*) \\ &= -\bar{w} - j2\omega(\bar{w}_e - \bar{w}_m \end{aligned}$$

where the dissipated heat energy per unit volume is found from the divergence of the *real* part of p.

which is the value of the dissipation in a unit volume, since, according to (112), the rate at which the electromagnetic field power increases within 1 cu m is equal to the watt loss. Equation (112) is an equation of continuity and based on the conservation of energy. With respect to the complex Poynting vector p we find, by dividing it by the velocity of propagation,

$$\frac{p}{v} = \frac{p}{1.06 \times 10^8} = 7.05 \times 10^{-11}\underbrace{\epsilon^{-1.33\times10^{-1}}}_{\cong 1.0} - j7.92 \times 10^{-15}\epsilon^{-1.33\times10^{-1}}$$

$$= \underbrace{7.05 \times 10^{-11}}_{\text{dissipated as heat}} - \underbrace{j7.92 \times 10^{-15}}_{\text{reactive energy}} \qquad \text{joule/m}^3$$

28. Remarks on Formulas Dealing with Field Energy Distributions. If $\mathcal{E}$ and H denote the field values $\mathcal{E}_0\epsilon^{j\omega t}$ and $H_0\epsilon^{j(\omega t-\varphi)}$, we have in τ cubic meters of space, $\int_\tau \frac{1}{2}\kappa\mathcal{E}^2\,d\tau$ and $\int_\tau \frac{1}{2}\mu H^2\,d\tau$ joules stored, since for a unit of volume

$$\tfrac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2) \qquad \text{joules/m}^3$$

represents the total instantaneous energy density. The electromagnetic power flowing out of τ cubic meters of such a field is

$$P = -\frac{\partial\left[\frac{1}{2}\int_\tau (\kappa\mathcal{E}^2 + \mu H^2)\,d\tau\right]}{\partial t} \qquad \text{watts} \tag{122}$$

since it must be equal to the *decrease* of the electromagnetic field energy in each unit volume. This negative time rate is for a pure dielectric field medium equal to the divergence of p (energy flow), *i.e.*, equal to the power in watts crossing perpendicularly 1-sq-m areas. The factor of $\frac{1}{2}$ enters since $\mathcal{E}$ and H are the volt per meter and amperes per meter values at an instant and the average value between $\mathcal{E} = 0$ and $\mathcal{E} = \mathcal{E}$, and $H = 0$ and $H = H$ is $\frac{1}{2}\mathcal{E}$ and $\frac{1}{2}H$, respectively. This is based on the integrals $\int_0^{\kappa\mathcal{E}} \underset{\cdot}{\mathcal{E}} \cdot d(\kappa\underset{\cdot}{\mathcal{E}})$ and $\int_0^{\mu H} \underset{\cdot}{H} \cdot d(\mu \underset{\cdot}{H})$ for the electric and magnetic volume densities in joules per cubic meter. We have then for the general case, where there is a time phase difference between the electric and magnetic fields,

$$\tfrac{1}{2}[\kappa\mathcal{E}_0^2\epsilon^{j2\omega t} + \mu H_0^2\epsilon^{j2(\omega t-\varphi)}] \qquad \text{instantaneous field energy/m}^3$$

with a time average energy of

$$\tfrac{1}{2}[\kappa\mathcal{E}_0^2 \underbrace{\cos^2 2\omega t}_{=\frac{1}{2}} + \mu H_0^2 \underbrace{\cos^2 (2\omega t - 2\varphi)}_{=\frac{1}{2}}] = \tfrac{1}{4}(\kappa\mathcal{E}_0^2 + \mu H_0^2)$$

$$= \tfrac{1}{2}(\kappa\mathcal{E}_e^2 + \mu H_e^2) \qquad \text{joules/m}^3 \tag{123}$$

which, for a volume of τ cubic meters of electromagnetic field space, yields

$$\frac{1}{4}\int_\tau (\kappa \mathcal{E}_0^2 + \mu H_0)^2\, d\tau = \frac{\kappa}{2}\int_\tau \mathcal{E}_e^2\, d\tau + \frac{\mu}{2}\int_\tau H_e^2\, d\tau \qquad \text{joules}$$

The corresponding power volume density P/τ and power P in τ cubic meters of electromagnetic field space then yield

$$\frac{P}{\tau} = \frac{1}{2}\frac{\partial[\kappa\mathcal{E}_0^2\epsilon^{j2\omega t} + \mu H_0^2\epsilon^{j2(\omega t-\varphi)}]}{\partial t} = \underbrace{\frac{1}{2}[j2\omega\kappa\mathcal{E}_0^2\epsilon^{j2\omega t} + j2\omega\mu H_0^2\epsilon^{j2(\omega t-\varphi)}]}_{\text{instantaneous power}}$$

where in the $\epsilon^{ja} = \cos(a) + j\sin(a)$ expression, only the *real* portion $\cos(a)$ stands for the instantaneous value. Therefore, the average power density becomes

$$\frac{P}{\tau} = j\omega\,[\kappa\mathcal{E}_0^2 \underbrace{\cos^2(2\omega t)}_{=\frac{1}{2}} + \mu H_0^2 \underbrace{\cos^2(2\omega t - 2\varphi)}_{=\frac{1}{2}}]$$

$$= j\omega\left(\frac{\kappa\mathcal{E}_0^2}{2} + \frac{\mu H_0^2}{2}\right)$$

$$= j\omega(\kappa\mathcal{E}_e^2 + \mu H_e^2)$$

$$= j2\omega\underbrace{\left(\frac{\kappa\mathcal{E}_e^2 + \mu H_e^2}{2}\right)}_{\text{mean sum}} \qquad \text{watts/m}^3 \quad (124)$$

The general field function of form $F_0^2\cos(2\omega t - 2\varphi)$, for the substitutions $\omega' = 2\omega$ and $\theta = 2\omega t = \omega' t$, yields $d\theta/dt = \omega'$ or $dt = (d\theta)/\omega'$ with a time average

$$F_{av} = \frac{1}{T'}\int_0^{T'} F_0^2\cos^2(\omega' t - 2\varphi)\,dt = \frac{1}{\omega' T'}\int_0^{2\pi} F_0^2\cos^2(\theta - 2\varphi)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} F_0^2\cos^2(\theta - 2\varphi)\,d\theta$$

$$= \frac{4F_0^2}{2\pi}\int_0^{\pi/2}\cos^2(\theta - 2\varphi)\,d\theta = \frac{2F_0^2}{\pi}\int_0^{\pi/2}\frac{1 + \cos(2\theta - 4\varphi)}{2}\,d\theta$$

$$= \frac{F_0^2}{\pi}|\theta|_0^{\pi/2} + \frac{F_0^2}{\pi}\int_0^{\pi/2}\cos(2\theta - 4\varphi)\,d\theta$$

$$= \underbrace{\frac{F_0^2}{\pi}\left(\frac{\pi}{2} - 0\right)}_{\frac{F_0^2}{2}} + \underbrace{\frac{F_0^2}{\pi}\left|\frac{1}{2}\sin(2\theta - 4\varphi)\right|_0^{\pi/2}}_{\frac{1}{2}[\sin(\pi - 4\varphi) - \sin(-4\varphi)] = 0}$$

The electromagnetic power in τ cubic meters becomes

$$P = j\omega\left(\kappa \iiint_\tau \mathcal{E}_e^2\, d\tau + \mu \iiint_\tau H_e^2\, d\tau\right)$$
$$= j\omega\left(\kappa \iiint_\tau \mathcal{E}_e^2\, d\tau + \mu \iiint_\tau \frac{\kappa_e}{\mu}\mathcal{E}_e^2\, d\tau\right)$$
$$= j\omega\left[\kappa \iiint_\tau \mathcal{E}_e^2\, d\tau + \iiint_\tau \left(\kappa - j\frac{\sigma}{\omega}\right)\mathcal{E}_e^2\, d\tau\right]$$
$$= \sigma \iiint_\tau \mathcal{E}_e^2\, d\tau + j2\omega\kappa \iiint \mathcal{E}_e^2\, d\tau \qquad \text{watts} \tag{125}$$

This outcome resembles the expression for the power in an alternating-current network with an impedance load $Z = R \pm jX$ ohms, where we have the instantaneous voltage $E = E_0 \cos \omega t$ across Z and the instantaneous current $I = I_0 \cos (\omega t \pm \varphi)$ which passes through Z. The instantaneous power then becomes

$$P_t = \underbrace{E_0 I_0 \cos \omega t \cos (\omega t \pm \varphi)}_{-[\cos(\omega t - \omega t \mp \varphi) + \cos(\omega t + \omega t \pm \varphi)]} = \underbrace{E_e I_e \cos (\mp \varphi)}_{E_e^2/R \text{ watts}} + \underbrace{E_e I_e \cos (2\omega t \pm \varphi)}_{\text{wattless}}$$

The second term is, as in (125), of double frequency and therefore wattless, while the first term represents the average power dissipated as is the case for the $\sigma\mathcal{E}_e^2$ triple integral. Since, for instance,

$$E_e I_e \cos (2\omega t - \varphi) = E_e I_e \underbrace{\cos \varphi}_{R/Z} \cos 2\omega t + E_e I_e \underbrace{\sin \varphi}_{X/Z} \sin 2\omega t$$

and

$$\underbrace{A \sin (2\omega t) + B \cos (2\omega t) = \sqrt{A^2 + B^2} \sin (2\omega t + \psi); \qquad \psi = \tan^{-1}\left(\frac{B}{A}\right)}_{\text{proved on p. 115 in connection with Fig. 24}}$$

we find for the wattless term also

$$E_e I_e \cos (2\omega t - \varphi) = \sqrt{\frac{E_e^2 I_e^2 R^2}{Z^2} + \frac{E_e^2 I_e^2 X^2}{Z^2}} \sin\left(2\omega t + \tan^{-1}\frac{R}{X}\right)$$
$$= E_e I_e \sin [2\omega t + \tan^{-1} (\cot \varphi)]$$
$$= \frac{E_e^2}{Z} \sin [2\omega t + \tan^{-1} (\cot \varphi)]$$

Applying (125) to dry rocky ground for $\mathcal{E}_e = 1$ volt/m, $\sigma = 10^{-5}$ mho/m, $\kappa = 7.083 \times 10^{-11}$ farad/m, and $f = 100$ Mc/sec, we find, for 1-cu-m space and the evaluations in Sec. 28, that we can simplify (125) to $P = \sigma\mathcal{E}_e^2 + j2\omega\kappa\mathcal{E}_e^2$ with the evaluation

$$P = 10^{-5} + j12.56 \times 10^8 \times 7.083 \times 10^{-11}$$
$$= \underbrace{10^{-5}}_{\text{lost power}} + \underbrace{j8.9 \times 10^{-2}}_{\text{reactive}} \text{ watt/m}^3$$

showing quite a favorable value for the actual field power. Matters would not be quite so fortunate in the case of a cubic meter of ocean water. Suppose we orient the unit cube such that through one square-meter face electromagnetic field energy enters perpendicular to the face, so that at the opposite parallel square-meter face any still available field power will escape also normal to the square-meter face. Suppose we specify the ocean water as $\sigma = 5$ mhos/m, $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m, and $\kappa = 80\kappa_0 = 80 \times 8.854 \times 10^{-12} = 7.1 \times 10^{-10}$ farad/m. The figure of merit of the ocean as a wave medium is then at 100 Mc/sec, $Q = \omega\kappa/\sigma = 6.28 \times 10^8 \times 7.1 \times 10^{-10}/5 = 8.9 \times 10^{-2} = 0.089$. This is a rather poor electromagnetic quality. Let us find the frequency f for which the ocean behaves at least as a semiconductor, *i.e.*, also a semi-dielectric, since then $Q = 1$ and the displacement current density numerically equal to the conduction current density. This means that $\sigma = \omega\kappa$ and $f = \sigma/(6.28\kappa) = 5/(6.28 \times 7.1 \times 10^{-10}) \cong 1{,}122 \times 10^6$ cycles/sec or 1,122 Mc/sec. From formulas (30) on page 53, we find

$$q = \sqrt{1+1} = \sqrt{2}; \qquad r = 1 - j;$$

$$Z_0 = \sqrt{\frac{1.257 \times 10^{-6}}{(1-j)7.1 \times 10^{-10}}} = R_0 + jX_0 \qquad \text{complex ohms}$$

$$R_0 = \sqrt{\frac{1.257 \times 10^{-6}}{7.1 \times 10^{-10}}}\sqrt{\frac{1+\sqrt{2}}{4}} \qquad \text{ohms}$$

$$X_0 = \sqrt{\frac{1.257 \times 10^{-6}}{7.1 \times 10^{-10}}}\,\frac{\sqrt{2/(1+\sqrt{2})}}{2\sqrt{2}} \qquad \text{reactive ohms}$$

and what is of most interest in this example,

$$\alpha = \frac{5}{2}\sqrt{\frac{1.257 \times 10^{-6}}{7.1 \times 10^{-10}}}\sqrt{\frac{2}{1+\sqrt{2}}} \qquad \text{nepers/m}$$

$$\beta = \underbrace{\omega}_{6.28 \times 1.122 \times 10^9}\sqrt{1.257 \times 10^{-6} \times 7.1 \times 10^{-10}}\sqrt{\frac{1+\sqrt{2}}{2}} \qquad \text{circular radians/m}$$

$$v = \frac{\omega}{\beta} \qquad \text{m/sec}$$

The attenuation constant α then yields

$$\alpha = 2.5 \times \underbrace{\sqrt{17.7 \times 10^2}}_{42} \cdot \underbrace{\sqrt{82.8 \times 10^{-2}}}_{9.15 \times 10^{-1}} = 96 \text{ nepers/m}$$

This is quite a large attenuation even for as high a frequency as 1,122 Mc/sec. Now when we evaluate the real component of the complex Poynting vector p of (118*a*) for the square-meter face across which the

electromagnetic wave enters the cubic meter and evaluate the real part of p for the square-meter face over which the remaining power leaves the cubic meter of ocean water, we find the power lost in the cubic meter by taking the difference between the respective real powers in watts per square meter. Let us take the case where we have the effective electric field $\mathcal{E}_e = \mathcal{E}_0/\sqrt{2}$ at the face over which electromagnetic power enters the unit volume; then according to (118*a*), we find

$$\text{Power dissipated in 1 cu m} = 7.96 \times 10^{-8} \frac{\beta \mathcal{E}_0^2 \epsilon^{-2\alpha\,0}}{\mu f^{(\text{Mc})}} - 7.96 \times 10^{-8} \frac{\beta \mathcal{E}_0^2 \epsilon^{-2\alpha\,1}}{\mu f^{(\text{Mc})}}$$

$$= 7.96 \times 10^{-8} \frac{\beta \cdot 2^2 \cdot 1}{\mu f^{(\text{Mc})}} - 7.96 \times 10^{-8} \frac{\beta \cdot 2^2 \epsilon^{-2\times 96}}{\mu f^{(\text{Mc})}}$$

$$= 4 \times 7.96 \times 10^{-8} \frac{\beta}{\mu f^{(\text{Mc})}} (1 - \underbrace{\epsilon^{-192}}_{\text{negligible}})$$

$$= 4 \times 7.96 \times 10^{-8} \frac{\beta}{\mu f^{(\text{Mc})}} \qquad \text{watts}$$

since even ϵ^{-6} yields already a value as small as only 0.00248 (only 0.248 per cent of the amount of the face over which power enters). Therefore, ϵ^{-192} must be negligible. We find for the phase constant

$$\beta = 6.28 \times 1.122 \times 10^{6} \underbrace{\sqrt{1.257 \times 10^{-6} \times 7.1 \times 10^{-10}}}_{2.983 \times 10^{-8}} \underbrace{\sqrt{\frac{2.414}{2}}}_{1.1} = 2.3 \times 10^2 \text{ radians/m}$$

The loss per cubic meter then becomes equal to

$$\frac{4 \times 7.96 \times 10^{-8} \times 2.3 \times 10^2}{1.256 \times 10^{-6} \times 1{,}122} = 5.19 \times 10^{-2} \text{ watt}$$

This does *not* mean that the watt loss due to joulean heat is uniformly distributed in the cubic meter of ocean. As a matter of fact, the electromagnetic wave hardly penetrates to any appreciable distance before all the energy is dissipated (on account of the large magnitude of α). Computations on penetration depth are given on pages 477–479. The field intensity of 1 volt effective per meter is likewise quite a large value, which is not likely to happen in the distant radiation region. Moreover, the magnetic H vector lags the electric $\mathcal{E}$ vector by 22.5 time degrees, since, according to page 45, $\sin \varphi = \alpha/\sqrt{\alpha^2 + \beta^2} = 96/\sqrt{96^2 + 230^2}$

$= 96/\sqrt{62{,}116} = 0.386$. *This relative time phase angle could also be* computed from Eq. (117). We find then that $\varphi = 0.5 \tan^{-1} (-\sigma/\omega\kappa)$ $= 0.5 \tan^{-1} (-1) = 0.5 \times 45 \text{ deg} = 22.5 \text{ deg}$.

29. Logarithmic Decrement, Power Factor, and Phase Difference in a Dissipative Wave Medium. According to the electric and magnetic field formulas of Eq. (117), page 124, we have a decay factor $\epsilon^{-\alpha x}$ multiplied with the field amplitude. We have, therefore, a logarithmic falling off of the respective electric and magnetic fields with the distance x, due to the medium dissipation. In Eqs. (120) and (121), we have energy relations in connection with the Poynting vector. The quantity $\mathcal{E}_i$ represents the impressed field, which when multiplied with the current density represents the impressed power that keeps the outgoing electromagnetic wave sustained. The fact that a field is so energized that besides the electromagnetic field energy also the medium losses are supplied does not mean that there is no amplitude decay due to medium dissipation. If we look at a wave train containing several wavelengths, we shall find that both field associations decrease by a definite amount for distances of 1 wavelength. The field positions 1 wavelength apart in space and along the direction of propagation correspond to the period $T = 1/f$ seconds, since it takes the wave front T seconds to move through a distance of 1 wavelength. If $\mathcal{E}_1$ and $\mathcal{E}_2$ volts per meter denote the instantaneous electric field values a distance of 1 wavelength λ apart along the energy transfer, we call $\delta = \log_\epsilon (\mathcal{E}_1\mathcal{E}_2)$ the logarithmic decrement. The reason for this can be seen from the following derivation based on the ratio $\mathcal{E}_1/\mathcal{E}_2$. We find then

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{\mathcal{E}_0\epsilon^{-\alpha x}\epsilon^{j(\omega t-\beta x)}}{\mathcal{E}_0\epsilon^{-\alpha(x+\lambda)}\epsilon^{j[\omega t-\beta(x+\lambda)]}} = \frac{\mathcal{E}_0\epsilon^{-\alpha x}\epsilon^{j\omega t}\epsilon^{-j\beta x}}{\mathcal{E}_0\epsilon^{-\alpha x}\epsilon^{-\alpha\lambda}\epsilon^{j\omega t}\epsilon^{-j\beta x}\epsilon^{-j\beta\lambda}} = \frac{1}{\epsilon^{-\alpha\lambda}\epsilon^{-j\beta\lambda}}$$

$$= \epsilon^{+\alpha\lambda}\epsilon^{+j\beta\lambda} = \epsilon^{\alpha\lambda}\epsilon^{j(2\pi/\lambda)\lambda} = \epsilon^{\alpha\lambda}[\underbrace{\cos(2\pi)}_{=1} + j\underbrace{\sin(2\pi)}_{=0}] = \epsilon^{\alpha\lambda}$$

a simple result, expressing that $\mathcal{E}_2 = \mathcal{E}_1\epsilon^{-\alpha\lambda}$. This means that the field amplitude falls off exponentially. Instead of using the exponential form $\mathcal{E}_1/\mathcal{E}_2 = \epsilon^{\alpha\lambda}$, we can also use the logarithmic form, and we have the comparisons

$$\delta = \log_\epsilon\left(\frac{\mathcal{E}_1}{\mathcal{E}_2}\right) = \alpha\lambda = 2.303 \log_{10}\left(\frac{\mathcal{E}_1}{\mathcal{E}_2}\right) \quad \text{and} \quad \frac{\mathcal{E}_1}{\mathcal{E}_2} = \epsilon^{\alpha\lambda} \qquad (126)$$

where δ is the logarithmic decrement per space wavelength λ along the direction of wave propagation due to medium losses. The quantity α is usually due to an effective conductivity σ of a medium, where the medium is assumed isotropic. We have α in nepers per meter wave travel. The logarithmic medium decrement $\delta = \alpha\lambda$ is in nepers.

Wave spread likewise causes wave decay as does destructive wave interference. Such attenuations *cannot* be expressed logarithmically.

Figure 25 shows how we can divide any finite area of S square meters into infinitesimal area elements dS (very small areas $\Delta S \to 0$). We have then a mosaic structure with vanishingly small H circulations, all curling in the same sense. The shaded elementary area dS is a representation with a closed boundary *abcda*. The elementary H circulations of neighboring closed areas such as *bafeb*, *adhga*, *dcmnd*, and *cbikc* interfere with each other. Thus at the common boundary *cd*, the $H\,dl$ effect is $+Hcd$ for the shaded area element, but $+H\,dc$ or $-Hcd$ for the neighboring *dcmnd* area element. Hence, respective magnetomotive forces result in $Hcd - Hcd$, which

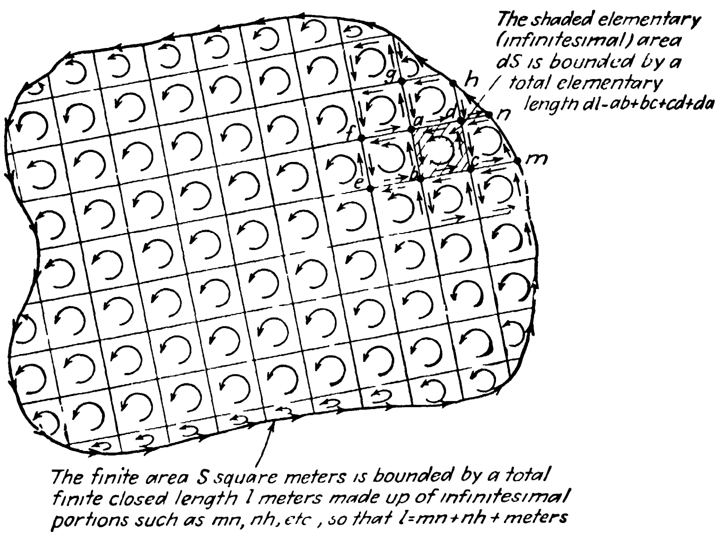


FIG. 25. How elementary $H\,dl$ circulations may cause in effect a finite Hl circulation (this is the integration or summing up of elementary magnetic dipole effects).

means a zero effect along the common boundary *cd*. This happens all around the boundary of the shaded area dS of contour *abcda*. It happens only *partially* for the neighboring area of contour *dcmnd*, because there is *no* opposing magnetomotive force for the *Hmn* amperes along the boundary *mn*. The very outside boundary forces along the elementary portions, *mn*, *nh*, etc., remain, and we have a circulating magnetomotive force $H(mn + nh + \cdots) = Hl$ amperes left at the very outside contour. This is the closed boundary of l meters, enclosing the finite area of S square meters. Hence, all the elementary closed $H\,dl$ circulations sum up to the finite Hl circulation. Since the elementary $H\,dl$ circulations normal to their respective areas dS also cause infinitesimal conduction current densities dJ and infinitesimal displacement current densities $d[\partial(\kappa\varepsilon)/\partial t]$ in amperes per square meter, we have a sum of effects as expressed by the curl H relation of Eqs. (1). For a finite area of 1 sq m, we have $\oint_l H\,dl = J_e + j\omega\kappa\varepsilon_e$ amperes per square meter, if J_e and $j\omega\kappa\varepsilon_e$ denote the average effective conduction and displacement current densities across the 1-sq-m area bounded by l meters. The average effective current densities refer to the general case where

due to skin effect or some other cause, the current across a certain enclosed area may not be uniformly distributed.

From the foregoing discussion, we note that across a finite area of S square meters, we deal with a conduction current density $J_e = \sigma\mathcal{E}_e$ amperes per square meter and the time quadrature displacement current density $j\omega\kappa\mathcal{E}_e$. The current flows are along electric lines of force, *i.e.*, along $\mathcal{E}_0\epsilon^{-\alpha x}\epsilon^{j(\omega t-\beta x)}$ variations, if wave propagation occurs along the positive X direction with a velocity of ω/β meters per second. We deal then with conduction currents $I_e = \sigma S\mathcal{E}_e$ and displacement currents $I_d = j\omega\kappa S\mathcal{E}_e$ in amperes if a finite H circulation encloses an area of S square meters. The resultant general space current $I_r = \sqrt{I_e^2 + I_d^2}$ has then also a time phase angle of $\varphi = \tan^{-1}(\omega\kappa/\sigma) = \tan^{-1}(Q)$. Now in wave propagation, the inphase component $I_r \cos\varphi$ represents a *loss* current in amperes, since the corresponding current density $(I_r \cos\varphi)/S = \mathcal{E}_e\sqrt{\sigma^2 + \omega^2\kappa^2}\cos\varphi$ in amperes per square meter causes the medium dissipation $\mathcal{E}_e\sigma\mathcal{E}_e = \sigma\mathcal{E}_e^2$ watts per cubic meter, because $\cos\varphi = \sigma/\sqrt{\sigma^2 + \omega^2\kappa^2}$ and the average value of the field heat loss is $P_h = \mathcal{E}_e \cdot J_r \cos\varphi$ watts per cubic meter, for $J_r = \sqrt{J_e^2 + J_d^2} = \sqrt{\sigma^2\mathcal{E}_e^2 + \omega^2\kappa^2\mathcal{E}_e^2} = \mathcal{E}_e\sqrt{\sigma^2 + \omega\kappa^2}$ amperes per square meter. We have then for the general isotropic medium

$$P = \underbrace{\mathcal{E}_eJ_r\cos\varphi}_{\text{dissipation}} + \underbrace{\mathcal{E}_eJ_d\sin\varphi}_{\text{useful field}} = \mathcal{E}_eJ_r(\underbrace{\cos\varphi}_{\frac{\sigma}{\sqrt{\sigma^2+\omega^2\kappa^2}}} + \underbrace{\sin\varphi}_{\frac{\omega\kappa}{\sqrt{\sigma^2+\omega^2\kappa^2}}})$$

$$= \mathcal{E}_e \cdot \mathcal{E}_e\sqrt{\sigma^2 + \omega^2\kappa^2}\left(\frac{\sigma}{\sqrt{\sigma^2 + \omega^2\kappa^2}} + \frac{\omega\kappa}{\sqrt{\sigma^2 + \omega^2\kappa^2}}\right)$$

$$= \underbrace{\sigma\mathcal{E}_e^2}_{\substack{\text{medium}\\\text{power}\\\text{loss}}} + \underbrace{\omega\kappa\mathcal{E}_e^2}_{\substack{\text{useful electro-}\\\text{magnetic power}}} \quad \text{watts/m}^3 \tag{127}$$

Since, in wave propagation, we are not interested in heating up the field region, we can use only the term $\omega\kappa\mathcal{E}^2$ in actual wave propagation. This means that we have all the more medium efficiency the larger the ratio $Q = \omega\kappa/\sigma$ of the dissipative medium is. Since

$$\cos\varphi = \frac{\sigma}{\sqrt{\sigma^2 + \omega^2\kappa^2}};\ \sin\varphi = \frac{\omega\kappa}{\sqrt{\sigma^2 + \omega^2\kappa^2}} = \cos\underbrace{(90 - \varphi)}_{\psi} = \cos\psi \tag{128}$$

we are *more* interested in the sin φ current component in the electromagnetic field than in the cos φ component. Therefore, it is the complementary angle $\psi = 90 - \varphi$, rather than φ that is of concern. The angle ψ is known as the "phase difference" if we deal with condenser fields. We can also use such a name for the electromagnetic radiation

field, since actually the dielectric constant κ in farads per meter refers to an instantaneous electric field energy in a cubic meter of medium such as would exist in a condenser dielectric with square-meter plates that are parallel and separated by a distance of 1 m.

Now let us proceed with useful orders of magnitude. First, trigonometric tables will show that the angle ψ of sine and tangent functions, when expressed in radians, is practically equal to $\sin\psi$ and $\tan\psi$, if $\psi \leqq 11$ deg, since $\sin 11° = 0.1908$, $\tan 11° = 0.1944$, or $\sin\psi \cong \tan\psi \cong 0.19$, and $\psi = 11/57.3 \cong 0.19$ radian. The trigonometric table also shows that for $\psi \leqq 11$ deg, we have $\cos\psi \cong 1$, since $\cos 11°$, which is the most severe limit, even yields exactly a value of 0.9816. Now $\psi = 11°$ would, according to $\psi = 11° = 90 - 11 = 79° = \varphi$, lead to $\varphi = 79° = \tan^{-1}(Q) = 5.1446$. Therefore, for Q values of 5.145 and higher, we have in (128) a condition for which $\cos\psi$ is essentially equal to unity, regardless of the operating frequency $\omega/6.28$ in $Q = \omega\kappa/\sigma$. To be very conservative, let us take the lower limit of the medium Q as 10, for which we have $\omega\kappa/\sigma = 10$ and $\varphi = \tan^{-1} 10 = 84.3$ deg. Then for Q values of 10 or more, the phase difference ψ of the dissipative wave medium is independent of the operating frequency. This can be seen from (128), since for the most severe limit $\sigma = \omega\kappa/10$ and

$$\cos\psi = \sin\varphi = \frac{\omega\kappa}{\sqrt{\sigma^2 + \omega^2\kappa^2}} = \frac{\omega\kappa}{\sqrt{\dfrac{\omega^2\kappa^2}{100} + \omega^2\kappa^2}} = \frac{\omega\kappa}{\sqrt{1.01\omega^2\kappa^2}} \cong 1$$

Now what does this convey? It expresses that for Q values equal to or even larger than 10, the customary power factor $\cos\varphi$ must practically vanish. That this is the case we note from the most severe limit of Q equal only to 10, for which $\varphi = 84.3$ deg and $\cos 84.3° = 0.0993$. This is 9.93 per cent of the value of the useful $\sin\varphi$ function. We deal, therefore, for the most severe limit of the Q value with a radiation efficiency of the dissipative medium of somewhat better than 90 per cent.

With respect to (127), we have also the following details:

$$\begin{aligned} P &= \mathcal{E}_e J_r(\cos\varphi + \sin\varphi) = \frac{\mathcal{E}_e^2}{\sqrt{\sigma^2 + \omega^2\kappa^2}}(\cos\varphi + \sin\varphi) \\ &= \mathcal{E}_e^2\left(\frac{\sigma}{\sigma^2 + \omega^2\kappa^2} + \frac{\omega\kappa}{\sigma^2 + \omega^2\kappa^2}\right) = \mathcal{E}_e^2\left[\frac{\sigma^2}{\sigma(\sigma^2 + \omega^2\kappa^2)} + \frac{\omega^2\kappa^2}{\omega\kappa(\sigma^2 + \omega^2\kappa^2)}\right] \\ &= \mathcal{E}_e^2\left(\frac{\cos^2\varphi}{\sigma} + \frac{\sin^2\varphi}{\omega\kappa}\right) = \mathcal{E}_e^2\left(\frac{\cos^2\varphi}{\omega\kappa/Q} + \frac{\sin^2\varphi}{\omega\kappa}\right) = \frac{\mathcal{E}_e^2}{\omega\kappa}(Q\cos^2\varphi + \sin^2\varphi) \\ &= \frac{\mathcal{E}_e^2}{\sigma}\left(\cos^2\varphi + \frac{\sin^2\varphi}{Q}\right) \qquad \text{watts/m}^3 \end{aligned} \tag{129}$$

Hence, for a displacement current ten times the value of the conduction current, we have

$$P = \frac{\mathcal{E}_e^2}{\sigma}(\cos^2\varphi + 0.1\sin^2\varphi) = \frac{\mathcal{E}_e^2}{\omega\kappa}(10\cos^2\varphi + \sin^2\varphi)$$

Now let us proceed with Q values equal to 10 or more. We find then, in the formulas of Eqs. (30), page 53,

$$\left.\begin{aligned}
&Q \geqq \underbrace{10}_{\text{or smaller}} = \frac{\omega\kappa}{\sigma}; \qquad q = \sqrt{1 - 0.01} \cong 1; \qquad r = 1 - j0.1 \\
&\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}} \text{ nepers/m}; \qquad \beta = \omega\sqrt{\mu\kappa} \text{ circular radians/m} \\
&v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\kappa}} \text{ m/sec}; \qquad \lambda = \frac{v}{f} = \frac{1}{f\sqrt{\mu\kappa}} \text{ m} \\
&\delta = \alpha\lambda = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\frac{1}{f\sqrt{\mu\kappa}} = \frac{\sigma}{2\kappa f} = \frac{\pi\sigma}{\omega\kappa} = \frac{\pi}{Q} \\
&\cos\varphi = \frac{\sigma}{\sqrt{\sigma^2 + \omega^2\kappa^2}} = \frac{1}{\sqrt{1 + Q^2}} \cong \frac{1}{Q} = \frac{\delta}{\pi} \quad \text{dissipation power factor} \\
&\sin\varphi = \frac{\omega\kappa}{\sqrt{\sigma^2 + \omega^2\kappa^2}} = \frac{1}{\sqrt{1 + \frac{1}{Q^2}}} \cong 1 = \cos\psi \quad \text{useful electromagnetic field factor} \\
&\sin\psi = \sin(90 - \varphi) = \cos\varphi = \frac{1}{Q} = \frac{\delta}{\pi} \cong \psi \quad \text{circular radians}
\end{aligned}\right\} \qquad (130)$$

The quantity δ is the logarithmic decrement per wavelength, ψ is the phase difference of the general wave propagation medium, and f is in cycles per second. We have, therefore, the following relations for the logarithmic medium decrement δ and the phase difference ψ of the medium

$$\delta = \frac{\pi}{Q} = \pi\cos\varphi = \pi\sin\psi \cong \pi\psi \qquad (130a)$$

When we compare the field energy dissipated in a certain volume of a general medium to supply the heat loss on account of the conductivity with the electromagnetic field energy, we can compute the electromagnetic field energy in joules per cubic meter either from the maximum electric field energy or from the maximum magnetic field energy. In order to obtain a relationship with the logarithmic decrement $\delta = \alpha\lambda$, we must make the computations with respect to a full-space wavelength of λ meters. This means that with respect to time, $T = 1/f$ seconds. We find then

$$\left.\begin{aligned} \frac{\text{Dissipated energy of medium, heat loss in joules}}{\text{Useful field energy in joules}} &= \frac{\sigma \mathcal{E}_e^2 \cdot T}{\kappa \mathcal{E}_e^2} \\ = \frac{\sigma}{\kappa f} = 2\delta = 2\pi\psi \qquad &\text{radians numeric for } f \text{ in cycles/sec} \end{aligned}\right\} \quad (131)$$

30. Summary of the Interdependence between Electromagnetic Fields and Currents. In the general wave medium, conduction and displacement currents exist by virtue of an electromagnetic field. For a pure dielectric wave medium, we have *only* displacement currents. When a wave strikes the earth crust or other obstacle, such obstacles generally exhibit a dissipative medium. It depends upon the operating frequency to what extent the obstacle is a dielectric and to what extent it is a conductive mirror (reflector) in effect. The figure of merit of a medium is given by the ratio $\omega\kappa/\sigma$ and *must* be a large numeric value for a good wave medium. The denominator is a measure of the conduction current (watt current), and the numerator is a measure of the wattless displacement current. The ratio of the electromagnetic field energy used up in joulean heat loss divided by the remaining field energy is equal to 6.28 times the power factor of the medium. It is also 6.28 divided by the Q value (figure of merit) of the medium. Electromagnetic wave fronts can be replaced by electromagnetic current sheets, even though magnetic current sheets have, so far, never been verified experimentally. An electric current sheet would be an infinitely thin sheet carrying finite current per unit length that is perpendicular to the lines of flow. Good conductors carry only VHF and UHF currents in extremely *thin* surface skins *next to the field* (outside surface of a solid conductor and inside surface of a conductive wave guide wall). We shall find that an advancing wave front is a separation between an electromagnetic field region and a region without a field. Such a surface of separation of discontinuity may be replaced by electric and magnetic current sheets. Electromagnetic wave diffraction, as well as reflection and refraction, may play a part for certain obstacles. Conditions in the lower troposphere may cause appreciable refraction and cause a radio beam to curve downward (and sometimes upward), depending on humidity and temperature gradients with respect to altitude. Atmospheric conditions may also trap electromagnetic wave energy and guide it efficiently over larger distances than expected from the customary classic ray theory.

CHAPTER II

SPACE ELECTROMAGNETIC FIELDS OF ELEMENTARY ELECTRIC AND MAGNETIC DIPOLES

Electric doublets are elementary radiators that, in effect, also behave like an infinitesimal linear current element which radiates electromagnetic field energy into space. We may imagine a linear radiator of finite total length as being made up of infinitesimal current elements. If we sum up the radiation effects of such elements at a space point, with respect to amplitude and relative time phase, we shall obtain the resultant field effect of the linear radiator. A radiator may be made up of linear sections, such as in the case of an inverted L antenna or a T antenna. *Magnetic doublets* are infinitesimally small current loops, since a small current loop, such as used for exciting wave guides, or for coupling purposes, produces at neighboring points a field similar to a small magnetic dipole. The field of such small current loops resembles the field of an electric dipole if we interchange the electric and magnetic fields.

A customary center-fed half-wave radiator is a finite electric dipole. Elementary electric and magnetic radiators of this type exhibit simple radiation patterns.

31. Curvilinear Orthogonal Coordinate System Applied to Maxwell's Field Equations. In many problems, electromagnetic field relations must be expressed in cylindrical as well as in spherical coordinates, instead of in ordinary Cartesian XYZ coordinates, for which a space point is fixed by the intersection of three perpendicular *planes*. Expressing field relations in a general coordinate system, known as the "reference orthogonal system" of Fig. 26, it is possible, by comparing factors, to express a relation in Cartesian, cylindrical, or spherical coordinates. The reference system determines a space point by the intersection of three *curved* surfaces, since two curved surfaces intersect along a space curve intersecting the remaining curved surface at the space point. The space curves Oa, Ob, and Oc formed by the intersections of respective pairs of curved surfaces are coordinate curves a, b, and c instead of the straight coordinates x, y, and z in an XYZ system. When such curved surfaces are chosen so as to make the coordinate axes at every space point mutually perpendicular, we deal with *orthogonal general coordinates*.

In the Cartesian system, the volume $d\tau$ of a space element as shown

in Fig. 27 is $dx\,dy\,dz$, since the elementary perpendicular edges dx, dy, and dz are straight. In the reference coordinate system, we have, as might be said, a distorted parallelepiped, since the edges are space curves

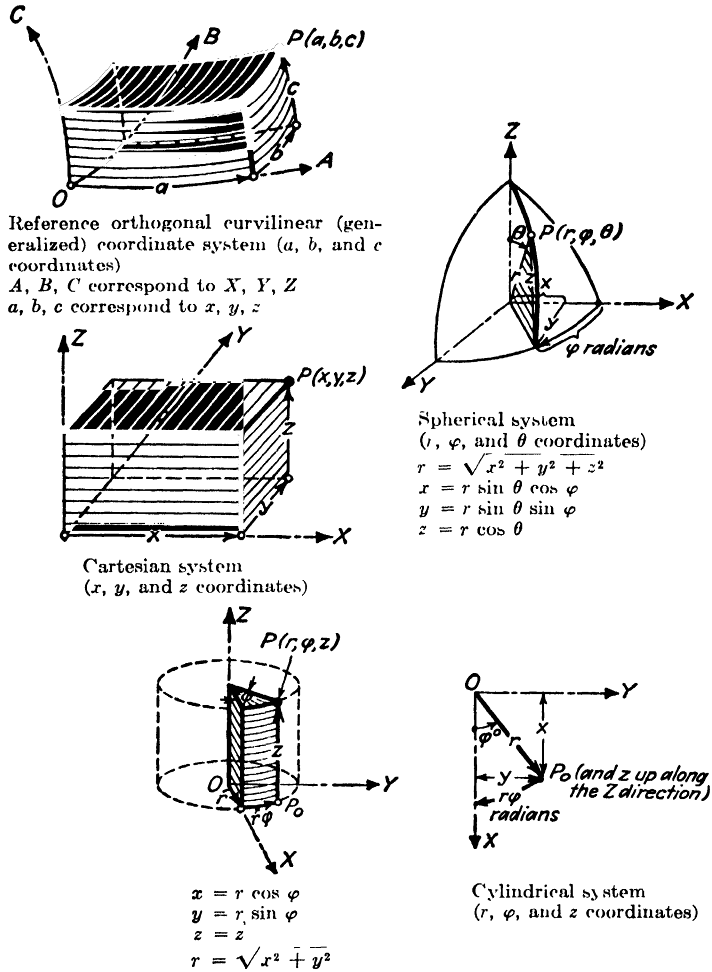
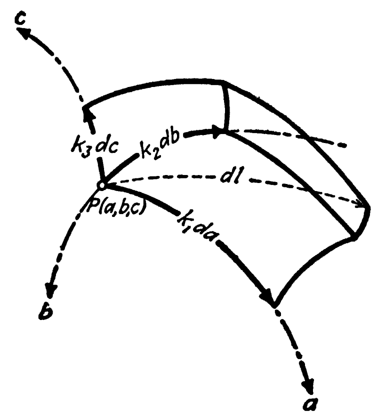


FIG. 26. Generalized coordinate system in comparison with the Cartesian, cylindrical, and spherical coordinate systems.

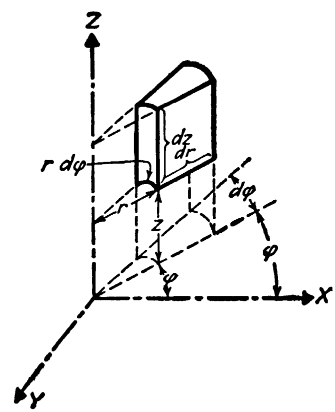
for curved surfaces. Therefore, we have to multiply the curved incremental edge da by a factor k_1 in order to have the equivalent of a straight edge, which is perpendicular to the equivalent straight edges $k_2\,db$ and $k_3\,dc$, if db and dc are the corresponding curved elementary edges. The values for k_1, k_2, k_3, a, b, and c can be found by inspection of the edges

of the elementary volumes shown in Fig. 27. On account of the orthogonal condition for all cases illustrated, for the infinitesimal line element dl of the Cartesian system compared with the reference system, we have

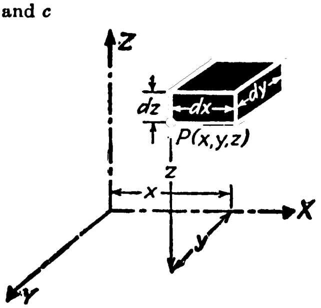
$$\left.\begin{array}{ll}(dl)^2 = (k_1\,da)^2 + (k_2\,db)^2 + (k_3\,dc)^2 & \text{for reference coordinates}\\ (dl)^2 = (dx)^2 \quad + (dy)^2 \quad + (dz)^2 & \text{for Cartesian coordinates}\end{array}\right\} \quad (132)$$



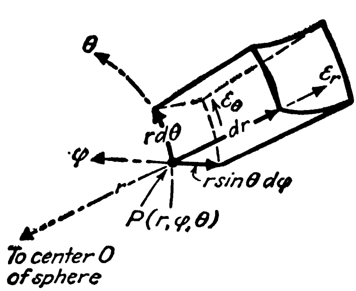
Orthogonal *reference volume* $d\tau = k_1k_2k_3\,da\,db\,dc$
Cross diagonal $dl = \sqrt{(k_1\,da)^2 + (k_2\,db)^2 + (k_3\,dc)^2}$
da, db, and dc are increments of a, b, and c



Cylindrical coordinates:
Since $k_1 = 1$, $k_2 = r$, $k_3 = 1$, and $a = r$, $b = \varphi$, $c = z$, we have $d\tau = r\,d\varphi\,dr\,dz$



Cartesian coordinates:
Since $k_1 = 1 = k_2 = k_3$ and $a = x$, $b = y$, $c = z$, we have $d\tau = dx\,dy\,dz$



Spherical coordinates:
Since $k_1 = 1$, $k_2 = r$, $k_3 = r\sin\theta$, and $a = r$, $b = \theta$, $c = \varphi$, we have
$d\tau = dr\;r\,d\theta \cdot r\sin\theta\,d\varphi$
$= r^2\sin\theta\,dr\,d\varphi\,d\theta$

FIG. 27. Details on the elementary volume dimensions for the generalized and customary volume elements.

since dl is the diagonal of respective systems. Comparing factors, we note that for Cartesian coordinates, $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $da = dx$, $db = dy$, and $dc = dz$ or $a = x$, $b = y$, and $c = z$.

For cylindrical and spherical coordinates, we have similarly

$$\left.\begin{array}{ll} (dl)^2 = (dr)^2 + (r\,d\varphi)^2 + (dz)^2 & \text{cylindrical coordinates} \\ (dl)^2 = (k_1\,da)^2 + (k_2\,db)^2 + (k_3\,dc)^2 & \text{reference coordinates} \\ (dl)^2 = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\cdot d\varphi)^2 & \text{spherical coordinates} \end{array}\right\} \quad (133)$$

Comparing factors as before against the reference expression, we have

$$\left.\begin{array}{l} k_1 = 1; \quad k_2 = r; \quad k_3 = 1; \quad a = r; \quad b = \varphi; \quad c = z \quad \text{for cylindrical coordinates} \\ k_1 = 1; \quad k_2 = r; \quad k_3 = r\sin\theta; \quad a = r; \quad b = \theta; \quad c = \varphi \quad \text{for spherical coordinates} \end{array}\right\} \quad (134)$$

The derivation of Maxwell's equations in reference coordinates is the same as for Cartesian coordinates[1] and results in the expressions

$$\left.\begin{array}{l} \dfrac{1}{k_2k_3}\left[\dfrac{\partial(k_3\mathcal{E}_c)}{\partial b} - \dfrac{\partial(k_2\mathcal{E}_b)}{\partial c}\right] = -\dfrac{\partial(\mu H_a)}{\partial t} \\ \dfrac{1}{k_1k_3}\left[\dfrac{\partial(k_1\mathcal{E}_a)}{\partial c} - \dfrac{\partial(k_3\mathcal{E}_c)}{\partial a}\right] = -\dfrac{\partial(\mu H_b)}{\partial t} \\ \dfrac{1}{k_1k_2}\left[\dfrac{\partial(k_2\mathcal{E}_b)}{\partial a} - \dfrac{\partial(k_1\mathcal{E}_a)}{\partial b}\right] = -\dfrac{\partial(\mu H_c)}{\partial t} \end{array}\right\} \text{for the curl } \mathcal{E} \text{ components} \quad (135)$$

$$\left.\begin{array}{l} \dfrac{1}{k_2k_3}\left[\dfrac{\partial(k_3H_c)}{\partial b} - \dfrac{\partial(k_2H_b)}{dc}\right] = \left[\sigma\mathcal{E}_a + \dfrac{\partial(\kappa\mathcal{E}_a)}{\partial t}\right] \\ \dfrac{1}{k_1k_3}\left[\dfrac{\partial(k_1H_a)}{\partial c} - \dfrac{\partial(k_3H_c)}{\partial a}\right] = \left[\sigma\mathcal{E}_b + \dfrac{\partial(\kappa\mathcal{E}_b)}{\partial t}\right] \\ \dfrac{1}{k_1k_2}\left[\dfrac{\partial(k_2H_b)}{\partial a} - \dfrac{\partial(k_1H_a)}{\partial b}\right] = \left[\sigma\mathcal{E}_c + \dfrac{\partial(\kappa\mathcal{E}_c)}{\partial t}\right] \end{array}\right\} \text{for the curl } H \text{ components} \quad (136)$$

For Cartesian coordinates (Fig. 27), we have $k_1 = k_2 = k_3 = 1$ and $a = x$, $b = y$, and $c = z$. Therefore, from (135) and (136), we find the expressions

[1] HUND, A., "Phenomena in High-frequency Systems," pp. 324–341, McGraw-Hill Book Company, Inc., New York, 1936. Consult also pp. 214–218 of this text.

for any variation	for harmonic variation of frequency $\omega/6.28$ in cycles/sec	
$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\frac{\partial(\mu H_x)}{\partial t}$	$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -j\omega\mu H_x$	volts/m² (voltage density)
$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\frac{\partial(\mu H_y)}{\partial t}$	$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -j\omega\mu H_y$	
$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\frac{\partial(\mu H_z)}{\partial t}$	$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -j\omega\mu H_z$	
$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \sigma\mathcal{E}_x + \frac{\partial(\kappa\mathcal{E}_x)}{\partial t}$	$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + j\omega\kappa)\mathcal{E}_x$	amp/m² (current density)
$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \sigma\mathcal{E}_y + \frac{\partial(\kappa\mathcal{E}_y)}{\partial t}$	$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = (\sigma + j\omega\kappa)\mathcal{E}_y$	
$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \sigma\mathcal{E}_z + \frac{\partial(\kappa\mathcal{E}_z)}{\partial t}$	$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (\sigma + j\omega\kappa)\mathcal{E}_z$	

(137)

For *cylindrical* coordinates, Fig. 27 shows that in (135) and (136) we must make the substitutions $k_1 = k_3 = 1$, $k_2 = a = r$, $b = \varphi$ and $c = x$, yielding the expressions

for any variations	for a harmonic variation
$\frac{\partial \mathcal{E}_z}{r\,\partial \varphi} - \frac{\partial \mathcal{E}_\varphi}{\partial z} = -\frac{\partial(\mu H_r)}{\partial t}$	$\frac{\partial \mathcal{E}_z}{\partial \varphi} - r\frac{\partial \mathcal{E}_\varphi}{\partial z} = -j\omega\mu r H_r$
$\frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} = -\frac{\partial(\mu H_\varphi)}{\partial t}$	$\frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} = -j\omega\mu H_\varphi$
$\frac{\partial(r\mathcal{E}_\varphi)}{\partial r} - \frac{\partial \mathcal{E}_r}{\partial \varphi} = -r\frac{\partial(\mu H_z)}{\partial t}$	$\frac{\partial(r\mathcal{E}_\varphi)}{\partial r} - \frac{\partial \mathcal{E}_r}{\partial \varphi} = -j\omega\mu r H_z$
$\frac{\partial H_z}{r\,\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = \sigma\mathcal{E}_r + \frac{\partial(\kappa\mathcal{E}_r)}{\partial t}$	$\frac{\partial H_z}{\partial \varphi} - r\frac{\partial H_\varphi}{\partial z} = r(\sigma + j\omega\kappa)\mathcal{E}_r$
$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \sigma\mathcal{E}_\varphi + \frac{\partial(\kappa\mathcal{E}_\varphi)}{\partial t}$	$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = (\sigma + j\omega\kappa)\mathcal{E}_\varphi$
$\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} = r\left[\sigma\mathcal{E}_z + \frac{\partial(\kappa\mathcal{E}_z)}{\partial t}\right]$	$\frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} = r(\sigma + j\omega\kappa)\mathcal{E}_z$

(138)

For *spherical* coordinates, Fig. 27 gives the values $k_1 = 1$, $k_2 = a = r$, $k_3 = r \sin\theta$, $b = \theta$, and $c = \varphi$, and (135) and (136) yield with such substitutions

for any variation	for a harmonic variation
$\frac{1}{r\sin\theta}\left[\frac{\partial(\mathcal{E}_\varphi \sin\theta)}{\partial\theta} - \frac{\partial\mathcal{E}_\theta}{\partial\varphi}\right] = -\frac{\partial(\mu H_r)}{\partial t}$	$\frac{\partial(\mathcal{E}_\varphi \sin\theta)}{\partial\theta} - \frac{\partial\mathcal{E}_\theta}{\partial\varphi} = -j\omega\mu r\, H_r \sin\theta$
$\frac{\partial\mathcal{E}_r}{\partial\varphi} - \sin\theta\,\frac{\partial(r\mathcal{E}_\varphi)}{\partial r} = -r\sin\theta\,\frac{\partial(\mu H_\theta)}{\partial t}$	$\frac{\partial\mathcal{E}_r}{\partial\varphi} - \sin\theta\,\frac{\partial(r\mathcal{E}_\varphi)}{\partial r} = -j\omega\mu r\, H_\theta \sin\theta$
$\frac{\partial(r\mathcal{E}_\theta)}{\partial r} - \frac{\partial\mathcal{E}_r}{\partial\theta} = -r\,\frac{\partial(\mu H_\varphi)}{\partial t}$	$\frac{\partial(r\mathcal{E}_\theta)}{\partial r} - \frac{\partial\mathcal{E}_r}{\partial\theta} = -j\omega\mu r H_\varphi$
$\frac{\partial(H_\varphi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\varphi} = r\sin\theta\left[\sigma\mathcal{E}_r + \frac{\partial(\kappa\mathcal{E}_r)}{\partial t}\right]$	$\frac{\partial(H_\varphi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\varphi} = (\sigma + j\omega\kappa)r\,\mathcal{E}_r \sin\theta$
$\frac{\partial H_r}{\partial\varphi} - \sin\theta\,\frac{\partial(rH_\varphi)}{\partial r} = r\sin\theta\left[\sigma\mathcal{E}_\theta + \frac{\partial(\kappa\mathcal{E}_\theta)}{\partial t}\right]$	$\frac{\partial H_r}{\partial\varphi} - \sin\theta\,\frac{\partial(rH_\varphi)}{\partial r} = (\sigma + j\omega\kappa)r\,\mathcal{E}_\theta \sin\theta$
$\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta} = r\left[\sigma\mathcal{E}_\varphi + \frac{\partial(\kappa\mathcal{E}_\varphi)}{\partial t}\right]$	$\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta} = (\sigma + j\omega\kappa)r\mathcal{E}_\varphi$

(139)

In many practical applications, we do not have to use all the components given for respective coordinate systems.

Considering first the case of an electromagnetic field that is *independent of* the *Z coordinate*, for a wave medium free of sources, we have $\partial/\partial z = 0$ and (137), (138), and (139) separate into two independent groups, since terms with $\partial/\partial z$ vanish. We find from (137) and (138) for a harmonic wave excitation

$$\left.\begin{aligned}&\frac{\partial \mathcal{E}_z}{\partial y} = -j\omega\mu H_x; \qquad \frac{\partial \mathcal{E}_z}{\partial x} = j\omega\mu H_y;\\ &\qquad (\sigma + j\omega\kappa)\mathcal{E}_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \qquad \text{for Cartesian coordinates}\\ &\frac{\partial \mathcal{E}_z}{\partial \varphi} = -j\omega\mu r H_r; \qquad \frac{\partial \mathcal{E}_z}{\partial r} = j\omega\mu H_\varphi;\\ &\quad r(\sigma + j\omega\kappa)\mathcal{E}_z = \frac{\partial(rH_\varphi)}{\partial r} - \frac{\partial H_r}{\partial \varphi} \qquad \text{for cylindrical coordinates}\end{aligned}\right\} \quad (140)$$

and

$$\left.\begin{aligned}&\frac{\partial H_z}{\partial y} = (\sigma + j\omega\kappa)\mathcal{E}_x; \qquad \frac{\partial H_z}{\partial x} = -(\sigma + j\omega\kappa)\mathcal{E}_y;\\ &\qquad j\omega\mu H_z = \frac{\partial \mathcal{E}_x}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial x} \qquad \text{for Cartesian coordinates}\\ &\frac{\partial H_z}{\partial \varphi} = r(\sigma + j\omega\kappa)\mathcal{E}_r; \qquad \frac{\partial H_z}{\partial r} = -(\sigma + j\omega\kappa)\mathcal{E}_\varphi;\\ &\qquad j\omega\mu r H_z = \frac{\partial \mathcal{E}_r}{\partial \varphi} - \frac{\partial(r\mathcal{E}_\varphi)}{\partial r} \qquad \text{for cylindrical coordinates}\end{aligned}\right\} \quad (141)$$

For a coaxial line, we have *circular symmetry.* The resultant field is, therefore, *independent of* the cylinder *coordinate* φ. This introduces the simplification $\partial/\partial\varphi = 0$. The angle φ is the angle in the XY plane and planes parallel thereto with the $+X$ direction. We then find from (138) and (139) the group

$$\left.\begin{aligned}&\frac{\partial \mathcal{E}_\varphi}{\partial z} = j\omega\mu H_r; \qquad \frac{\partial(r\mathcal{E}_\varphi)}{\partial r} = -j\omega\mu r H_z;\\ &\qquad (\sigma + j\omega\kappa)\mathcal{E}_\varphi = \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \qquad \text{cylindrical coordinates}\\ &\frac{\partial(\mathcal{E}_\varphi \sin\theta)}{\partial \theta} = -j\omega\mu r\, H_r \sin\theta; \qquad \frac{\partial(r\mathcal{E}_\varphi)}{\partial r} = j\omega\mu r H_\theta;\\ &\qquad r(\sigma + j\omega\kappa)\mathcal{E}_\varphi = \frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \qquad \text{spherical coordinates}\end{aligned}\right\} \quad (142)$$

and the segregated group

$$\left.\begin{aligned}&\frac{\partial H_\varphi}{\partial z} = -(\sigma + j\omega\kappa)\mathcal{E}_r; \qquad \frac{\partial(rH_\varphi)}{\partial r} = r(\sigma + j\omega\kappa)\mathcal{E}_z;\\ &\qquad j\omega\mu H_\varphi = \frac{\partial \mathcal{E}_z}{\partial r} - \frac{\partial \mathcal{E}_r}{\partial z} \qquad \text{cylindrical coordinates}\\ &\frac{\partial(H_\varphi \sin\theta)}{\partial \theta} = r\sin\theta(\sigma + j\omega\kappa)\mathcal{E}_r; \qquad \frac{\partial(rH_\varphi)}{\partial r} = -r(\sigma + j\omega\kappa)\mathcal{E}_\theta;\\ &\qquad j\omega\mu r H_\varphi = \frac{\partial \mathcal{E}_r}{\partial \theta} - \frac{\partial(r\mathcal{E}_\theta)}{\partial r} \qquad \text{spherical coordinates}\end{aligned}\right\} \quad (143)$$

Now, considering the transverse electromagnetic power transfer (TEM waves) along the coaxial line (along the Z axis) that is possible from the lowest frequency on upward, we also have $\mathcal{E}_z = 0$. The cylindrical coordinate system applies for a coaxial line, and the first of Eqs (143) simplifies for $\mathcal{E}_z = 0$ to

$$\frac{\partial H_\varphi}{\partial z} = -(\sigma + j\omega\kappa)\mathcal{E}_r; \qquad \frac{\partial \mathcal{E}_r}{\partial z} = -j\omega\mu H_\varphi \tag{144}$$

since $\partial(rH_\varphi)/\partial r$ vanishes for $\mathcal{E}_z = 0$ for which no displacement currents along the Z axis (axis of coax) are possible, in addition to the conduction current along the inside surfaces of the concentric line. The latter current causes only circular magnetomotive forces along the Z axis (distributed along this axis). These forces do not depend on the distance from the common axis of the concentric line.

There are also engineering applications for which the electromagnetic field is independent of two coordinates, for instance the Cartesian X and Y coordinates, and Maxwell's equations of (137) simplify because of $\partial/\partial x = 0$ and $\partial/\partial y = 0$ to

$$\left.\begin{aligned} &\frac{\partial \mathcal{E}_x}{\partial z} = \frac{d\mathcal{E}_x}{dz} = -j\omega\mu H_y; \qquad \frac{d\mathcal{E}_y}{dz} = j\omega\mu H_x; \qquad \mathcal{E}_z = 0 \\ &\frac{dH_x}{dz} = (\sigma + j\omega\kappa)\mathcal{E}_y; \qquad \frac{dH_y}{dz} = -(\sigma + j\omega\kappa)\mathcal{E}_x; \qquad H_z = 0 \end{aligned}\right\} \tag{145}$$

Applications of (140) to (145) appear throughout this text

Moreover, for space points of the electromagnetic field with continuous electromagnetic properties, Maxwell's equations read

$$\left.\begin{aligned} \nabla \times \mathcal{E} + \frac{\partial(\mu H)}{\partial t} &= 0 \\ \nabla \times H - \frac{\partial(\kappa\mathcal{E})}{\partial t} &= J \end{aligned}\right\} \text{where } \nabla \times \mathcal{E} \equiv \text{curl } \mathcal{E} \text{ and } \nabla \times H \equiv \text{curl } H$$

for the current density J, and we have in the orthogonal reference coordinate system

$$\left.\begin{aligned} \frac{\partial(k_2k_3\mu H_a)}{\partial a} + \frac{\partial(k_1k_3\mu H_b)}{\partial b} + \frac{\partial(k_1k_2\mu H_c)}{\partial c} &= 0 \\ \frac{\partial(k_2k_3\kappa\mathcal{E}_a)}{\partial a} + \frac{\partial(k_1k_3\kappa\mathcal{E}_b)}{\partial b} + \frac{\partial(k_1k_2\kappa\mathcal{E}_c)}{\partial c} &= k_1k_2k_3\rho \end{aligned}\right\} \tag{146}$$

for the average charge ρ in coulombs per cubic meter of the medium through which electromagnetic waves pass.

32. Notes on Magnetic and Electric Shells in Connection with Magnetic and Electric Dipoles. We have electric as well as magnetic currents

in the Maxwellian main equations (1), since we deal with variations of $\mathcal{E}$ and H. We must distinguish between electric polarization and magnetization. We have then electric and magnetic current moments. A Hertzian doublet has an electric charge moment $m = q\,dl$ meter · coulombs, since it consists of equal $\pm q$ charges in coulombs an infinitesimal distance dl apart. These charges are $\epsilon^{j\omega t}$ variations. An infinitesimally short linear radiator $i\,dl$ has an electric current moment $m_e = I\epsilon^{j\omega t}\,dl$ meter · amperes. Small current loops act like magnetic dipoles with a magnetic moment. The moment is equal to the volume integral of its magnetization.

We have also electric and magnetic dipole effects when dealing with atoms, since, according to present-day atomic theory, electric charges contained in matter are in random motion. Such motions of no regularity can be affected by electromagnetic fields, whether we deal with a medium with an abundance of freely moving electrons as in conductors or with constraint electrons as in dielectrics. Constraint electrons are checked by strong forces and known as "closed-shell negative elementary charges" or "bound charges," while in conductors we deal with loose electrons or free charges, known as "valence electrons."

For constraint charges, *i.e.*, bound charges in perfect dielectrics, we still have random motions, except that a particular number of electrons is tied in with a particular atom whether only in random motions or also acted upon by an electromagnetic field. This is also true, whether only a magnetomotive force or just an electromotive force acts as a strong external force. Therefore, with dielectrics having constraint electrons in closed shells only, we have random motions as in conductors, but the equal positive charge of the nucleus requires these electrons, even though in motion, to remain in the orbits of the atom. The interatomic forces are strong, and an external field force can cause only *distortion* of the random motions but *not* an electronic *escape* from the atom. In conductors, electrons may escape from the atom owing to an external force, without other electrons having to take the place of the escaping negative elementary charges, since the external force causes different potentials at places from which electrons escape to places where additional electrons arrive.

It is the *distortion* of the random motions that concerns us here. The disturbance or the distortion of random motions can be a flow (or a drift), as in a conductor where electrons constitute a current flow along the conductor if, for instance, an electromotive force causes such a flow of valence electrons. This is also the case in a thermionic vacuum tube. Here the electrons liberated at the hot cathode are pulled toward the anode as a convection current when an electric field between the plate and the cathode renders the plate potential positive with respect to the hot cathode, which evaporates electrons. It can also be a two-way drift or flow as in a tube with positive ionization present. This would be drifts of negative and positive charges moving in opposite directions owing to an external force. Instead of a drift, we may also have elementary negative charges cycling around the nucleus. This can also be caused by an external force due to a forced orientation of the axis through the nucleus about which electronic circulations occur. Hence, this is another form of random motion distortion. If we have only one atom and a steady electron rotation about its nucleus, this rotation of a negative charge acts like a magnetizing current (like a current flowing around a loop). The convection magnetizing current due to the circulating electronic charge causes, in turn, along the axis of this loop, according to

the corkscrew rule, a magnetic flux, which, according to Eqs. (1) may be called a "magnetic current," expressed in volts. Therefore, if an external force distorts the random motions in atoms in such a way that the axes of the various atoms are no longer oriented at random but are parallel with each other, we have, as it were, magnetic current effects along these axes, while the electronic circulations or rotations about respective parallel axes cause elementary magnetizing currents.

Such distortions may cause minute separations between the electronic charges and the compensating positive charge of the proton of the atom due to the action of the external force. If Δl denotes the separation and q the equal negative charge, $q\,\Delta l$ denotes the atomic dipole moment in meter · coulombs. If the effects of all such doublets are added for a definite volume element and divided by the volume, we obtain a vector, acting at the symmetry center of the volume in a definite direction and giving a measure of the *electric polarization.*

In a similar way, we can relate the electron spins about atom axes to elementary magnets. If several of such magnets are parallel and oriented by an external force, they give a measure of the magnetization. Each individual electron circulation so oriented then produces a magnetic dipole effect. This means that we have elementary magnets, when external forces change the course of the electronic charges of an atom from random orbits to circulations about a common axis. This axis[1] is the direction along which the so-called "magnetic current" flows (measured in volts).

If we deal with small electric dipoles and small current loops we may have a current distribution in a small loop so that only the magnetic dipole moment plays a part.[2] Under such conditions, the current loop causes at neighboring points, a field like a small magnetic dipole. Figure 30 shows that a small *horizontal* loop (*magnetic* dipole) has field effects similar to a small *vertical electric* dipole. The magnetic dipole has then H_θ and H fields in place of the $\mathcal{E}_\theta$ and $\mathcal{E}_r$ fields, which holds for the electric

[1] A clear exposition of such magnetization actions is given in R. W. P. King, "Electromagnetic Engineering," Vol. I, Fundamentals, pp. 47–52, McGraw-Hill Book Company, Inc., New York, 1945, and M. Mason and W. Weaver, "The Electromagnetic Field," University of Chicago Press, Chicago, 1929. J. A. Stratton, *op. cit.*, pp. 233–237, solves the problem by means of the magnetic vector potential.

[2] The retarded Hertz integral is evaluated by a series expansion, as is shown in detail in the book by J. A. Stratton, "Electromagnetic Theory," pp. 431–434, McGraw-Hill Book Company, Inc., New York, 1941. The entire field may then be expressed as the sum $\sum_{n=0}^{n=\infty} \prod^{(n)} (x, t)$ if the wavelength of the harmonic electromagnetic field is many times greater than the largest dimension of the region occupied by a current ($\beta\rho \ll 1$, *i.e.*, the phase constant $2\pi/\lambda$ multiplied by the largest dimension). Electric and magnetic multipoles at the origin then satisfy the series. The first term of the sum is then a partial field $\prod^{(0)}$, due to an electric dipole moment m; the second term yields a partial field $\prod^{(1)}$, due to a magnetic dipole of moment m_m and an electric quadrupole. The moments of a quadrupole of a charge distribution are given by the components of a tensor. There is also a good description of the magnetic dipole radiator in J. C. Slater, "Microwave Transmission," pp. 232–234, McGraw-Hill Book Company, Inc., New York, 1942.

dipole. The magnetic dipole has also an $\mathcal{E}_\varphi$ in place of an H_φ field. The field of the small loop radiator starts out with the simplest TE wave, while the Hertzian doublet radiator starts out a field with the simplest TM wave. Respective dipoles are, therefore, complementary as slot and metal radiators are complementary. For the magnetic dipole, the impedance is low and small loops can be used efficiently as electromagnetic field pickups and as radiators *without* the tuning that would be needed for small electric dipoles.

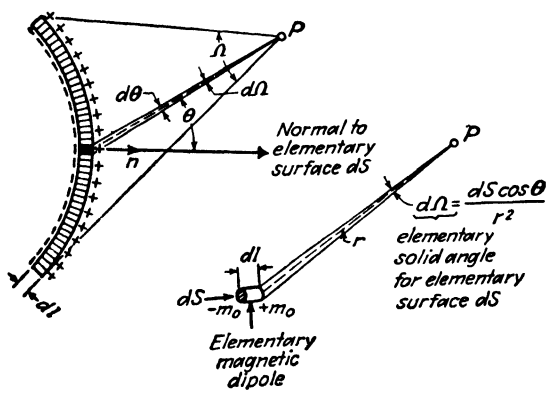


FIG. 28. Magnetic shell (a double layer of positive and negative magnetic masses m_0) Ω represents the solid angle of the total surface S sq m.

The magnetization M of a certain space region that carries a current can be expressed by its equivalence, the magnetic moment m_m per unit volume. If $d\tau$ is a space element, expressed in cubic meters, then $M = dm_m/d\tau$ is in amperes per meter. The quantity M is then a vector volume density of magnetization, and we have a continuous vector point function throughout a certain finite volume τ. We deal then with an *axial* vector for every point of the space τ. The direction of the vector is along the axis of magnetization in the positive direction of the right-hand corkscrew, while the absolute amount of the vector expresses the average rotation of electric charge about the axis in $d\tau$ near the space point. The equivalence of a magnetic dipole also upholds the concept of amperian current whirls (circulating currents), since a magnetized volume is, in effect, the same as a current distribution of density $J = \text{curl } M$ (also written as $\nabla \times M$).

In equivalence, a linear current element $i\,dl$ meter · amperes, for the instantaneous current value $i = I\epsilon^{j\omega t}$, acts like a *magnetized shell.* Such a shell is made up of a great number of elementary magnets, which can be imagined as placed sidewise, where each elementary magnet has a length dl expressed in meters. Such a shell is a thin sheet of magnetic

matter of opposite polarity on respective faces as is indicated in Fig. 28, where P is a space point. The solid angle $d\Omega$ of an elementary portion of a shell, with elementary area dS square meters, is as given in this figure. For an elementary *electric* dipole of length dl and charges $\pm q$ at respective ends, the relation $m = q\,dl$ meter · coulombs stands for the electric doublet or charge moment. Then similarly, we have for an elementary magnetic dipole, the magnetic moment $m_m = m_0\,dl$, where in each case the moment is along the dl axis. The moment toward the field point P is $m_m \cos\theta$. In Fig. 28, this field point is a distance of r meters away from the symmetry center of the elementary magnetic dipole.

Just as for an electric dipole, the magnetic potential dU at field point P, due to an elementary magnetic dipole, is for the permeability μ henrys per meter of the medium,

$$dU = \frac{m_m \cos\theta}{4\pi\mu r^2} \qquad \text{amp}$$

if $r \gg dl$. If ψ is the intensity of the shell, *i.e.*, the magnetic moment per unit area of respective shell surfaces S in square meters, the magnetic moment of the elementary magnetic dipole is $m_m = \psi\,dS$ and

$$dU = \frac{\psi\,dS}{4\pi\mu r^2}\cos\theta = \psi\,d\Omega \qquad \text{amp}$$

The resultant potential due to the entire shell is then

$$U = \frac{1}{4\pi\mu}\int_\Omega \psi\,d\Omega \qquad \text{amp}$$

If σ_m denotes the magnetic surface density, for an elementary magnetic dipole, we have the magnetic matter $m_0 = \sigma_m\,dS$ with a corresponding moment $m_m = \sigma_m\,dS \cdot dl = \psi\,dS$ making $\psi = \sigma_m\,dl$.

Any path carrying a current I amperes can be broken up as in Fig. 29 into infinitely small current meshes. Similar to the description given in connection with Fig. 25, the magnetic field effect due to the mosaic of elementary current circulations, with the current I in the same direction and of the same magnitude, is equivalent to the magnetic field of the single outside I circulation surrounding all these meshes (currents along the common boundaries due to adjacent circulations cancel each other, except at the very outer boundary). The outside boundary need not be in a plane, and yet each *elementary* mesh circulation forms a current loop in its own plane. The magnetic effect of a circulating elementary mesh current therefore is known. It is equivalent to that of a *magnetic shell* with an intensity proportional to current I. By addition

of the elementary magnetic effects, we find the same magnetic effect as caused by a single magnetic shell, coinciding with the surface of which the outside contour is the boundary current circuit. This boundary is known as the "equivalent magnetic shell." Hence, a current flowing in any closed circuit produces the same magnetic field as the equivalent magnetic shell with the current path as its boundary and a uniform strength proportional to the current flow.

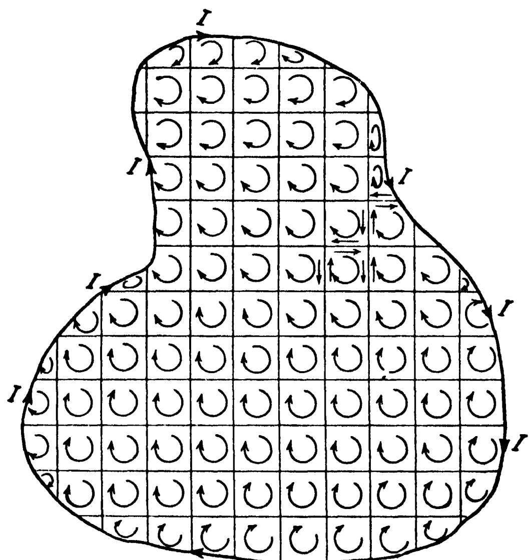


FIG. 29. Shows that all the current circulations result in an outside unidirectional instantaneous current flow I similar to the conditions shown in connection with Fig. 25. Outside contour or I path need not be in a plane.

The magnetic moment of a resultant elementary linear current, for each elementary circulation, is $dm_m = I \cdot \eta \, dS$, where η denotes the unit vector normal to the surface element dS of the plane current loop forming the elementary mesh. We note that only the area and current value play a part, but *not* the *shape* of the loop. An ordinary electromagnet and keeper act like a magnetic shell. A small loop can be used to map out and explore magnetic field distributions as a small electric dipole is used to explore an electric field distribution.

The total radiation effect of linear antennas, can be found by dividing the entire antenna length into $i\,dl$ current elements, or equivalent elementary electric dipoles, (which have radiation characteristics as given in Sec. 20) and summing up all field effects of the radiator elements with respect to magnitude and phase. Hence, a closed loop can also be divided

into infinitesimally small current loops (meshes) which act as elementary *magnetic* dipoles and their field effects summed up with respect to magnitude and relative phase.

The magnetic induction B due to an elementary loop of area dS is identical with the electric field caused by an electric dipole of moment $I\,dS$, where I is the elementary loop current. Hence, the magnetic induction caused by the finite current loop must be the same as the electric field due to a uniform electric double layer of strength I, whose boundary coincides with the actual finite closed current path.

If we have the magnetic H lines of force due to an electric circuit, we pass over the same trace without meeting a magnetic discontinuity. This corresponds to an *electric* double layer. The circulating currents are, in one respect, not the same as customary currents, since there is no power dissipation, and therefore we can take the amperian circulations as though they were flowing in *super* conductors.

The magnetic moment of an elementary curren circulation is $dm_m = \mu I\,dS$ if μ is in henrys per meter, the current I in amperes, and the area dS of the elementary loop is in square meters. For the magnetic flux density B in webers per square meter, we then have the energy associated with an elementary current circulation as dm_mB/μ joules, since $dm_mB/\mu = dm_mH = \mu I\,dS \cdot H$, where μ is in henrys per meter or (volt · sec)/(amp · m); I in amperes, dS in square meters, and H in amperes per meter, rendering a dimension of volt · ampere · seconds, or joules.

With respect to an electric double layer, the positive and negative magnetic matters of Fig. 28 then refer to electric surface charges $\pm q$ coulombs per square meter. As for the magnetic shell, we can divide the double layer into elementary Hertzian doublets of electric charge moments $dm = q_0\,dl\,dS$. Calling $\varphi = q_0\,dl$ the charge moment per unit area, we find the electric potentials

$$dV = \frac{\varphi \cos\theta}{4\pi\kappa r^2}\,dS \qquad \text{volts} \qquad \text{for an elementary electric doublet}$$

$$V = \frac{1}{4\pi\kappa}\int\!\!\int_S \frac{\varphi\cos\theta}{r^2}\,dS = \frac{1}{4\pi\kappa}\int \varphi\,d\Omega \qquad \text{volts} \qquad \text{for entire electric double layer}$$

if κ is the dielectric constant in farads per meter of the medium in which the field point P of Fig. 28 is located and Ω is the solid angle indicated in Fig. 28.

Inasmuch as between the double layers we have the electric field intensity $-q_0/\kappa$, we have for the potential across the electric double layer $V' = q_0\,dl/\kappa$ volts, yielding $V = (\frac{1}{4}/\pi)\int V'\,d\Omega$ volts, just as we have for the magnetic potential $U' = m_0\,dl/\mu$ across the magnetic double layer, yielding $U = (\frac{1}{4}/\pi)\int U'\,d\Omega$ amperes.

Inasmuch as an infinitesimally small loop area dS in square meters carrying a current of I amperes, corresponds to an elementary magnetic double layer of moment $dm_m = \mu I\, dS$, for a harmonic current variation $i = I\epsilon^{j\omega t}$, we have, in effect, a magnetic current element with a moment $\mu\, dS(\partial i/\partial t) = j\omega\mu I\, dS$. In Sec. 33, the formulas are given for $\mathcal{E}_\varphi$ volts per meter and H_θ and H_r in amperes per meter, at any distance from an elementary loop radiator, showing that at a great distance from an elementary loop for $\beta = 2\pi/\lambda$, $Z_0 = \sqrt{\mu/\kappa} \cong 377$ ohms (in the case of atmosphere), we have essentially only H_θ and $\mathcal{E}_\varphi$ field associations with the loop in the XY plane and $\mathcal{E}_\varphi/H_\theta = 377\,\Omega$. For λ meters $= 300/f$ megacycles per second

$$H_\theta = -\frac{\pi\, dSI\epsilon^{-j\beta r}\sin\theta}{r\lambda^2} \quad \text{amp/m} \tag{147}$$

where r is the distance in meters from the field point P to the elementary current circulation. This relation holds for the radiation region. Next to the loop radiator (close to it), we have an inverse cube law of the distance r for both the H_θ and H_r components and an inverse square law for the $\mathcal{E}_\varphi$ electric field component. These formulas hold for unobstructed space and, therefore, require modification when applied to operation inside a wave guide.

The radiated power of an elementary current circulation, *i.e.*, an elementary magnetic dipole, is obtained in the same way as for an elementary electric dipole (a Hertzian doublet), by forming the complex Poynting radiation vector and utilizing the useful field value. The integration is then taken over a spherical surface whose radius is of no concern (as far as its magnitude is concerned), since the field flow does not depend upon it, because the same energy flux streams outward and across all concentric spherical surfaces in the case of a nondissipative medium ($\sigma = 0$). The total power flow in watts is then, for a field impedance $Z_0 = \sqrt{\mu/\kappa}$ ohms and a velocity v of propagation equal to $1/\sqrt{\mu\kappa}$ meters per second,

$$\left.\begin{aligned} P &= \frac{\beta^4}{12\pi} Z_0 m_m^2 && \text{for the magnetic elementary dipole} \\ P &= \frac{\mu\omega^4}{12\pi v} m^2 && \text{for the electric elementary dipole} \end{aligned}\right\} \begin{array}{l}(m \text{ and } m_m \text{ are} \\ \text{maximum in-} \\ \text{stantaneous} \\ \text{moment values)}\end{array} \tag{148}$$

where $\omega/6.28$ is in cycles per second, $\beta = \omega/v$ in circular radians per meter, and $2\pi/\beta$ equal to λ in meters. The quantity m then represents the Hertzian doublet moment, which is[1] also $m = \int i\, dl\, dt = \int I\, dl \cdot \epsilon^{j\omega t}\, dt =$

[1] Details on pp. 161, 169, 241–242, and 252–253.

$i\,dl/(j\omega) = -jm_e/\omega$ for the elementary current moment m_e of an $i\,dl$ radiator. For customary atmosphere, we have $\mu \cong \mu_0 = 1.257 \times 10^{-6}$ henry/m, $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m, $v \cong c = 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec, $Z_0 = \sqrt{\mu/\kappa} \cong \sqrt{\mu_0/\kappa_0} = 377$ ohms. For f in megacycles per second we find then for the total power radiated from elementary dipoles

$$\begin{aligned} P^{(\text{watts})} &= 194 \times 10^{-8} f^4 m_m^2 \quad \text{for magnetic elementary dipole} \\ P^{(\text{watts})} &= 173 \times 10^{9} f^4 m^2 \quad \text{for electric elementary dipole} \end{aligned} \tag{149}$$

Assume a circular current loop of radius ρ meters, so that the enclosed area $S = 3.14\rho^2$ square meters. Also assume that the current flow $I\epsilon^{j\omega t}$ is essentially uniform along any point of the circular conductor. For such a condition the magnetic moment is $m_m = SI\epsilon^{j\omega t}$ in equivalence, since the current multiplied by the enclosed area S is the strength of the magnetic shell. Using this value in Eq. (149), we find for the total radiated power of a magnetic dipole (a small current loop)

$$P_m = 1.54 \times 10^5 I^2 \frac{\rho^4}{\lambda^4} = 1.91 \times 10^{-5} I^2 f^4 \rho^4 \qquad \text{watts} \tag{150}$$

Since the effective current $I_e = I/\sqrt{2}$ amp and for the radiation resistance R_r in ohms, we have also $P_m = R_r I_e^2 = R_r I^2/2$ watts, and we find the formula for the radiation resistance for such a small current loop radiator as

$$R_r = 3.08 \times 10^5 \left(\frac{\rho}{\lambda}\right)^4 = 3.82 \times 10^{-5} (\rho f)^4 \qquad \text{ohms} \tag{151}$$

It can be seen that the fourth power of the frequency enters, as well as the fourth power of the radius of the small-loop radiator.

Inasmuch as in (149), the elementary moment m refers to the Hertzian doublet (electric charge) moment $m = q\,dl$ meter · coulombs and the elementary linear current radiator $i\,dl$ has equivalent radiation effects, we can use its instantaneous current moment $m_e = dlI\epsilon^{j\omega t}$ meter · amperes, by means of the conversion $m = -jm_e/\omega$, where $\omega/6.28$ is the frequency f in cycles per second. If the frequency is expressed in megacycles per second, we have $m = -jm_e/(2 \times 10^6 \pi f)$. Introducing the square of the amplitude value in the electric dipole relation of (149), we have for the total power radiated from such a dipole

$$P_e = 173 \times 10^9 f^4 \frac{(I\,dl)^2}{4\pi^2 10^{12} f^2} = 4.39 \times 10^{-3} I^2\,dl^2 f^2 = 395 I^2 \left(\frac{dl}{\lambda}\right)^2 \quad \text{watts} \tag{152}$$

Since $P_e = R_r I_e^2 = R_r I^2/2$, the Hertzian doublet has a radiation resistance

$$R_r = 789.5\left(\frac{dl}{\lambda}\right)^2 = 8.78 \times 10^{-3}(f\,dl)^2 \qquad \text{ohms} \tag{153}$$

Comparing the results of the magnetic dipole [Eq. (151)] with Eq. (153) for the electric dipole, we note that for the *electric* dipole the radiation resistance is proportional only to the square of the frequency compared with the fourth-power law of a small current loop radiator.

We must find out the limitations of formulas (150) and (151), since unlike for the P_e and corresponding R_r relations (152) and (153), we have made assumptions. These assumptions are contained in the statement that the closed-circuit radiator, circular or one of any other shape, must be small compared with the wavelength. The assumptions must also take care of the theoretical outcomes on pages 245–249, which show that we have to deal, for either type of radiator, with field wave impedances that are generally, for a distance r meters from the radiator,

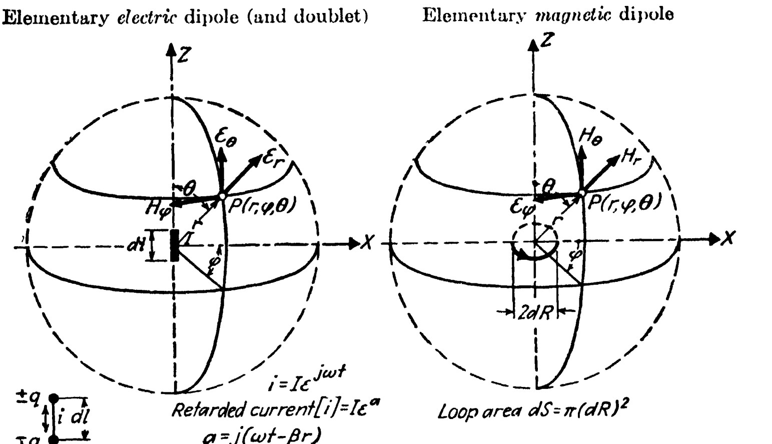
$$\left.\begin{aligned} Z &= -\frac{\mathcal{E}_\varphi}{H_\theta} = j\omega\mu r + \underbrace{Z_0(\beta r)^4}_{\text{radiation}} \quad &&\text{for small current loop radiator} \\ Z &= \frac{\mathcal{E}_\theta}{H_\varphi} = \frac{1}{j\omega\kappa r} + j\omega\mu r + \overbrace{Z_0(\beta r)^2}^{\text{resistance term}} \quad &&\text{for Hertzian doublet} \end{aligned}\right\} \text{ohms} \tag{154}$$

Therefore the wave impedance Z is by no means always $Z_0 = \sqrt{\mu/\kappa}$ ohms and pure resistive even though in a nondissipative medium ($\sigma = 0$). We have a series in each case. It depends, therefore, also on how close we are to a radiator and to what extent we can neglect the remaining terms of these series relations. We note that even for the few terms given above (these terms are always needed, since the respective series must include at least the radiation term), we have a complex wave impedance Z. But for the small-loop radiator, it is an inductive reactance $j\omega\mu r$ ohms, which increases with the frequency $\omega/6.28$ at a certain distance from the loop, which is also effective, besides the resistive term $Z_0(\beta r)^4 = Z_0(\omega r/v)^4 = Z_0(6.28fr)^4\mu^2\kappa^2 = K_1 f^4$. Since the factors representing K_1 are constant, this impedance also discloses that radiation occurs proportional to the fourth power of the frequency f. On the other hand, an elementary electric dipole exhibits, besides a square-law resistance term $K_2 f^2 = Z_0(\omega r\sqrt{\mu\kappa})^2 = \mu\kappa r^2 6.28^2 f^2$ ohms, both inductive $j\omega\mu r$ and capacitive $-j/(\omega\kappa r)$ terms in ohms. A small-loop radiator then acts as a rule with a positive reactance unless neighboring effects cause capacitive effects. The theory shows that for meeting such assumptions, the value $\beta\rho$ must be chosen smaller than unity, where ρ is the radius of the loop in meters and ρ/λ is the radius in terms of the operating wavelength λ

in the relation $\beta\rho = 6.28(\rho/\lambda) < 1$. Applying this approximation relation, we note that λ/ρ must be greater than 6.28. Let us take the case where the wavelength λ in the μ, κ, $\sigma = 0$ medium is eight times the radius ρ of the current loop. This gives for $\lambda = 8\rho$, a ratio $\rho/\lambda = \frac{1}{8}$ and a radiation resistance due to the magnetic dipole effect of

$$R_r = 3.08 \times 10^5 \left(\frac{1}{8^4}\right) = \frac{30.8 \times 10^4}{4.096 \times 10^3} = 75.3 \qquad \text{ohms}$$

33. Notes on Relations of Electric and Magnetic Dipoles. Figure 30 shows such elementary radiators in typical space orientations that correspond to each other.



Infinitely small *electric* dipole of constant I distribution and length dl has an electric current moment of $m_e = i\,dl = I\epsilon^a\,dl$ meter · amp

Hertzian doublet has charge moment $m = q\,dl$ meter · coulombs

$$m = \frac{i\,dl}{j\omega} = \frac{m_e}{j\omega}$$

Infinitely small *magnetic* dipole (loop) of constant current and phase distribution and radius dR has a magnetic moment $m_m = j\omega\mu I\epsilon^a\,dS$ meter · volts

FIG. 30. Electric and magnetic elementary dipoles.

In each case, the current-carrying element is so small that either the length dl or the circumference $6.28dR$ has fixed current distributions along their respective radiator lengths. We can also use a linear finite current path l or a circular path of finite length $6.28R$ if we provide phase distributions of the *same* polarity (for instance, three equal-phased half-wave distributions). But we then have to use the effective linear length l_e or the effective circumference $(6.28R)_e$, which are shorter but have equivalent fixed-current distributions. For an elementary $i = I\epsilon^{j\omega t}\,dl$ radiator, the quantity $i\,dl = m_e$ denotes the instantaneous current moment in meter · amperes. It is a vector along respec-

tive $\pm dl$ directions. This moment can also be related to the electric charge moment $m = q\,dl$ meter · coulombs, where q is an oscillating charge, *i.e.*, subject to $\epsilon^{j\omega t}$ variations. We have then $m = m_e/(j\omega)$, where m and m_e may also be taken as the amplitude or the effective value and instantaneous values, respectively.

In a similar way, we have, for an infinitely small plane current loop of area $dS = \pi(dR)^2$ expressed in square meters, a magnetic doublet moment $\mu I\,dS$ in meter · volt · seconds, according to the equivalent magnetic shell effect. This can be put equal to $m_0\,dl$ in order to compare it with the electric doublet moment $q\,dl$ which is in meter · ampere · seconds, since coulombs is in ampere · seconds. Our doublet moment is also in this case such that the magnetic matter m_0 is a variable time function of $\epsilon^{j\omega t}$. Just as the electric doublet moment $m = q\,dl$ produces the same field effects as an elementary linear $I\epsilon^{j\omega t}\,dl$ radiator of electric current moment $m_e = i\,dl$ meter · amperes, the magnetic doublet moment $m_0\,dl$ causes the same field effects as a elementary magnetic current moment $i_m\,dl$ meter · volts, where $i_m = I_m\epsilon^{j\omega t}$ denotes the instantaneous magnetic current in volts. We need an expression for this moment and find

$$m_m = i_m\,dl = \frac{\partial(\mu i\,dS)}{\partial t} = \mu\,dS\,\frac{\partial i}{\partial t} = j\omega\mu\,dSI\epsilon^{j\omega t} \qquad \text{meter · volts} \quad (155)$$

since the current in the elementary current loop is $i = I\epsilon^{j\omega t}$ amperes. It is this value for m_m which is to be used in the field relations of an infinitesimal loop radiator just as we use the value $m_e = I\,dl\epsilon^{j\omega t}$ for the field relations of an elementary electric dipole. To be strict, we must in either case use the retarded time function ϵ^a, where $a = j(\omega t - \beta r)$ and $\omega/6.28$ is the frequency in cycles per second. The quantity β is the phase constant $2\pi/\lambda$ with the space wavelength λ in meters In nondissipative space ($\sigma = 0$), μ in henrys per meter, and κ in farads per meter we have

$$\lambda^{(\text{m})} = \frac{1}{\sqrt{\mu\kappa}\,f^{(\text{Mc/sec})}} \quad (156)$$

For dissipative propagation space (σ mhos per meter also effective), we have instead of κ, which is the actual dielectric constant, the equivalent dielectric constant $\kappa_e = \kappa - j\sigma/\omega$ farads per meter with $\omega/6.28$ in cycles per second. In customary propagation (air), we have λ meters = $300/f^{(\text{Mc/sec})}$. We have then the comparisons of Table 2 for the electric and magnetic dipole. They are the outcomes of derivations in Chap. III.

When the specific conductivity σ is also effective in a medium, we have space dissipation. The retarded scalar potential V and vector potential Λ for the electric dipole (elementary doublet) are affected by the velocity $v = \omega/\beta$ meters per second, as given in Eq. (30). We must

TABLE 2
(Nondissipative medium)

Elementary electric dipole ($i\,dl$ and $q\,dl$ radiators) Causes a TM Wave	Elementary magnetic dipole ($2\pi\,dR\ i$ radiator, infinitely small i-loop radiator) Causes a TE Wave	Remarks
$\mathcal{E}_\theta = j\dfrac{0.5Z_0 I\,dl}{r\lambda}A$ (transverse field)	$H_\theta = -\dfrac{0.25\beta^2 I\,dS}{\pi r}A$ $A = \left[1 - \dfrac{j}{\beta r} - \dfrac{1}{(\beta r)^2}\right]P\sin\theta$	$\omega/6.28$ is in cycles/sec r m is distance from radiation center to field point P $Z_0 = \sqrt{\dfrac{\mu}{\kappa}} \cong 377$ ohms for air
$\mathcal{E}_r = \dfrac{0.5Z_0 I\,dl}{\pi r^2}B$ (longitudinal field)	$H_r = j\dfrac{0.5\beta I\,dS}{\pi r^2}B$ $B = \left(1 - \dfrac{j}{\beta r}\right)P\cos\theta$	λ is in m I in amp $\beta = 2\pi/\lambda$ μ in henrys/m $\cong 1.257 \times 10^{-6}$ henrys/m and $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m for air
$H_\varphi = j\dfrac{0.5I\,dl}{r\lambda}C$ (transverse field)	$\mathcal{E}_\varphi = -\dfrac{0.25Z_0\beta^2 I\,dS}{\pi r}C$ $C = \left(1 - \dfrac{j}{\beta r}\right)P\sin\theta$	$P = \epsilon^a$, $a = j(\omega t - \beta r)$, and θ angle with Z axis as in Fig. 30 $\sigma = 0$ mhos/m
For *any distance* r		$\mathcal{E}$ is in volts/m H in amp/m $I = I\epsilon^a$ amp $dS = \pi\,dR^2$
$\mathcal{E} = \sqrt{\mathcal{E}_\theta^2 + \mathcal{E}_r^2}$ (transverse-longitudinal field)	$H = \sqrt{H_\theta^2 + H_r^2}$ (transverse-longitudinal field)	
$\mathcal{E}_\theta = j\dfrac{0.5Z_0 I\,dl}{r\lambda}P\sin\theta$ $\mathcal{E}_r = 0$ and $H_\varphi = \dfrac{\mathcal{E}_\theta}{Z_0}$ } only transverse fields exist now in effect	$H_\theta = -\dfrac{0.25\beta^2 I\,dS}{\pi r}P\sin\theta$ $= -\dfrac{\pi I\,dS}{r\lambda^2}P\sin\theta$ $H_r = 0$ and $\mathcal{E}_\varphi = -Z_0 H_\theta$	
For *distant radiation region* (radiation region for 1.5λ and more distance)		
$\mathcal{E}_\theta = -j\dfrac{0.5Z_0 I\,dl}{\beta^2 r^3\lambda}P\sin\theta$ (transverse)	$H_\theta = \dfrac{0.25I\,dS}{\pi r^3}P\sin\theta$	
$\mathcal{E}_r = -j\dfrac{0.5Z_0 I\,dl}{\pi\beta r^3}P\cos\theta$ (longitudinal)	$H_r = \dfrac{0.5I\,dS}{\pi r^3}P\cos\theta$	
$H_\varphi = \dfrac{0.5I\,dl}{\beta r^2\lambda}P\sin\theta$ (transverse)	$\mathcal{E}_\varphi = -j\dfrac{0.25\omega\mu I\,dS}{\pi r^2}P\sin\theta$	
$\mathcal{E} = \sqrt{\mathcal{E}_\theta^2 + \mathcal{E}_r^2}$ (transverse-longitudinal)	$H = \sqrt{H_\theta^2 + H_r^2}$	
Next to elementary *radiator*		

then use the propagation constant

$$\Gamma = \alpha + j\beta = j\omega\sqrt{\mu\kappa_c} = \overset{\text{nepers/m}}{\frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}\sqrt{\frac{2}{1+q}}} + \overset{\text{circular radians/m}}{j\omega\sqrt{\mu\kappa}\sqrt{\frac{1+q}{2}}};$$

$$q = \sqrt{1 + \frac{\sigma^2}{\omega^2\kappa^2}} \qquad (157)$$

because the space attenuation constant α is no longer zero. Since $\omega/6.28$ denotes the frequency in cycles per second, we note that we have a *dispersive* medium depending on the frequency. In many cases, wave propagation is then limited to short distances only, as a matter of fact often to very short distances. These distances may be only so-called "skin deep" at very high and ultrahigh frequencies in some cases. The *magnetic* vector potential to be used is then

$$A^{(\text{volt sec/m})} = \mu\frac{I\,dl}{4\pi r}\epsilon^{j\left(\omega t - \frac{\Gamma}{j}r\right)} = \frac{\mu I\,dl}{4\pi r}\epsilon^{j\omega(t - r\sqrt{\mu\kappa_c})} \qquad (158)$$

for an elementary electric dipole (or a Hertzian doublet). Similarly, we have generalized conditions for the elementary loop radiator. We find then for $b = j\omega(t - r\sqrt{\mu\kappa_c}) = j\omega[t - r\sqrt{\mu(\kappa - j\sigma/\omega)}]$, $Q = \epsilon^b$, $Z = \sqrt{\mu/\kappa_c}$,

$$\left.\begin{aligned} \mathcal{E}_\theta &= j\frac{0.25\omega\mu I\,dl}{\pi r}\left[1 + \frac{1}{r\Gamma} + \frac{1}{(r\Gamma)^2}\right]Q\sin\theta \\ \mathcal{E}_r &= \frac{0.5ZI\,dl}{\pi r^2}\left(1 + \frac{1}{r\Gamma}\right)Q\cos\theta \\ H_\varphi &= \frac{0.25I\,dl\Gamma}{\pi r}Q\sin\theta \end{aligned}\right\} \text{for the elementary electric dipole radiator} \qquad (159)$$

For the elementary current loop radiator (elementary magnetic dipole), we must use the *electric* retarded vector potential, which, owing to the electric shell equivalence, yields

$$A_e = \kappa_c\frac{J_m\,dlQ}{4\pi r} = \frac{j\omega\kappa_c\mu I\,dS}{4\pi r}Q \qquad \text{amp}\cdot\text{sec/m} \qquad (160)$$

From Table 2 it is seen that at short distances, which are less than 1 m, the higher power terms of r are most significant while the inverse distance term becomes a quantity of only little importance. The relations for field points next to the radiator are then the well-known results as found by the Biot-Savart law.

For the distant radiation region, Table 2 shows that we have the well-known classic TEM wave ($\mathcal{E}$ and H for either dipole are in space

quadrature and in time phase). For regions where the longitudinal (radial) components also play a part (radiation and induction fields are superimposed), we *no longer* have space quadrature and time inphase condition for the resultant electric and magnetic fields of respective dipole radiators. A small loop, an electric pole, or a pole *will not point correctly* when too close to the electromagnetic source.[1]

34. Applications of Electric and Magnetic Doublet Formulas. For most work, the distant radiation field only is of interest, and customary atmosphere is the medium through which wave propagation occurs. Then $0.5Z_0 = 376.7/2 = 188.35$ and $0.25\beta^2 = (\pi/\lambda)^2$ in the respective dipole relations of Table 2. We then have for respective field contributions $d\mathcal{E}$ and dH in the distant radiation region,

$$\left.\begin{aligned} d\mathcal{E}_\theta^{(\text{volts/m})} &= \frac{188.35 I\,dl}{r\lambda} P \sin\theta \\ &= \frac{6.28 \times 10^{-1} f I\,dl}{r} P \sin\theta; \\ dH_\varphi^{(\text{amp/m})} &= \frac{d\mathcal{E}_\theta}{376.7} \end{aligned}\right\} \text{for } \textit{electric} \text{ doublet (or elementary linear radiator). Fields are proportional to } I \text{ and } dl \text{ and frequency}$$

$$\left.\begin{aligned} dH_\theta^{(\text{amp/m})} &= \frac{3.14 I\,dS}{r\lambda^2} P \sin\theta \\ &= \frac{3.49 \times 10^{-5} f^2 I\,dS}{r} P \sin\theta; \\ d\mathcal{E}_\varphi^{(\text{volts/m})} &= -376.7\, dH_\theta \end{aligned}\right\} \text{for } \textit{magnetic} \text{ doublet (or elementary loop radiator). Fields are proportional to } I \text{ and } dS \text{ and the square of the frequency} \qquad (161)$$

where λ is in meters, f is in megacycles per second, I is in amperes, the distance r to the field point is in meters, $P = \epsilon^a$, $a = j(\omega t - \beta r)$, $\omega/6.28$ is in cycles/sec, $\beta = 6.28/\lambda$ radians per meter, θ is the angle that the elementary length dl of the electric dipole makes with the r direction and the angle that r makes with the normal to the elementary loop area dS of the elementary magnetic dipole (Fig. 30).

Suppose we assume sinusoidal current distributions along a conductor that radiates into unobstructed space, causing so-called "free-field intensities." Figure 31 depicts what will happen for a half-wave electric dipole and a very short electric dipole for which we still have finite lengths for the poles but such that the total length Δl is only 1 per cent

[1] The time phase is due to the fact that the general formulas given in Table 2 and holding for any distance are complex, *i.e.*, contain real and imaginary terms. As a matter of fact, we have terms making up the actual radiated power and fields that are positive j terms (positive reactive power) and negative j terms (negative reactive power). This is brought out in detail on pp. 245–249. Respective dipoles will indicate, however, the correct resultant field.

in terms of the wavelength operating along the dipole. Therefore, the total dipole length $\Delta l = 0.01\lambda$ meters. The current distribution from the central feeder gap decreases practically linearly toward respective open ends of the dipole. According to Fig. 31, this results in a total effective length $l_e = 0.5\Delta l = 0.005\lambda$ meters. The above formula then yields the electric field contribution

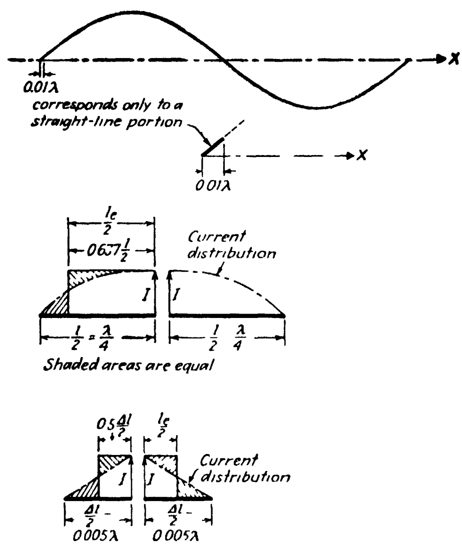


FIG. 31. Graphical solution for the equivalent length l_e of a thin linear radiator with a sinusoidal current distribution. Electric half-wave dipole of total length $l = \lambda/2$ has an effective length $l_e = \frac{2}{\pi} l = 0.637l$ meters with an equivalent fixed current value of I amp. Very short total length $\Delta l = 0.01\lambda$ yields effective or equivalent length $l_e = 0.5\Delta l = 0.01 \times 0.5\lambda = 0.005\lambda$.

$$\Delta\mathcal{E}_\theta = \frac{188.35Il_e}{r\lambda} P \sin\theta = \frac{188.35 \times 0.005I}{r} P \sin\theta$$

$$= \frac{9.417 \times 10^{-1}I}{r} F \sin\theta \qquad \text{volts/m}$$

Even though this numerical result no longer directly includes the wavelength, since Δl as well as l_e are expressed in terms of the wavelength, we must still relate the distance r to the wavelength, besides the fact that this result applies only to the pure radiation field. For distances $r \geqq 10\lambda$, we are not only well in the true radiation region but often far enough from such a short electric dipole as to make Δl negligible compared with the distance r. For $r = 10\lambda$ meters and a 200-Mc wave excitation, we have $\lambda = {}^{300}/_{200} = 1.5$ m and $r = 10\lambda = 10 \times 1.5 = 15$ m, which for

a 5-amp effective input current at the dipole gap would cause an effective electric field intensity of

$$\Delta \mathcal{E}_\theta = \frac{5 \times 9.417 \times 10^{-1}}{15} \sin \theta = 3.139 \times 10^{-1} \sin \theta \qquad \text{volts/m}$$

and a perpendicular effective magnetic field intensity

$$\Delta H_\varphi = \frac{3.139 \times 10^{-1}}{376.7} \sin \theta = 8.3 \times 10^{-4} \sin \theta \qquad \text{amp/m}$$

It is permissible, in this case, to use the elementary dipole directivity of $\sin \theta$, since the total radiator length is only 1 per cent of the wavelength. For longer dimensions, the directivity D_θ instead of $\sin \theta$ has to be used as is described in detail in Secs. 102–105, 108–112, and 113–122.

Now let us see what happens when we bend the total Δl length of the electric dipole into a closed circular loop in order to obtain a small magnetic dipole effect. We have $2\pi\, \Delta R = \Delta l = 0.01\lambda$ and, therefore, a corresponding small radius $\Delta R = 0.01\lambda/6.28 = 1.592 \times 10^{-3}\lambda$. For such a condition we have a magnetic dipole moment, and we can use the dH_θ and $d\mathcal{E}_\varphi$ formulas of Eq. (161). The corresponding enclosed area is then

$$\Delta S = \pi\, \Delta R^2 = \pi \left(\frac{0.01\lambda}{2\pi}\right)^2 = \frac{10^{-4}\lambda^2}{12.56} = 7.96 \times 10^{-6}\lambda^2 \qquad \text{sq m}$$

which could also be found from $\pi(1.592 \times 10^{-3}\lambda)^2$. We find then at a distance of $r = 30$ m and an effective current of 5 amp

$$\Delta H_\theta = -\frac{3.14 I\, \Delta S \sin \theta}{30\lambda^2} = -\frac{5 \times 3.14 \times 7.96 \times 10^{-6}}{30} \sin \theta$$
$$= -4.16 \times 10^{-6} \sin \theta \qquad \text{amp/m}$$

as against $4.15 \times 10^{-4} \sin \theta$ amperes per meter in the case of the ΔH_φ value of the electric dipole of same conductor length. The minus sign does not matter as far as the effective field value of ΔH_θ is concerned. Therefore, we have essentially only 1 per cent of the field intensity (whether electric or magnetic, they are merely interlinked by Z_0) at the same field points at 30 m from the radiation center in the case of the small loop radiator, as found for the electric dipole. The Poynting power flow is then

$$p_\theta = Z_0 \cdot \Delta H_\theta^2 = 376.7 \times (4.16 \times 10^{-6} \sin \theta)^2 = 6.55 \times 10^{-9} \sin^2 \theta \qquad \text{watts/m}^2$$

or only $0.00655 \sin^2 \theta$ microwatts per square meter. The associated effective electric field is then found as

$$\Delta\mathcal{E}_\varphi = 376.7 \cdot \Delta H_\theta = 376.7 \times 4.16 \times 10^{-6} \sin\theta = 1.565 \times 10^{-3} \sin\theta \quad \text{volts/m}$$

If we had chosen the largest space dimension (extension in space) of the circular[1] radiator equal to 0.01λ, or $\Delta R = 0.005\lambda$ and $\Delta S = \pi(0.005\lambda)^2$ we should have found an effective magnetic field intensity

$$\Delta H_\theta = \frac{3.14 I\,\Delta S \sin\theta}{r\lambda^2} = \frac{(0.005\pi)^2 I \sin\theta}{r} = \frac{25 \times 9.86 \times 10^{-6} \times 5 \sin\theta}{30}$$
$$= 4.1 \times 10^{-5} \sin\theta \quad \text{amp/m}$$

Hence, we find a field value that is 10.1 per cent of the electric dipole effect if the largest space extension is the same in either case.

It is now of interest to learn how electric and magnetic dipoles act in the neighborhood of such radiators and in the region between the near field and the distant radiation field.

35. Numerical Data for Total Field Effects in the Neighborhood and the Between-field Region of Electric and Magnetic Dipoles. As brought out in connection with Table 2, page 163, we deal with longitudinal $\mathcal{E}$ waves, called "transverse magnetic" waves (TM waves) in the case of an electric dipole, and with longitudinal H waves, also called "transverse electric" waves (TE waves) in the case of a very small[2] loop radiator. It is only for the radiation region that we deal with TEM waves, *i.e.*, waves for which both the electric and magnetic fields are entirely transverse to the direction of wave propagation. In the neighborhood of a radiator, we have generally neither inphase $\mathcal{E}$ and H_φ fields nor H_φ normal to the resultant electric $|\mathcal{E}| = \sqrt{|\mathcal{E}_\theta^2| + |\mathcal{E}_r^2|}$ field in the case of an electric dipole. Similarly, we find in this near region such conditions for the $\mathcal{E}_\varphi$ and the resultant $|H| = \sqrt{|H_\theta^2| + |H_r^2|}$ field in the case of a small-loop radiator.

It is convenient to distinguish between three regions, namely, the distant radiation region (usually a great distance), the region in the neighborhood of the radiator, and the region between the neighborhood field and the true distant radiation field.

The general formulas holding for any distance and given in Table 2 have the advantage that the factors A, B, and C contain the same terms, regardless of whether we deal with an electric or with a magnetic dipole. We find also that $P \sin\theta$ and $P \cos\theta$ occur in corresponding dipole formulas in the same way. On the other hand, the formulas of Eqs. 164, as derived on pages 169–174, have the factors expressed as $(1/r_e)$,

[1] The radiator does not have to be circular in order to cause magnetic dipole effect.

[2] By very small is meant very small with respect to the operating wavelength, since we require essentially same current phase for all points of the circular radiator.

$(1/r_e)^2$, and $(1/r_e)^3$ terms, where

$$r_e = \beta r = 2\pi r/\lambda$$

stands for the electrical distance from radiation center to field point.

Since for an elementary electric dipole, the doublet charge moment $m = q\,dl$ can be expressed in terms of the electric current moment $m_e = I\,dl\,\epsilon^a = i\,dl$ of the elementary linear current radiator, for the retarded exponent $a = j(\omega t - \beta r) = j(\omega t - r_e)$, we have the conversion

$$m = \frac{m_e}{j\omega} \tag{162}$$

Introducing the substitutions

$$C = \frac{\beta^3}{4\pi\omega\epsilon} \qquad \text{and} \qquad D = \frac{\beta^2 \sin\theta}{4\pi} \tag{163}$$

for the electric doublet, we find

$$\left.\begin{array}{ll}
\mathcal{E}_\theta^{(\text{volts/m})} = m_e C \sin\theta \left(\dfrac{j}{r_e} + \dfrac{1}{r_e^2} - \dfrac{j}{r_e^3}\right) & \text{for elementary linear current radiator } i\,dl \\
\mathcal{E}_\theta^{(\text{volts/m})} = m\omega C \sin\theta \left(-\dfrac{1}{r_e} + \dfrac{j}{r_e^2} + \dfrac{1}{r_e^3}\right) & \text{for Hertzian doublet radiator } q\,dl \\
\mathcal{E}_r^{(\text{volts/m})} = 2m_e C \cos\theta \left(\dfrac{1}{r_e^2} - \dfrac{j}{r_e^3}\right) & \text{for } i\,dl \text{ radiator} \\
\mathcal{E}_r^{(\text{volts/m})} = 2m\omega C \cos\theta \left(\dfrac{j}{r_e^2} + \dfrac{1}{r_e^3}\right) & \text{for } q\,dl \text{ radiator} \\
H_\varphi^{(\text{amp/m})} = m_e D \left(\dfrac{j}{r_e} + \dfrac{1}{r_e^2}\right) & \text{for } i\,dl \text{ radiator} \\
H_\varphi^{(\text{amp/m})} = m\omega D \left(-\dfrac{1}{r_e} + \dfrac{j}{r_e^2}\right) & \text{for } q\,dl \text{ radiator}
\end{array}\right\} \tag{164}$$

These expressions now lend themselves readily in determining what happens in the neighborhood of radiators, the between region as well as the distant radiation region. Only the terms within the parentheses are used to decide on the respective regions, although also the ϵ^a factor of m_e with the exponent $a = j(\omega t - r_e)$ and $r_e = \beta r = 6.28r/\lambda$ is of importance. That these expressions are the same as the formulas of Table 2 can be seen from the following:

For instance, the θ component reads

$$\mathcal{E}_\theta = j\,\frac{0.5 Z_0 I\,dl}{r\lambda} A; \qquad A = \left(1 - \frac{j}{r_e} - \frac{1}{r_e^2}\right)\epsilon^a \sin\theta$$

$$a = j(\omega t - r_e); \qquad r_e = \beta r$$

Therefore

$$\mathcal{E}_\theta = j\frac{\overbrace{Z_0 I\, dl\epsilon^a}^{m_e} \sin\theta}{2r\lambda}\left(1 - \frac{j}{r_e} - \frac{1}{r_e^2}\right)$$
$$= \frac{\sqrt{\mu/\kappa}\, m_e \sin\theta}{2r\lambda}\left(j + \frac{1}{r_e} - \frac{j}{r^2}\right)$$
$$= \frac{\sqrt{\mu/\kappa}\, m_e \sin\theta}{2r\lambda}\left(\frac{j}{r_e} + \frac{1}{r_e^2} - \frac{j}{r_e^2}\right)\beta r$$

The multiplier in front of the parenthesis is then

$$\frac{\sqrt{\frac{\mu}{\kappa}}\, m_e \sin\theta}{2\lambda}\beta = \frac{\sqrt{\frac{\mu}{\kappa}}\, m_e\beta^3 \sin\theta}{2\lambda\beta^2} = \frac{\sqrt{\frac{\mu}{\kappa}}\,\beta^3 m_e \sin\theta}{2\lambda\omega^2\mu\kappa} = \frac{\overbrace{\sqrt{\frac{1}{\mu\kappa}}}^{c'}\,\beta^3 m_e \sin\theta}{\underbrace{2\lambda f}_{c'}2\pi\omega\kappa}$$
$$= \frac{\beta^3 m_e \sin\theta}{4\pi\omega\kappa}$$

Therefore,

$$\mathcal{E}_\theta = \frac{\beta^3 m_e \sin\theta}{4\pi\omega\kappa}\left(\frac{j}{r_e} + \frac{1}{r_e^2} - \frac{j}{r_e^3}\right)$$

which checks the expression of Eq. (161). In the same way, $\mathcal{E}_r$ and H_φ of Eq. (164) can be checked for the elementary $i\,dl$ radiator, for which $i = I\epsilon^a$.

Inasmuch as in Table 2 the values in the parentheses hold for the electric and magnetic doublet, this must also be true for the parenthesis values given in formulas (164). We proceed with the first of expressions (164), since we actually deal with electric currents in practical radiators, and the $i\,dl$ radiator with its m_e moment at any field point gives the same field effect as the $q\,dl$ or Hertzian doublet radiator.

Table 3 gives computations for the $1/r_e$, $1/r_e^2$, and $1/r_e^3$ values, in terms of the r/λ value, *i.e.*, for the distance r_e to the field point in wavelength.[1] The electrical distance $r_e = \beta r = 2\pi r/\lambda$ is in radians, since for a distance r equal to the space wavelength λ, the wave front has advanced 1 wavelength, which gives $r_e = 2\pi \times 1$, or 6.28 radians (or 360 electrical space degrees). These degrees, in this case, are along the direction of wave propagation (normal to the advancing wave front).

At many places in this text, we speak of the induction field, which is always interlinked with the radiator, in order to distinguish it from the

[1] In many radiation problems, it is convenient wherever possible to express dimensions in terms of the wavelength operating along the particular dimension. It is then possible to obtain expressions holding for any desired wavelength.

TABLE 3. EQUIVALENT ELECTRIC DISTANCE EFFECT

$\frac{r}{\lambda}$	$r_e = \beta r$	$\frac{1}{r_e}$	$\frac{1}{r_e^2}$	$\frac{1}{r_e^3}$	$\frac{r}{\lambda}$	r_e	$\frac{1}{r_e}$	$\frac{1}{r_e^2}$	$\frac{1}{r_e^3}$
0.010	0.0628	15.92	254	4,060	1.1	6.9	0.145	0.0210	0.00304
0.012	0.0752	13.3	177	2,350	1.2	7.52	0.133	0.0177	0.00235
0.014	0.088	11.4	129	1,470	1.3	8.15	0.123	0.0151	0.00185
0.016	0.1002	10.0	100	1,000	1.4	8.8	0.114	0.0129	0.00147
0.018	0.113	8.85	78.1	691	1.5	9.4	0.1065	0.01136	0.001205
0.020	0.1256	7.98	63.5	508	1.6	10.02	0.100	0.01	0.001
0.022	0.138	7.23	52.5	380	1.7	10.68	0.0935	0.00871	0.000815
0.024	0.151	6.63	44.0	291	1.8	11.3	0.0885	0.00781	0.00691
0.026	0.1635	6.12	37.5	230	1.9	11.95	0.084	0.00703	0.00059
0.028	0.176	5.68	32.3	184	2.0	12.56	0.0798	0.00635	0.000508
0.030	0.188	5.31	28.2	151	2.2	13.8	0.0723	0.00525	0.00038
0.033	0.206	4.86	23.5	114	2.4	15.1	0.0663	0.0044	0.000291
0.036	0.226	4.43	19.6	87	2.6	16.35	0.0612	0.00375	0.00023
0.040	0.251	3.98	15.8	63	2.8	17.6	0.0568	0.00323	0.000184
0.045	0.281	3.55	12.6	41.8	3.0	18.8	0.0531	0.00282	0.000151
0.050	0.314	3.19	10.2	32.5	3.2	20.1	0.0497	0.00246	0.000123
0.055	0.345	2.90	8.4	21.3	3.4	21.4	0.0468	0.00218	0.000102
0.06	0.377	2.65	7.0	18.6	3.6	22.6	0.0443	0.00196	0.000087
0.07	0.44	2.27	5.18	11.8	3.8	23.85	0.0420	0.00177	0.000074
0.08	0.502	1.99	3.95	7.9	4.0	25.1	0.0398	0.00158	0.000063
0.09	0.563	1.77	3.15	5.6	4.2	26.4	0.0379	0.00144	0.000055
0.10	0.628	1.592	2.54	4.06	4.4	27.6	0.0362	0.00131	0.000047
0.12	0.752	1.33	1.77	2.35	4.6	28.9	0.0347	0.0012	0.000042
0.15	0.94	1.065	1.136	1.205	4.8	30.1	0.0332	0.0011	0.000037
					5.0	31.4	0.0319	0.00102	0.000032
0.1592	1.0000	1.000	1.000	1.000	5.2	32.6	0.0306	0.00094	0.000029
					5.4	33.9	0.0295	0.00087	0.000026
0.20	1.256	0.798	0.635	0.508	5.6	35.15	0.0284	0.00081	0.000023
0.25	1.565	0.640	0.407	0.26	5.8	36.5	0.0274	0.00075	0.000020
0.30	1.88	0.531	0.282	0.151	6.0	37.7	0.0265	0.0007	0.000019
0.35	2.19	0.457	0.208	0.096	6.2	39.0	0.0256	0.00066	0.000017
0.40	2.51	0.398	0.158	0.063	6.4	40.1	0.0249	0.00062	0.000015
0.50	3.14	0.319	0.102	0.0325	6.6	41.5	0.0241	0.00058	0.000014
0.60	3.77	0.265	0.07	0.0186	6.8	42.6	0.0235	0.00055	0.000013
0.65	4.06	0.246	0.0607	0.0149	7.0	44.0	0.0227	0.00052	0.000012
0.70	4.4	0.227	0.0518	0.0118	7.2	45.2	0.0221	0.00049	0.000011
0.8	5.02	0.199	0.0395	0.0079	7.4	46.5	0.0215	0.00046	0.000010
0.9	5.63	0.177	0.0315	0.0056	7.6	47.75	0.0210	0.00044	0.000009
					7.8	49.0	0.0204	0.00042	0.000008
1.0	6.28	0.1592	0.0254	0.00406	8.0	50.24	0.0199	0.00039	0.000008

Remarks: For $r/\lambda = 0.1592$ we have $r_e = \beta r = 1 = 1/r_e = 1/r_e^2 = 1/r_e^3$ and the radiation field equal to the induction field. For $r/\lambda = 10$ we have $r_e = 62.8$ and $1/r_e = 0.01592$, $1/r_e^2 = 0.000254$, and $1/r_e^3 = 0.00000406$. Hence for distances r larger than 10λ, only the $1/r_e$ term is significant.

radiation field, which escapes from the radiator into space with a velocity of $1/\sqrt{\mu\kappa}$ meters per second in a nondissipative medium ($\sigma = 0$). The induction field is in time quadrature with the radiator current, and therefore, on the average, it cannot do any work. Time quadrature components are $+j$ and $-j$ terms and refer to wattless power. On the other hand, real terms, whether in phase or in antiphase with the radiator current, refer to power in watts. If we have terms that are in antiphase with the radiator current, work must be done by the voltage that impresses the current on the radiator element against such real field terms. Therefore, we have a means of finding the relative magnitudes of such terms, since the formulas of both Eq. (164) and Table 2 contain $\pm j$ terms as well as $\pm$ real terms. We note from the dipole expressions of Eq. (164) that it is simpler to separate real and imaginary terms for the φ field component. This is also seen in Table 2, since the H_φ field for the electric dipole and the $\mathcal{E}_\varphi$ field for the magnetic dipole contain only one real and one imaginary term. Suppose we take the first H_φ expression of Eq. (164), which reads

$$H_\varphi = DI\,\epsilon^a\,dl\left(\frac{j}{r_e} + \frac{1}{r_e^2}\right) = j\,DI\,\epsilon^a\,dl\left(\frac{1}{r_e} + \frac{1}{jr_e^2}\right) = j\,DI\,\epsilon^a\,dl\left(\frac{1}{r_e} - \frac{j}{r_e^2}\right)$$

The real as well as the imaginary term of the parenthesis is multiplied by $j\,DI\,\epsilon^a\,dl$, *i.e.*, by the radiator current of amplitude I. The multiplication of the real term $1/r_e$ refers to the radiated field, while the multiplication by the imaginary term j/r_e^2 stands for the induction field. When the respective magnitudes of these terms are equal, the radiation field as well as the induction field must also be equal. This happens when $1/r_e = 1/r_e^2$ or $r_e = \beta r = r_e^2 = (\beta r)^2$. This occurs when $\beta = 1/r$, since then $\beta r = 1 = (\beta r)^2$. But $\beta = 2\pi/\lambda$ and $r = \lambda/(2\pi)$. Hence,

For $r = 0.1592\,\lambda$ meters, radiation and induction fields are equal (165)

For distances less than 0.1592λ, the induction field is predominant; and for distances greater than 0.1592λ, the radiation field becomes more and more pronounced as r increases. The same result can be used for either dipole, the Hertzian doublet or the infinitely small current loop, since, according to Table 2, we have

$$H_\varphi^{(\text{amp/m})} = j\,\frac{\epsilon^{j(\omega t - r_e)}}{2\lambda}\sin\theta\left(\underbrace{\frac{I\,dl}{r}}_{\text{active}} - j\,\underbrace{\frac{I\,dl}{\beta r^2}}_{\text{reactive}}\right) \qquad \text{electric dipole of length } dl \text{ m}$$

$$\mathcal{E}_\varphi^{(\text{volts/m})} = \frac{Z_0\beta^2\epsilon^{j(\omega t - r_e)}}{4\pi}\sin\theta\left(\overbrace{\frac{I\,dS}{r}}^{\text{active}} - j\,\overbrace{\frac{I\,dS}{\beta r^2}}^{\text{reactive}}\right) \qquad \text{magnetic dipole or loop of enclosed area } dS \text{ sq m} \tag{166}$$

The inphase active and the time quadrature reactive factors are again equal, when $r = \beta r^2$ or $\beta r = r_e$ equal to unity. Hence, $1/r_e = 1/r_e^2 = 1/r_e^3$ is also equal to unity. Table 3 gives the absolute magnitudes of these values $1/r_e$ for the first, second, and third power.

From Eq. (166), we note that the active radiation field H_{ra} of H_φ is proportional to $I\,dl/(r\lambda)$ for an electric dipole, while the reactive induction or j field H_j is proportional to $I\,dl/(\lambda\beta r^2)$. Hence,

$$\frac{\text{Radiation field } H_{ra}}{\text{Induction field } H_j} = \beta r = r_e \tag{167}$$

For the magnetic dipole, *i.e.*, the loop radiator of area dS, the radiation field $\mathcal{E}_{ra}$ is proportional to $Z_0\beta^2 I\,dS/r$ and the induction field $\mathcal{E}_j$ of the $\mathcal{E}_\varphi$ field is proportional to $Z_0\beta I\,dS/r^2$ with exactly the same result as in Eq. (167) for the ratio $\mathcal{E}_{ra}/\mathcal{E}_j$. This shows again that

$$\underbrace{H_{ra} = r_e H_j}_{\text{electric doublet}} \qquad \text{and} \qquad \underbrace{\mathcal{E}_{ra} = r_e \mathcal{E}_j}_{\text{magnetic doublet}} \tag{167a}$$

for the respective H_φ amperes per meter and $\mathcal{E}_\varphi$ volts per meter values. Hence, for either dipole, the radiation field is larger than the interlinked induction field only when the electrical distance $r_e = \beta r = 6.28r/\lambda$ from the radiation center is larger than unity.

The total magnetic field intensity is then generally $H = H_{ra} + H_j$ amperes per meter for the electric dipole, and $\mathcal{E} = \mathcal{E}_{ra} + \mathcal{E}_j$ volts per meter for the magnetic dipole is the total electric field intensity.

Now in the neighborhood of radiators for which $r_e \ll 1$, we are essentially in a relatively strong induction field and close to oscillating charges ρ and current densities J. We have also

$$\epsilon^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \frac{a^4}{24} + \cdots = \epsilon^{j\omega t}\epsilon^{-jr_e}$$
$$= \epsilon^{j\omega t}\left(1 - jr_e + \frac{r_e^2}{2} - j\frac{r_e^3}{6} + \frac{r_e^4}{24} - \cdots\right) \simeq \epsilon^{j\omega t}$$

since the retarded effect is negligible. We have, therefore, $H_j \cong H$ and $\mathcal{E}_j \cong \mathcal{E}$, which confirms the well-known quasi-stationary condition solution.

What we call $r_e \ll 1$ is now of interest from a practical point of view. Often, 1 per cent of a field intensity can be taken as negligible compared with the full value. Then we can express the above limitation by $r_e \leqq 0.01$ for the region where, for instance, the Biot-Savart law essentially confirms the measured magnetic field effect in the neighborhood of a current-carrying conductor. But

$$r_e = \frac{2\pi}{\lambda} r = \frac{2\pi f}{f\lambda} r = \frac{\omega}{v} = \omega \sqrt{\mu\kappa}\, r$$

and we can limit the zone of the neighborhood field (near field) of a radiator by the condition

$$r\omega \sqrt{\mu\kappa} \leqq 0.01$$

Inasmuch as for $r_e > 1$ we have H_{ra} and $\mathcal{E}_{ra}$ more and more pronounced in comparison with the respective induction fields H_i and $\mathcal{E}_i$ the larger the value r in $r_e = \beta r$ is, we can, for practical purposes, limit the distance r to not less than one hundred times the value that r has for equal radiation and induction fields (for which $r_e = 1$). This is written as $r\omega \sqrt{\mu\kappa} \geqq 100$ for a nondissipative medium ($\sigma = 0$). On such a practical basis, we then have the formulas

$$\left.\begin{array}{lll}
r^{(m)} \leqq \dfrac{1.592 \times 10^{-9}}{f^{(Mc)} \sqrt{\mu\kappa}} & \text{for a nondissipative medium of } \mu \text{ henrys/m and } \kappa \text{ farads/m} & \text{near or neighborhood region (induction zone)} \\
r^{(m)} \leqq \dfrac{4.77 \times 10^{-1}}{f^{(Mc)}} & \text{for vacuum and practically so for air} & \\
r^{(m)} \geqq \dfrac{1.592 \times 10^{-5}}{f^{(Mc)} \sqrt{\mu\kappa}} & \text{for } \mu \text{ and } \kappa \text{ medium} & \text{pure radiation region} \\
r^{(m)} \geqq \dfrac{4.77 \times 10^{3}}{f^{(Mc)}} & \text{for air and vacuum} &
\end{array}\right\} \quad (168)$$

since for air $1/\sqrt{\mu\kappa} \cong 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec.

Suppose we have a 100-Mc current in a radiator; then customary alternating-current laws (based only on voltage and current relations, but not strictly on Maxwell's field relations satisfying boundary conditions) apply for distances $r \leqq 4.77 \times 10^{-1}/100 \leqq 4.77 \times 10^{-3}$ m. This is equal to or less than 4.77 mm. This is also equal to or less than $4.77 \times 3.937 \times 10^{-2} = 0.188$ in. (less than about $\frac{3}{16}$ in.). For a customary standard broadcast frequency of, say, 1,000 kc/sec = 1 Mc/sec, we had $r \leqq 0.477$ m $= 39.37 \times 0.477 = 18.8$ in. For a 60-cycle house current, *however*, we had $r \leqq 4.77 \times 10^{-1}/(60 \times 10^{-6}) \leqq 7.95$ km. This is equal to or less than $7.95 \times 0.6214 = 4.95$ statute miles. This shows very clearly how the frequency affects the reach of the induction field. Thus for an ordinary house current, we can safely say that elementary current elements will practically cause only induction fields that fall off very rapidly with the distance r from the conductor.

For the practically pure radiation field, formula (168) shows that for a 100-Mc radiator current, we are limited to distances

$$r \geqq \frac{4.77 \times 10^3}{100} \geqq 47.7 \text{ m}$$

i.e., distances longer than $47.7 \times 3.281 \cong 156$ ft. Inasmuch as $\lambda = {}^{300}\!/_{100} = 3$ m, the hundredfold minimum limit would require a distance of at least $47.7/3 = 15.9$ wavelengths from the radiation center. Table 4 gives an idea of the practical pure radiation and near-field zones of a radiator. We note that even down to a frequency belonging to the standard broadcast range and if $f = 1$ Mc/sec, the essential induction field region computed for quasi-stationary conditions limits the maximum distance r from the radiation center to only 18.8 in. For the UHF range of frequencies, the near-field region is limited to only a few millimeters distance (25.4 mm = 1 in.) from the radiating conductor. But when we deal with a 60-cycle commercial alternating current, the field about the conductor is still subject to quasi-stationary conditions up to 4.95 miles from the conductor.

TABLE 4

Frequency f, Mc/sec	Pure radiation zone			Near-field zone	
	$\geqq r$ m	$\geqq r$ ft	$\lambda = \dfrac{300}{f}$ m	$\leqq r$ m	$\leqq r$ in.
100,000	0.0477	0.156	0.003	4.77×10^{-6}	0.000188
30,000	0.159	0.522	0.001	1.59×10^{-5}	0.000628
10,000	0.477	1.56	0.03	4.77×10^{-5}	0.00188
3,000	1.59	5.22	0.01	1.59×10^{-4}	0.00628
1,000	4.77	15.6	0.3	4.77×10^{-4}	0.0188
700	6.814	22.5	0.128	6.81×10^{-4}	0.027
500	9.54	31.3	0.6	9.54×10^{-4}	0.0375
300	15.9	52.2	1.0	1.59×10^{-3}	0.0628
200	23.85	78.2	1.5	2.38×10^{-3}	0.094
100	47.7	156	3.0	4.77×10^{-3}	0.188
50	95.4	313	6.0	9.54×10^{-3}	0.375
30	159	522	10.0	1.59×10^{-2}	0.628
10	477	1,560	30.0	4.77×10^{-2}	1.88
1	4,770 (= 4.77 km)	15,600	300.0	4.77×10^{-1}	18.8
6×10^{-5} (house current)	79,500,000 (= 79.5 megameters)	(= 49,500 miles)	5×10^6 (= 5 km)	7.95×10^3 (= 7.95 km)	(= 4.95 miles)

Remarks. For air propagation 1 megameter = 10^6 m = 10^3 km = 621.4 statute miles.
1 m = 3.281 ft
= 39.37 in.

Note: For a 100 000-Mc current, the pure radiation region starts as close as $0.156 \times 12 = 18.72$ in., while for a house current it would require $79.5 \times 621.4 \cong 49{,}500$ miles

On the other hand, the practically pure radiation region occurs for distances greater than 0.156 ft at 100,000 Mc/sec, greater than 0.522 ft at 30,000 Mc/sec, etc. But for the mid-frequency of the standard broadcast range, say for $f = 1$ Mc/sec, we must be 15,600 ft, *i.e.*, as much as 15,600/5,280 = 2.98 miles, away from the current-carrying conductor before a pure radiation field is realized. For a 60-cycle house current, the radiation field would theoretically not set in before a distance of 49,500 miles is reached. This is a very great distance, indeed, since the equatorial radius of the earth is only 3,963.34 miles with a circumference of about 6.28 times this value. This explains why commercial alternating currents, even with sufficient radiator dimensions to bring about a suitable phase distribution, essentially would cause a mixed field and not a radiation field of the pure type. The between-field zone is between the near-field zone and the pure radiation zone.

In formulas (164), the j terms add in quadrature with the real terms. Hence, if we take the case for $r = 1.5\lambda$, Table 3 gives us the values $1/r_e = 0.1065$, $1/r_e^2 = 0.01136$, and $1/r_e^3 = 0.001205$. For the Hertzian doublet expression, we find then for $G = m\omega C \sin\theta$

$$\mathcal{E}_\theta^{(\text{volts/m})} = G(-0.1065 + j0.01136 + 0.001205)$$
$$= G(-0.105295 + j0.01136)$$
$$= G\sqrt{0.105295^2 + 0.01136^2}\;\Big/\tan^{-1}\left(-\frac{0.01136}{0.105295}\right)$$
$$= 0.1053G\;\underline{/173.82^\circ}$$

We have, therefore, an amplitude value $0.1053G$ and a relative time phase $\psi_\theta = 173.82$ deg.

For the radial component, according to Eq. (164), and for the substitution $K = 2m\omega C \cos\theta$, we have

$$\mathcal{E}_r^{(\text{volts/m})} = K\left(\frac{j}{r_e^2} + \frac{1}{r_e^3}\right) = K(j0.01136 + 0.001205)$$
$$= K\sqrt{\underbrace{0.01136^2}_{0.00012996} + \underbrace{0.001205^2}_{0.00000144}}\;\Big/\tan^{-1}\frac{0.01136}{0.001205}$$
$$= 0.0114K\;\underline{/83.95^\circ}$$

For the only magnetic field, according to (164), and the substitution $M = m\omega D$, we find

$$H_\varphi^{(\text{amp/m})} = M\left(-\frac{1}{r_e} + \frac{j}{r_e^2}\right) = M(-0.1065 + j0.01136)$$
$$= M\sqrt{0.1065^2 + 0.01136^2}\Big/\tan^{-1}\frac{0.01136}{-0.1065} = 0.1065M\underline{/173.9^\circ}$$

It is therefore seen that for the $r = 1.5\lambda$ distance, the inverse $1/r_e$ terms are not much affected by the highest power term of $1/r_e$ within a certain bracket.

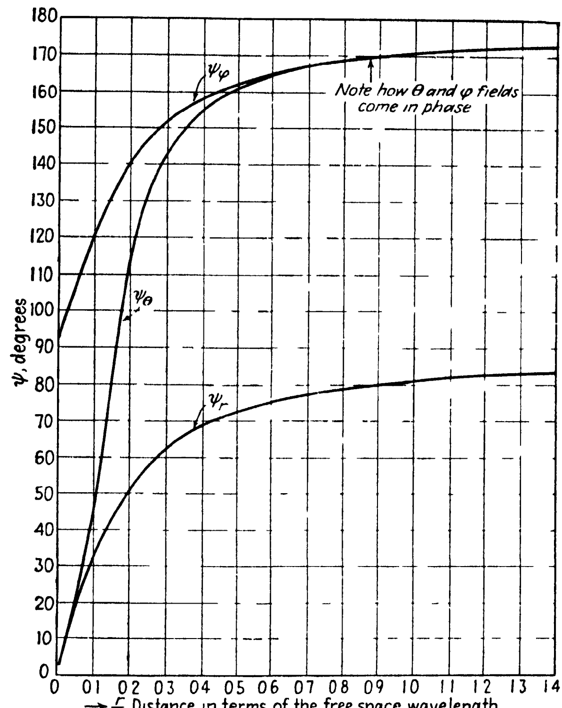


FIG 32. Relative time phase angles of the $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ fields in the near field region (details in Table 5).

In the same way, we can also find the magnitudes and relative phases for the magnetic dipole. Using, for instance, the H_r formula of Table 2 and putting $N = jP \cos\theta \cdot 0.5\beta I\, dS/(\pi r^2)$, we find for $r = 1.5\lambda$

$$H_r^{(\text{amp/m})} = N\left(1 - \frac{j}{r_e}\right) = N(1 - j0.1065)$$

$$= N\sqrt{1 + 0.1065^2}\,/\underline{\tan^{-1}(-0.1065)}$$

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$$= 1.02N/\underline{173.9^\circ}$$

Figure 32 gives the relative ψ_θ, ψ_r, and ψ_φ phases for the $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ fields computed by means of Eq. (164) for a range of $r = 0.01\lambda$ to 1.4λ. This then covers the near or neighborhood field and an important portion of the between field The actual pure radiation region cannot

TABLE 5

$\frac{r}{\lambda}$*	$r_e = \beta r = 2\pi \frac{r}{\lambda}$	Absolute values		
		A_θ	A_r	A_φ
0.01	0.0628	4015	4030	254
0.02	0.1256	501	508	64
0.04	0.251	61	64	16.2
0.06	0.377	17.6	19.7	7.7
0.08	0.502	7.1	8.9	4.45
0.10	0.628	3.51	4.8	3.0
0.12	0.752	2.57	3.56	2.44
0.15	0.94	1.14	1.67	1.6
0.2	1.256	0.71	0.803	1.02
0.25	1.565	0.547	0.475	0.75
0.3	1.88	0.476	0.32	0.6
0.4	2.51	0.37	0.17	0.425
0.5	3.14	0.3	0.106	0.33
0.6	3.77	0.26	0.0725	0.27
0.7	4.4	0.218	0.053	0.23
0.8	5.02	0.2	0.041	0.202
0.9	5.63	0.176	0.032	0.18
1.0	6.28	0.1575	0.026	0.16
1.2	7.52	0.131	0.018	0.133
1.4	8.8	0.111	0.013	0.113
1.5	9.4	0.1053	0.0114	0.106

Remarks (Table 3 is made use of): Refers to Eq. (164)

$$\mathcal{E}_\theta^{(\text{volts/m})} = k_\theta G \left(-\frac{1}{r_e} + \frac{j}{r_e^2} + \frac{1}{r_e^3} \right)$$

$$G = m\omega C \sin\theta; \qquad m = q\,dl; \qquad C = \frac{\beta^3}{4\pi\omega\kappa}$$

$$k_\theta = \cos(\omega t - r_e + \psi_\theta)$$

$$\mathcal{E}_r^{(\text{volts/m})} = k_r K \left(\frac{j}{r_e^2} + \frac{1}{r_e^3} \right)$$

$$K = 2m\omega C \cos\theta; \qquad k_r = \cos(\omega t - r_e + \psi_r)$$

$$H_\varphi^{(\text{amp/m})} = k_\varphi M \left(-\frac{1}{r_e} + \frac{j}{r_e^2} \right)$$

$$M = m\omega D; \qquad D = \frac{\beta^2}{4\pi} \sin\theta$$

$$k_\varphi = \cos(\omega t - r_e + \psi_\varphi)$$

$$A_\theta = \sqrt{\left(\frac{1}{r_e^3} - \frac{1}{r_e}\right)^2 + \frac{1}{r_e^4}}; \qquad A_r = \sqrt{\frac{1}{r_e^6} + \frac{1}{r_e^4}}$$

$$A_\varphi = \sqrt{\frac{1}{r_e^2} + \frac{1}{r_e^4}}$$

* The $r/\lambda = 1$ condition is already in a strong radiation field region, since for $r/\lambda = 0.1592$ we have equal induction and radiation fields.

be realized before both ψ_θ and ψ_φ have a relative phase of 180 deg (time degrees) and ψ_r has reached the 90-deg value.

Table 5 gives the computations of the corresponding amplitude values. The charge q in the Hertzian doublet moment $m = q\,dl$ is oscillating and generally governed by the retarded function ϵ^d, where $d = j(\omega t + \psi - r_e)$ with a real component cos $(\omega t + \psi - r_e)$, where ψ is the relative time phase. In the pure radiation field $\psi_\theta = \psi_\varphi = 180$ time degrees; *i.e.*, $\mathcal{E}_\theta$ and H_φ are in time phase.

The general drift of the factors A_θ, A_r, and A_φ that control the amplitude values of $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ shows that for the largest r/λ given in this table, namely, $r/\lambda = 1.5$, we already obtain factors of 0.1053, 0.0114, and 0.106, which are little affected by the highest power of $1/r_e$, since r_e is larger than unity ($r_e = 9.4$). Even the respective additional relative phases $\psi_\theta = 173.82$ deg, $\psi_r = 83.95$ deg, and $\psi_\varphi = 173.9$ deg are already close enough to true radiation field values $\psi_\theta = \psi_\varphi = 180$ time degrees and $\psi_r = 90$ time degrees. As a matter of fact, the $\mathcal{E}_\theta$ and the H_φ fields are practically in time phase. On the other hand, for r_e values much smaller than unity, the lowest power term of $1/r_e$ within parentheses has only little effect. Table 3 shows that for $r/\lambda = 1.5$, we have $1/r_e = 0.1065$, $1/r_e^2 = 0.01136$, and $1/r_e^3 = 0.001205$. Hence, each higher power term of $1/r_e$ is about 10 per cent of the preceding one. Neglecting 10 per cent is not a serious error in this case, since we deal with quadrature additions, and neglecting a 10 per cent quadrature term is less than 10 per cent error, since $\sqrt{1^2 + (0.1)^2} = \sqrt{1.01} = 1.004988$ as against 1.000000 or only (0.004988)100 = 0.4 per cent. If we think in terms of actual field intensity measurements, they are in many cases not more than 20 per cent reliable, and a 10 per cent quadrature term can be neglected.

The formulas of Table 2, holding for any distance, have the advantage of giving expressions directly in measurable quantities, such as the current I amperes, whether we deal with an electric or a magnetic dipole effect. The parenthesis terms $A' = A/(P \sin\theta)$, $B' = B/(P\cos\theta)$, and $C/(P\sin\theta) = B'$ hold for either dipole and have the advantage for practical applications, since one term is always unity. We have then the absolute values of the parenthesis terms

$$A' = \sqrt{\left(1 - \frac{1}{r_e^2}\right)^2 + \frac{1}{r_e^2}} \qquad \text{and} \qquad B' = \sqrt{1 + \frac{1}{r_e^2}} \tag{169}$$

where B' holds not only for the electric as well as the magnetic dipole but also, in either case, for the longitudinal r component and the transverse φ component. This is timesaving when computing amplitude values. The relative phases due to the parenthesis terms are

$$\left.\begin{aligned} \psi &= \tan^{-1}\left[\frac{-1/r_e}{1 - (1/r_e^2)}\right] = \tan^{-1}\left(\frac{r_e}{1 - r_e^2}\right) \\ \psi_r^\circ &= \psi_\varphi^\circ = \tan^{-1}\left(-\frac{1}{r_e}\right) = \left[180 - \tan^{-1}\left(\frac{1}{r_e}\right)\right]^\circ \end{aligned}\right\} \tag{170}$$

Hence, also the relative ψ_r and ψ_φ time phases are identical for the parentheses of Table 2. Table 6 then gives detailed computations holding for the small-loop radiator as well as the small electric dipole. The phase angles of respective parenthesis values A' and B' are again plotted in Fig. 33 against the ratio r/λ. If we compare these plots with the plots of Fig. 32, which are based on the Hertzian doublet formulas of Eq. (164), we note that the ψ_θ plots are identical in either figure. However, for the formulas of Table 2 and computations in Table 6, we have identical ψ_r and ψ_φ plots in Fig. 33 covering the 90- to 180-deg range. The 180-deg condition is then the one into which the ψ_θ, ψ_r, and ψ_φ pass for larger and larger distances. For both Figs. 32 and 33, we note that from about $r/\lambda =$ 0.7 on, the ψ_θ and ψ_φ plots coincide, even though the ideal condition is only approximated (167.2 instead of 180 deg).

The reason why the ψ_r and ψ_φ plots in Fig. 32 are between 0 to 90 deg and 90 to 180 deg is because all j terms in formulas (164) are inside respective parentheses that determine the relative time phase. For the expressions given in Tables 2 and 6, we have identical corresponding parenthesis values for the electric and magnetic dipole at the expense of having partially j factors before the parentheses of the $\mathcal{E}_\theta$, H_φ and H_r relations. This j factor is another 90-deg phase shift of the φ component with respect to the r component. Hence, the time-phase curves of Fig. 32 should not be confused with the curves of Fig. 33. The curves refer to the parenthesis values in either case.

36. Essential Features of Elementary Electric and Magnetic Dipoles. Figure 17 shows that the magnetic vector potential of an elementary vertical $i\,dl$ radiator at a space point $P(r, \varphi, \theta)$ is also vertical, with a value Λ_z. The resultant vector potential Λ_z as well as the resultant electric field intensity $\mathcal{E}_z$ can be resolved into a meridian θ component and a radial r component. Similar conditions prevail for the magnetic dipole (Fig. 30), except that we deal with an electric vector potential Λ_e, as brought out in Chap. III. With the elementary current loop (elementary magnetic dipole) in the XY plane, we have H_θ, H_r, and $\mathcal{E}_\varphi$ fields instead of $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ fields. We can then use the general formulas of Table 2.

It is now of interest to learn how close a field intensity meter can be brought to a transmitter before the measured value changes, say, by about 5 per cent. The formulas of Table 2 show that both $\mathcal{E}_\theta$ of the electric dipole and H_θ of the magnetic dipole are proportional to the bracket value

$$B_\theta = 1 - \frac{j}{r_e} - \frac{1}{r_e^2} = 1 - j\rho - \rho^2 \tag{171}$$

if ρ denotes the reciprocal of electric distance $r_e = 6.28\ r/\lambda$. We also note that the respective radial components for $\mathcal{E}_r$ in the case of an electric

dipole and H_r in the case of the magnetic dipole are proportional to the common parenthesis value

$$B_r = 1 - jp \tag{172}$$

For both the H_φ field of the electric dipole and the $\mathcal{E}_\varphi$ field of the magnetic dipole, we have the proportionality with the parenthesis value

$$B_\varphi = 1 - jp \tag{173}$$

Hence, B_r and B_φ are identical parenthesis values.

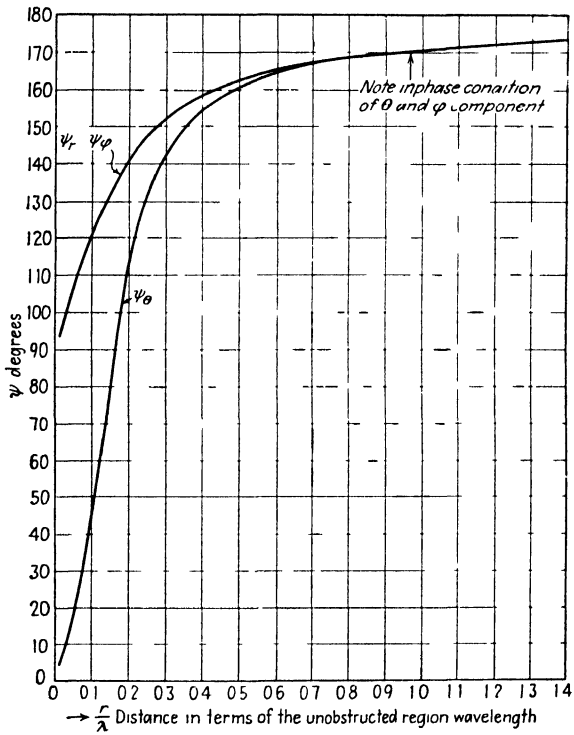


FIG. 33. Relative time phase angle (details in Table 6).

We are now in a position to analyze the effect on the field intensity meter. Such a meter is set for a maximum response on the indicating instrument. The formulas given by the manufacturer refer then to conditions prevailing in the true radiation field for which a pickup loop would point with its plane to the transmitting antenna. This assumes that the induction fields are negligible, if present at all, in the region where readings are taken. The term 1 in the B_θ value represents the effect of the radiation field in respective $\mathcal{E}_\theta$ and H_θ expressions of Table 2,

TABLE 6

$\frac{r}{\lambda}$	$r_e = \beta r = 2\pi \frac{r}{\lambda}$	r_e^4	$\frac{1}{r_e}$	$\frac{r}{1 - r^4}$	$\measuredangle\theta^\circ = \tan^{-1}\left(\frac{r_e}{1 - r_e^4}\right)$	$\measuredangle\psi_r^\circ = \measuredangle\varphi^\circ = \left[180 - \tan^{-1}\left(\frac{1}{r_e}\right)\right]^\circ$	$\frac{1}{r}$	$\left(1 - \frac{1}{r_e^4}\right)^2$	$\left(1 - \frac{1}{r_e^2}\right)^2 + \frac{1}{r_e} = (A')^2$	A'	$1 + \frac{1}{r_e^2} = (B')^2$	B'
0 01	0 0628	0 00394	15 92	0 063	3 6	93 55	254	64 009	64 263	253 6	255	15 97
0 012	0 0752	0 00565	13 3	0 0754	4 3	94 3	177	30 976	31 153	176 5	178	13 34
0 014	0 088	0 00774	11 4	0 0885	5 05	95	129	16 384	16 513	128 4	130	11 4
0 016	0 1002	0 0100	10 0	0 101	5 8	95 7	100	9 801	9 901	99 5	101	10 2
0 018	0 113	0 0128	8 85	0 115	6 6	96 4	78 1	5 944	6 022 1	77 8	79 1	8 85
0 02	0 126	0 0159	7 98	0 1275	7 25	97 3	63 5	3 906	3 969 5	63	64 5	8 02
0 024	0 151	0 0228	6 63	0 1545	8 8	98 6	44	1 849	1 893	43 4	45	6 71
0 03	0 188	0 0353	5 31	0 195	11 05	100 7	28 2	739 8	768	27 71	29 2	5 4
0 036	0 226	0 0511	4 43	0 239	13 45	102 7	19 6	346	365 6	19 1	20 6	4 55
0 04	0 251	0 063	3 98	0 268	15	104 1	15 8	219	234 8	15 35	16 8	4 1
0 045	0 281	0 079	3 55	0 306	17	105 7	12 6	135	147 6	12 15	13 6	3 7
0 05	0 314	0 0986	3 19	0 348	19 2	107 4	10 2	84 6	94 8	9 73	11 2	3 35
0 06	0 377	0 1421	2 65	0 441	23 8	110 7	7 0	36	43	6 56	8 0	2 83
0 07	0 44	0 1936	2 27	0 545	28 4	113 8	5 18	17 47	22 65	4 75	6 18	2 486
0 08	0 502	0 252	1 99	0 675	34	116 7	3 95	8 702	12 652	3 55	4 95	2 225
0 09	0 563	0 317	1 77	0 825	39 5	119 5	3 15	4 622	7 772	2 79	4 15	2 037
0 10	0 628	0 394	1 592	1 03	45 85	122 1	2 54	2 372	4 912	2 21	3 54	1 881
0 12	0 752	0 565	1 33	1 73	60	126 9	1 77	0 593	2 363	1 54	2 77	1 664
0 15	0 94	0 884	1 065	8 1	83	133 2	1 136	0 0196	1 156	1 08	2 136	1 463
0 1592	1 0	1 0	1 0	∞	90	135	1 0	0 0000	1 0000	1 00	2 000	1 410
0 20	1 256	1 588	0 798	−2 14	115	141 4	0 631	0 1369	0 7669	0 88	1 63	1 277
0 25	1 556	2 4336	0 64	−1 095	132 4	147 4	0 407	0 3516	0 7586	0 87	1 407	1 19
0 30	1 88	3 5344	0 531	−0 745	143 3	152	0 282	0 5151	0 797	0 882	1 282	1 13
0 35	2 19	4 7961	0 457	−0 577	150	155 4	0 208	0 6273	0 8353	0 91	1 208	1 1
0 4	2 51	6 3001	0 398	−0 475	154 6	158 3	0 158	0 709	0 867	0 934	1 158	1 08
0 5	3 14	9 8596	0 319	−0 354	160 5	162 3	0 102	0 8064	0 9084	0 953	1 102	1 05
0 6	3 77	14 213	0 265	−0 285	164 1	165 11	0 07	0 865	0 935	0 963	1 07	1 03
0 7	4 4	19 36	0 227	−0 24	166 5	167 2	0 0518	0 899	0 9508	0 974	1 0518	1 025
0 8	5 02	25 2004	0 199	−0 207	168 3	168 7	0 0395	0 922	0 9615	0 98	1 0395	1 02
0 9	5 63	31 6969	0 177	−0 184	169 6	169 95	0 0315	0 9351	0 9666	0 981	1 0315	1 015
1 0	6 28	39 438	0 1592	−0 163	170 7	170 95	0 0254	0 951	0 9764	0 984	1 0254	1 012

1.1	6.9	47.61	0.145	−0.147	171.6		171.75	0.021	0.9584	0.9794	0.987	1.021	1.01
1.2	7.52	56.5504	0.133	−0.136	172.3		172.4	0.0177	0.9643	0.9820	0.99	1.0177	1.008
1.3	8.15	66.4225	0.123	−0.124	172.9		173	0.0151	0.9702	0.9853	0.991	1.0151	1.007
1.4	8.8	77.44	0.114	−0.115	173.4		173.5	0.0129	0.9742	0.9871	0.93	1.0129	1.006
1.5	9.4	88.36	0.1065	−0.107	173.9		173.9	0.01136	0.978	0.9894	0.995	1.0114	1.005

Remarks

$$\mathcal{E}_\theta^{(\text{volts/m})} = j\,\frac{0.5 Z_0 I\,dl}{r\lambda} A'\epsilon^a \sin\theta$$

$$\mathcal{E}_r^{(\text{volts/m})} = \frac{0.5 I\,dl}{\pi r^2} B\,\epsilon^a \cos\theta$$

elementary electric dipoles $i\,dl$, $i = I\epsilon^a$

$$H_\varphi^{(\text{amp/m})} = j\,\frac{0.5 I\,dl}{r\lambda} B\,\epsilon^a \sin\theta$$

$$H_\theta^{(\text{amp/m})} = -\frac{0.25\beta\, I\,dS}{\pi r} A\,\epsilon^a \sin\theta$$

$$H_r^{(\text{amp/m})} = j\,\frac{0.5\beta I\,dS}{\pi r} B\,\epsilon^a \cos\theta$$

elementary magnetic dipole $i\,dS$ (current loop enclosing area dS)

$$\mathcal{E}_\varphi^{(\text{volts/m})} = \frac{0.25 Z_0 \beta^2 I\,dS}{\pi r} B'\epsilon^a \sin\theta$$

$$A = \sqrt{\left(1 - \frac{1}{r_e}\right) + \frac{1}{r}}\ \underline{\angle\psi_\theta}$$

$$B' = \sqrt{1 + \frac{1}{r^2}}\ \underline{\angle\psi_r = \psi_\varphi}$$

$$\psi_\theta = \tan^{-1}\left(\frac{r}{1 - r_e}\right)$$

$$\psi_r = \psi_\varphi = \left[180 - \tan^{-1}\left(\frac{1}{r_e}\right)\right]^\circ$$

For details consult Table 3

$$r_e = \beta r = 6.28r \qquad a = j(\omega t - r_e)$$

$$Z_0 = \sqrt{\frac{\mu}{\kappa}}\ \text{ohms}$$

ψ_θ and $\psi_r = \psi_\varphi$ values are plotted in Figs. 32 and 33

holding for any distance of r meters from the transmitter. The remaining terms have effects only in the near- and between-field region. Hence, when we divide the actual field intensity $\mathcal{E}_\theta$ for the electric and actual

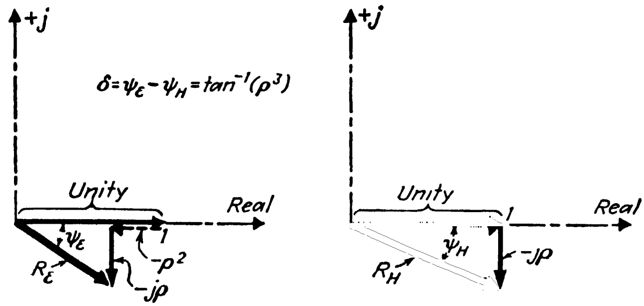


FIG. 34. Graphical solution for the absolute values (magnitudes) of $R_\mathcal{E}$ and R_H

intensity H_θ for the magnetic dipole by the portion $\mathcal{E}_{rad}$ and H_{rad}, respectively, existing in the true radiation field only, we find

$$\left.\begin{aligned} R_\mathcal{E} &= \frac{\mathcal{E}_\theta}{\mathcal{E}_{rad}} = B_\theta; \qquad R_H = \frac{H_\varphi}{H_{rad}} = B_\varphi \quad \text{for electric dipole} \\ R'_H &= \frac{H_\theta}{H_{rad}} = B_\theta; \qquad R'_\mathcal{E} = \frac{\mathcal{E}_\varphi}{\mathcal{E}_{rad}} = B_\varphi \quad \text{for magnetic dipole} \end{aligned}\right\} \tag{174}$$

where for propagation in air

$$\left.\begin{aligned} \mathcal{E}_\theta^{(\text{volts/m})} &= j\frac{188.35 I\,dl}{r\lambda} B_\theta \epsilon^{j(\omega t - 1/\rho)} \sin\theta; \\ \mathcal{E}_\varphi^{(\text{volts/m})} &= \frac{1{,}182.81 I\,dS}{r\lambda^2} B_\varphi \epsilon^{j(\omega t - 1/\rho)} \sin\theta, \\ H_\theta^{(\text{amp/m})} &= -\frac{3.14 I\,dS}{r\lambda^2} B_\theta \epsilon^{j(\omega t - 1/\rho)} \sin\theta; \\ H_\varphi^{(\text{amp/m})} &= j\frac{0.5 I\,dl}{r\lambda} B_\varphi \epsilon^{j(\omega t - 1/\rho)} \sin\theta \end{aligned}\right\} \tag{174a}$$

The product $I\,dl$ stands for the maximum electric dipole moment, and $I\,dS$ is the characteristic product for the elementary current loop enclosing the area dS. We note, therefore, that the ratio of actual field to radiation content is, in each case, equal to the corresponding parenthesis value B. It does not matter therefore whether we analyze the electric or the magnetic elementary dipole, since $R'_\mathcal{E} = R_H = B_\varphi$ and $R'_H = R_\mathcal{E} = B_\theta$. We can then proceed with the B_θ and B_φ values and find the magnitude as well as the relative phase.

Suppose we deal with the electric dipole where the absolute magnitude $R_\mathcal{E}$ and relative phase are

$$R_\mathcal{E} = \sqrt{1 - \rho^2 + \rho^4}\ \underline{/\psi_\mathcal{E}} = \tan^{-1}\left(\frac{-\rho}{1 - \rho^2}\right);$$

$$R_H = \sqrt{1 + \rho^2}\ /\psi_H = \tan^{-1}\rho \qquad (175)$$

since we have complex quantities. Figure 34 gives the graphical solutions for the absolute value of $R_\mathcal{E}$ and R_H. The time phase difference $\delta = \psi_\mathcal{E} - \psi_H = \tan^{-1}\rho^3$ may vary over quite a wide range in the near- and between-field regions, since only in the true radiation region do we have the inphase condition ($\psi_\mathcal{E} = \psi_H$). This expression is based on the derivation

$$\tan\delta = \tan(\psi_\mathcal{E} - \psi_H) = \frac{\tan\psi_\mathcal{E} - \tan\psi_H}{1 + \tan\psi_\mathcal{E}\tan\psi_H} = \frac{\rho/(\rho^2 - 1) - \rho}{1 + [\rho/(\rho^2 - 1)]\rho} = \rho^3$$

We can therefore compute the respective field ratios from

$$\left.\begin{aligned} R_\mathcal{E} = \sqrt{1 - \rho^2 + \rho^4}; \qquad R_H = \sqrt{1 + \rho^2}; \qquad \delta = \tan^{-1}\rho^3 \\ \rho = \frac{0.1592}{r/\lambda} \end{aligned}\right\} \qquad (176)$$

This is applied in Table 7 and plotted in Fig. 35. This figure shows

TABLE 7

$\frac{r}{\lambda}$	$R_\mathcal{E} = \frac{\mathcal{E}_\theta}{\mathcal{E}_{rad}}$	$R_H = \frac{H_\varphi}{H_{rad}}$	δ time phase
0.01	252	15.92	$\cong$ 90°
0.02	62.7	8	$\cong$ 90°
0.1	2.2	1.88	76°
$1/2\pi = 0.1592$	1.0000	1.41	45°
0.18	0.9101	1.335	34°20′
$\sqrt{2}/(2\pi) = 0.2251$	0.866 (min)	1.227	19°30′
0.25	0.8712	1.185	14°30′
$2/2\pi = 0.3184$	0.9014	1.12	7°10′
0.5	0.955	1.05	1°10′
1.0	0.987	1.0126	13.8′
2.0	0.9918	1.003	1.7′
10.0	0.99184	1.00012	0

Remarks (electric dipole). These formulas hold also for the magnetic dipole when $R_H' = H_\theta/H_{rad}$ is used in place of $R_\mathcal{E} = \mathcal{E}_\theta/\mathcal{E}_{rad}$ and $R_\mathcal{E}' = \mathcal{E}_\varphi/\mathcal{E}_{rad}$ in place of $R_H = H_\varphi/H_{rad}$.

again that for a value of $r \geqq 1.5\lambda$, we have about $R_\mathcal{E} \cong R_H = 1$ and time inphase condition prevails between the electric $\mathcal{E}_\theta$ and magnetic H_φ fields. For the magnetic dipole, we have correspondingly $R_H' \cong R_\mathcal{E}' = 1$. It can also be seen that for values of $r/\lambda = p_1$ to 1.5, the H_{rad} experiences an increase and the $\mathcal{E}_{rad}$ a decrease due to the near-region and between-

region effect. *For values of $r/\lambda < p_1$, an increase occurs in each case.* This is then in the neighborhood region of the radiator. For a value of $r/\lambda = 2/(2\pi) = 0.2251$ a minimum value of 0.866 occurs for $R_\mathcal{E}$ with $R_H = 1.227$ and a relative time phase angle of 19 deg 30 min between $\mathcal{E}_\theta$ and H_φ. The same relative time phase exists between $\mathcal{E}_\varphi$ and H_θ, in the case of a magnetic dipole. For a value of $r/\lambda = 1/(2\pi) = 0.1592$, we have $R_\mathcal{E} = 1$ and $R_H = 1.41$ but a phase angle $\delta = 45$ time degrees.

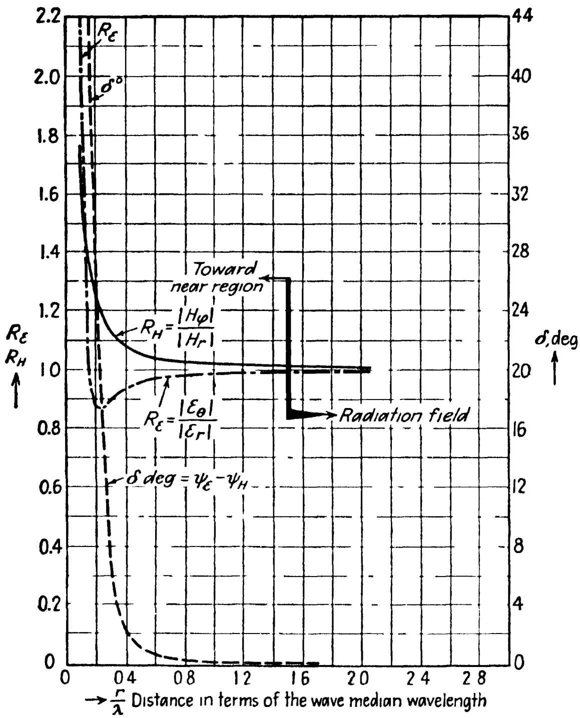


FIG. 35. $R_\mathcal{E}$, R_H, and δ relations as a function of r/λ (details in Table 7).

When we deal with field points close to a radiator (immediate neighborhood field), we find for air as a medium from the corresponding expressions of Table 2

$$\frac{\mathcal{E}_\theta}{\mathcal{E}_r} = 188.35 \tan \theta; \qquad \frac{\mathcal{E}_\theta}{H_\varphi} = -j\frac{376.7}{r_\iota} = -j376.7\rho$$

Hence,

$$\mathcal{E}_r = \frac{\mathcal{E}_\theta}{188.35} \cot \theta \ \text{ volts/m}; \qquad H_\varphi = j\frac{r_\rho \mathcal{E}_\theta}{376.7} \ \text{ amp/m} \qquad (177)$$

Therefore, the radial $\mathcal{E}_r$ component close to the radiator would be for a 45-deg direction with the elementary linear radiator equal to $\mathcal{E}_\theta/188.35$

volts/m. Since $r_e = \beta r = 6.28r/\lambda$ is a small value (r_e is smaller than unity), the magnetic field H_φ close to the elementary linear radiator (holds also for finite linear radiators) cannot be strong, since $r_e \ll 1$ and $\mathcal{E}_\theta$ and H_φ are in time quadrature. An open linear antenna serves then as a more sensitive pickup in such a field region.

On pages 159–160, we have found that the radiation resistance of elementary dipoles is

$$\left.\begin{aligned} R_r &= 789.5\left(\frac{dl}{\lambda}\right)^2 && \text{for the electric dipole of length } dl \\ R_r &= 3.08 \times 10^5\left(\frac{\rho}{\lambda}\right)^4 && \text{for a small current loop of radius } \rho \end{aligned}\right\} \text{ohms} \qquad (178)$$

where in each case uniform current phase is assumed. These relations are the outcome of the total radiated power in unobstructed space and given by

$$\left.\begin{aligned} P &= 2.195 \times 10^{-3} I_e^2\, dl^2 f^2 && \text{for the electric dipole} \\ P &= 9.55 \times 10^{-6} I_e^2 \rho^4 f^4 && \text{for the magnetic dipole} \end{aligned}\right\} \text{watts/surface in m}^2 \qquad (179)$$

If I_e denotes the effective radiator current in amperes and f the frequency in megacycles per second. For a finite length of a radiating conductor, the radiation increases very rapidly on account of the dl^2 and ρ^4 laws. Hence, when a linear radiator of effective length l_e in meters is used, we have $R_r = 789.5(l_e/\lambda)^2$ ohms. The actual length l of the radiator requires, for efficient space radiation, that equal phasing is provided if $l > \lambda/2$. This is taken up in Secs. 105–106 and 108.

37. Field Effects of Two Elementary Electric Dipoles. Suppose we have two elementary electric-current elements of maximum electric-current moments $m_1 = I_1\, dl$ and $m_2 = I_2\, dl$, respectively, expressed in meter · amperes and spaced a distance s meters apart; then we have, according to Table 2 and Eq. (159), at the location of the $I_2\, dl$ current element, due to the $I_1\, dl$ field effect, $\sin\theta = 1$ and the electric field contribution

$$d\mathcal{E}_{1,2}^{(\text{volts/m})} = j\,\frac{0.25\omega\mu\, I_1\, dl}{\pi s}\left[1 + \frac{1}{s\Gamma} + \frac{1}{(s\Gamma)^2}\right]\epsilon^{j\omega(t - s\sqrt{\mu\kappa_e})}$$

since $I_2\, dl$ is parallel with $I_1\, dl$ and s is the distance to the $I_1\, dl$ source. *All* terms in the brackets must be used, since in most cases the two current elements may be close to each other. For wave propagation in air, we have the propagation constant $\Gamma = j\beta = j\,\dfrac{\omega}{c}$ and $\sqrt{\mu\kappa_e} = \sqrt{\mu_0\kappa_0} = 1/c = 10^{-8}/3$, and

$$d\mathcal{E}_{1,2} = \frac{I_1\, dl\epsilon^{j\omega\left(t-\frac{s}{c}\right)}}{j\omega 4\pi\kappa_0 s^3} + \frac{I_1\, dl\epsilon^{j\omega\left(t-\frac{s}{c}\right)}}{4\pi c\kappa_0 s^2} + \frac{j\omega I_1\, dl\epsilon^{j\omega\left(t-\frac{s}{c}\right)}}{4\pi c^2\kappa_0 s} \quad \text{volts/m} \quad (180)$$

and the corresponding voltage $dE_{1,2}$ induced in the $I_2\, dl$ radiator is

$$dE_{1,2} = (d\mathcal{E}_{1,2})\, dl = \frac{(dl)^2}{4\pi\kappa_0 s^3} I_1\epsilon^{j(\omega t-\psi)} \underbrace{\left(\frac{1}{j\omega} + \frac{s}{c} + \frac{j\omega s^2}{c^2}\right)}_{\left(-\frac{j}{\omega}\right)(1+j\psi-\psi^2)} \quad \text{volts}$$

for $\psi = \beta s = (\omega/c)s$. Hence,

$$dE_{1,2} = \frac{(dl)^2 I_1}{4\pi\kappa_0\omega s^3}[\cos\underbrace{(\omega t - \psi)}_{\delta} + j\sin\delta](-j + \psi + j\psi^2)$$

$$= \frac{(dl)^2 I_1}{4\pi\kappa_0\omega s^3}(-j\cos\delta + \sin\delta + \psi\cos\delta + j\psi\sin\delta + j\psi^2\cos\delta - \psi^2\sin\delta)$$

$$= \underbrace{\frac{(dl)^2 I_1}{4\pi\kappa_0\omega s^3}[\psi\cos\delta + (1-\psi^2)\sin\delta]}_{\substack{\text{active induced volts in}\\ I_2\,dl\text{ due to }I_1\,dl\text{ radiator}}} + \underbrace{j\frac{(dl)^2 I_1}{4\pi\kappa_0\omega s^3}[(\psi^2-1)\cos\delta + \psi\sin\delta]}_{\text{reactive volts due to } I_1\,dl \text{ radiator}}$$

It is only the active voltage (real voltage) that can do *work* in the $I_2\, dl$ elementary radiator owing to the $I_1\, dl$ source. Since the real part, which is then of concern, is composed of *a* cos δ and *a* sin δ function, we have for the amplitude of the induced active voltage

$$dE_{1,2} = \frac{(dl)^2 I_1}{4\pi\kappa_0\omega s^3}\sqrt{\psi^2 + (1-\psi^2)^2} \quad \text{maximum volts} \quad (181)$$

The $I_2\, dl$ radiator affects the $I_1\, dl$ radiator in exactly the same way, and we find the induced active voltage in radiator 1 due to radiator 2 as

$$dE_{2,1} = \frac{(dl)^2 I_2}{4\pi\kappa_0\omega s^3}\sqrt{\psi^2 + (1-\psi^2)^2} \quad \text{maximum volts;}$$

$$\psi = \beta s = \frac{2\pi}{\lambda} s = \frac{\omega}{c} s \quad (182)$$

If the two elementary radiators were not coupled, we would have

$$\left.\begin{aligned} dP_s^{(\text{watts})} &= dP_1 = R_1\frac{I_1^2}{2} \quad \text{for the } I_1\, dl \text{ radiator} \\ dP_s^{(\text{watts})} &= dP_2 = R_2\frac{I_2^2}{2} \quad \text{for the } I_2\, dl \text{ radiator} \end{aligned}\right\} \begin{array}{l} R_1 \text{ and } R_2 \\ \text{are radia-} \\ \text{tion resist-} \\ \text{ances} \end{array} \quad (183)$$

The coupling, *i.e.*, the interaction between respective elementary radi-

ators, causes an increase of the energy flux radiated from each elementary radiator. Assuming that a phase difference ψ_1 also exists between the 1 and the 2 elementary radiator currents, we have,

$$I_1\epsilon^{j\omega t} \qquad \text{and} \qquad I_2\epsilon^{j(\omega t-\psi_1)}$$

and

$$\left.\begin{aligned}
dP_{1,2} &= d\mathcal{E}_{1,2}I_2\,dl = dE_{1,2}I_2 = \frac{(dl)^2I_1}{4\pi\kappa_0\omega s^3}[\psi\cos(\omega t-\psi) \\
&\qquad + (1-\psi^2)\sin(\omega t-\psi)]I_2\cos(\omega t-\psi_1) \\
dP_{2,1} &= d\mathcal{E}_{2,1}I_1\,dl = dE_{2,1}I_1 = \frac{(dl)^2I_2}{4\pi\kappa_0\omega s^3}[\psi\cos(\omega t-\psi-\psi_1) \\
&\qquad + (1-\psi^2)\sin(\omega t-\psi-\psi_1)]I_1\cos\omega t \\
&\left.\begin{aligned}
&\frac{1}{T}\int_0^T \cos(\omega t-\psi)\cos(\omega t-\psi_1)\,dt = \cos(\psi-\psi_1) \\
&\frac{1}{T}\int_0^T \sin(\omega t-\psi)\cos(\omega t-\psi_1)\,dt = -\sin(\psi-\psi_1) \\
&\frac{1}{T}\int_0^T \cos(\omega t-\psi-\psi_1)\cos\omega t\,dt = \cos(\psi+\psi_1) \\
&\frac{1}{T}\int_0^T \sin(\omega t-\psi-\psi_1)\cos\omega t\,dt = -\sin(\psi+\psi_1)
\end{aligned}\right\}\ \text{average values of time functions}
\end{aligned}\right\} \quad (184)$$

therefore, the average values of the mutual additional energy flux contributions

$$\left.\begin{aligned}
&\left.\begin{aligned}
dP_{1,2} &= \frac{(dl)^2I_1I_2}{4\pi\kappa_0\omega s^3}[\psi\cos(\psi-\psi_1) - (1-\psi^2)\sin(\psi-\psi_1)] \\
dP_{2,1} &= \frac{(dl)^2I_1I_2}{4\pi\kappa_0\omega s^3}[\psi\cos(\psi+\psi_1) - (1-\psi^2)\sin(\psi+\psi_1)]
\end{aligned}\right\}\ \text{watts} \\
&I = I_{\text{eff}}\sqrt{2} \qquad \text{amp} \\
&\psi = \beta s = \frac{2\pi}{\lambda}s = \frac{\omega}{c}s = \omega s\sqrt{\mu_0\kappa_0} = \frac{\omega s}{3\times10^8}
\end{aligned}\right\} \quad (185)$$

In (183), we found that the respective *self*-radiation powers $dP_s = dP_1$ and $dP_s = dP_2$ are $dP_1 + dP_2$ watts if there were no *mutual* field interference, *i.e.*, the $I_1\,dl$ radiator would not affect the $I_2\,dl$ radiator and vice versa. This total self-radiation flux has to be added to the mutual energy flux

$$dP_m = dP_{1,2} + dP_{2,1} = \frac{(dl)^2 I_1 I_2}{4\pi\kappa_0\omega s^3} \{\psi\underbrace{[\cos(\psi+\psi_1) + \cos(\psi-\psi_1)]}_{2\cos\psi\cos\psi_1} - (1-\psi^2)\underbrace{[\sin(\psi+\psi_1)+\sin(\psi-\psi_1)]}_{2\sin\psi\cos\psi_1}\}$$

$$= \frac{(dl)^2 I_1 I_2 \cos\psi_1}{2\pi\kappa_0\omega s^3} [\psi\cos\psi - (1-\psi^2)\sin\psi]$$

$$= I_1 I_2 \cos\psi_1 \underbrace{\left\{[\psi\cos\psi - (1-\psi^2)\sin\psi]\frac{(dl)^2}{2\pi\kappa_0\omega s^3}\right\}}_{R_m = R_{1,2} + R_{2,1} \text{ ohms}} \text{ watts} \quad (186)$$

Since $dP_m = I_{1\text{eff}} I_{2\text{eff}} \cos\psi_1 R_m = \frac{1}{2} I_1 I_2 \cos\psi_1 R_m$ is the *mutual energy transfer*, we have $R_m = R_{1,2} + R_{2,1}$ ohms as the *total mutual* radiation resistance of the $I_1\, dl$ and $I_2\, dl$ radiators, spaced a distance of s meters. Since $\psi = 2\pi s/\lambda$, $\kappa_0 = 8.854 \times 10^{-12}$ farad/m; $\omega = 6.28f = 6.28c/\lambda = 6.28 \times 3 \times 10^8/\lambda$, we find for the mutual radiation resistance

$$R_m^{(\text{ohms})} = 9.56\frac{(dl)^2\lambda}{s^3}\left\{6.28\frac{s}{\lambda}\cos\left(\frac{s}{\lambda}360°\right) - \left[1 - \left(6.28\frac{s}{\lambda}\right)^2\right]\sin\left(\frac{s}{\lambda}360°\right)\right\} \quad (187)$$

where the wavelength along the radiator is also in meters. It is noted that the mutual radiation resistance R_m is *not* affected by the phase ψ_1 between the currents I_1 and I_2 in the respective elementary radiators and *also not* by the magnitudes of I_1 and I_2 for the maximum time values. The relative phase and the magnitude enter, however, in the power relation for dP_m.

We have for a very small spacing s, in terms of the wavelength, for

$$x = 6.28\frac{s}{\lambda} \ll 1$$

$$R_m = 9.56\frac{(dl)^2}{s^3}[x\cos x - (1-x^2)\sin x] \cong 1{,}580\left(\frac{dl}{\lambda}\right)^2 \quad \text{ohms} \quad (188)$$

since

$$\cos x = 1 - \frac{x^2}{2!} + \underbrace{\frac{x^4}{4!} - \cdots}_{\text{neglected}} \cong 1 - \frac{x^2}{2}; \qquad \sin x = x - \frac{x^3}{3!} + \underbrace{\frac{x^5}{5!} - \cdots}_{\text{neglected}} \cong x - \frac{x^3}{6}$$

$$x\cos x - (1-x^2)\sin x = x\left(1 - \frac{x^2}{2}\right) - (1-x^2)\left(x - \frac{x^3}{6}\right)$$

$$= x - \frac{x^3}{2} - x + \frac{x^3}{6} + x^3 - \underbrace{\frac{x^5}{6}}_{\text{neglected}} \cong -\frac{x^3}{2} + \frac{x^3}{6} + x^3 \cong (1.167 - 0.5)x^3 = 0.667x^3$$

$$9.56\frac{(dl)^2\lambda}{s^3}\,0.667\left(\frac{6.28s}{\lambda}\right)^3 = 1{,}580\left(\frac{dl}{\lambda}\right)^2$$

We note that for $6.28s/\lambda \ll 1$, the radiation resistance due to the combined mutual effect is essentially equal to the self-radiation resistance of both elementary $I\,dl$ radiators, since each $I\,dl$ radiator has a radiation resistance of $80\pi^2(dl/\lambda)^2$.

38. Application of the Radiation Theory of Two Parallel Elementary Radiators to the Lecher Wire Feeder. The results of the foregoing section can be simply applied to a parallel-line system for which the length l in meters does not exceed one-half wavelength along the line. If such a length[1] is not exceeded, the go and the return conductor may be taken

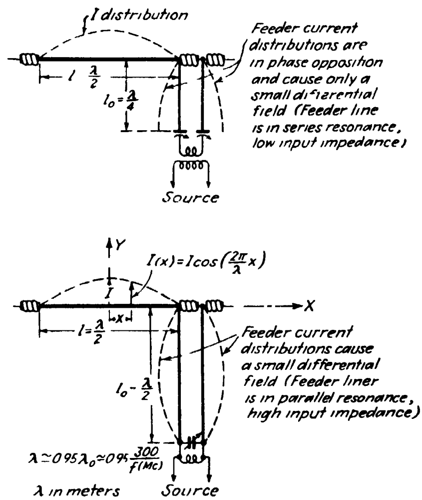


FIG. 36. Excitations of half-wave linear radiators by means of quarter-wave and half-wave Lecher wire feeders.

as though they were elementary parallel radiators if the effective length $l_e = (1/I)\int I(x)\,dx$ is used in place of the elementary length dl. Lecher wire feeders with wave distributions are used for causing a definite phase at feeder points due to the phase lag βl of the line if the actual length of the line is l meters and $\beta = 2\pi/\lambda$ is the phase constant of the feeder with λ acting along it. The wavelength λ along a feeder is about 5 per cent shorter than the free-space wavelength λ_0. When a half-wave linear radiator is excited by a Lecher feeder, this can be done by using an open-end quarter-wavelength feeder ($l = \frac{1}{4}\lambda$). One open end is then connected to one end of the radiator and the other open end is free for both the feeder and the radiator (Fig. 36). Since for such a condition, a current maximum exists at the input side ($l = 0$) of the Lecher feeder, we have a cosinoidal current distribution toward the open end, and

[1] If we had several equal-phase half-wave distributions along a line, we could also use the procedure described in this section without mentioned limitation.

$$l_e = \frac{1}{I}\int_{x=0}^{x=l=\lambda/4} I\cos\frac{2\pi}{\lambda}x\,dx = \frac{1}{I}\left|I\frac{\lambda}{2\pi}\sin\frac{2\pi}{\lambda}x\right|_{x=0}^{x=\lambda/4} = \frac{\lambda}{2\pi}\underbrace{(\sin 90^\circ - \sin 0)}_{=1}$$

$$= \frac{4l}{2\pi} = 0.636l$$

Now the self-radiation resistance of an elementary radiator is $R_r = 789.5(dl/\lambda)^2$ ohms. Therefore, for the go conductor, we have $R_1 = 789.5(l_e/\lambda)^2$ ohms and for the return conductor the same value as far as the self-effects are concerned, since by the effective length is meant a length that in equivalence has a fixed current distribution I. In unobstructed[1] radiation space

$$R_1 = R_2 = 789.5\left(\frac{l_e}{\lambda}\right)^2 = 789.5\left(\frac{0.636l}{\lambda}\right)^2 = 789.5\left(\frac{0.636\,\frac{\lambda}{4}}{\lambda}\right)^2$$
$$= 789.5 \times 0.159^2 \cong 20\Omega$$

The current in the go conductor along the equivalent length l_e is then in effect $+I_1$ amperes and in the return conductor in effect $-I_2$ amperes, where I_1 and I_2 have same magnitudes, namely, I maximum amperes. If I_{eff} is the corresponding effective value, the self-radiation effects of both conductors yield

$$P_s = P_1 + P_2 = R_1I_{\text{eff}}^2 + R_2(-I_{\text{eff}})^2 = (R_1 + R_2)I_{\text{eff}}^2$$

which would be $2 \times 20I_{\text{eff}}^2$ watts if the 789.5 constant instead of the 1,579 constant is used in the elementary radiation resistance formula. According to (185) and (186), the mutual radiation resistance R_m is the sum of $R_{1,2}$ and $R_{2,1}$ ohms, since the go conductor affects the return conductor and vice versa. The mutual radiation power is, according to (186), if expressed in rms value

$$P_m = R_mI_{\text{eff}}I_{\text{eff}}\cos\psi_1 \text{ watts} = -R_mI_{\text{eff}}^2 \qquad \text{watts}$$

since $\cos\psi_1 = \cos 180 \text{ deg} = -1$, because the go current is in antiphase with the return current. Therefore, the total radiated power from the $l = \lambda/4$ Lecher wire feeder is

$$P_r = P_s - P_m = \underbrace{(2 \times 20 - R_m)}_{R_r}I_{\text{eff}}^2 \qquad \text{watts}$$

[1] We deal now with a radiator of actual physical length l which is expressed in terms of the equivalent electric current moment l_eI meter · amperes just like $I\,dl$ meter · amperes for the maximum time value of the elementary linear radiator, for which in unobstructed space $R_r = 789.5(dl/\lambda)^2$. If, however, the radiator acts also with its image effect (standing on a perfect reflector), then we have $R_r = 2 \times 789.5 = 1{,}579(dl/\lambda)^2$ ohms and, therefore, also twice the value for the $\lambda/4$ radiator (2×20 ohms).

The value of R_m is computed from (187) by using $l_e = 0.636l = 0.636\lambda/4$ in place of the elementary length dl.

For a spacing of 10 cm between centers of the Lecher wires, we have $s = 0.1$ m or 10^{-1} m and $s^3 = 10^{-3}$. For a 100-Mc excitation in the feeder, we have a free-space wavelength $\lambda_0 = {}^{300}\!/_{100} = 3$ m and a wavelength λ along the Lecher feeder of about $0.95 \times 3 = 2.85$ m (since it is about 5 per cent shorter). We find then $s/\lambda = 10^{-1}/2.85 = 3.5 \times 10^{-2}$, $6.28s/\lambda = 2.2 \times 10^{-1}$, $(6.28s/\lambda)^2 = 4.84 \times 10^{-2}$, $360s/\lambda = 360 \times 3.5 \times 10^{-2} = 12.6$ deg, $l_e = 0.636\lambda/4 = 0.159\lambda$, $l_e^2 = 2.53 \times 10^{-2}\lambda^2$, $l_e^2\lambda = 2.53 \times 10^{-2}\lambda^3 = 2.53 \times 10^{-2} \times 2.85^3 = 2.53 \times 10^{-2} \times 23.15 = 5.88 \times 10^{-1}$. Therefore,

$$R_m = 9.56 \frac{(l_e)^2\lambda}{s^3} \left\{ 6.28 \frac{s}{\lambda} \cos\left(\frac{s}{\lambda} 360°\right) - \left[1 - \left(6.28 \frac{s}{\lambda}\right)^2\right] \sin\left(\frac{s}{\lambda} 360°\right) \right\}$$

$$= 9.56 \frac{5.88 \times 10^{-1}}{10^{-3}} \underbrace{[\underbrace{2.2 \times 10^{-1} \underbrace{\cos 12.6°}_{0.9759}}_{0.214} - \underbrace{\underbrace{(1 - 0.0484)}_{0.9516} \underbrace{\sin 12.6°}_{0.2181}}_{0.2075}]}$$

$$= 9.56 \times 588 \times 6.5 \times 10^{-3} = 36.5\Omega$$

The total radiation resistance is then $R_r = R_1 + R_2 - (R_{m12} + R_{m21}) = 2 \times 20 - 36.5 = 3.5\Omega$, which is a noticeable value. If the effective current impressed at the input of the quarter-wavelength feeder were 1 amp, the feeder radiation loss would be 3.5 watts. The best way of reducing this loss would be to decrease the spacing between go and return conductors to a suitable value. The spacing of this example was $s = 0.1$ m, or 0.1×39.37 in., or 3.937 in. If we take half this spacing, we find $s = 0.05 = 5 \times 10^{-2}$, $s^3 = 1.25 \times 10^{-4}$, $s/\lambda = 5 \times 10^{-2}/2.85 = 1.76 \times 10^{-2}$, $6.28s/\lambda = 6.28 \times 1.76 \times 10^{-2} = 1.1 \times 10^{-1}$, $(6.28s/\lambda)^2 = 1.21 \times 10^{-2}$, $360s/\lambda = 360 \times 1.76 \times 10^{-2} \cong 6.3$ deg, $\sin 6.3$ deg $= 0.1097$, $\cos 6.3$ deg $= 0.994$, and

$$R_m = 9.56 \frac{5.88 \times 10^{-1}}{1.25 \times 10^{-4}} \underbrace{[\underbrace{1.1 \times 10^{-1} \times 0.994}_{0.1093} - \overbrace{(\underbrace{1 - 0.0121}_{0.9879})0.1097}^{0.108384})]}_{9 \times 10^{-4}} = 40.5\Omega$$

and $R_r = 40 - 40.5 = 0.5$ ohm. Hence, still a smaller spacing has to be used to cut down the radiation loss to a negligible value at 100 Mc/sec.

39. Radiation from a Half-wave Lecher Feeder. When twice the frequency $f = 200$ Mc/sec is employed for the same actual length l of the double line and likewise with a half-wave linear radiator as the load, we have a half-wave distribution along the Lecher feeder. One end of the open-end parallel wire line is then again connected to one end of the linear $\lambda/2$ radiator. The driving voltage at the input end of the Lecher feeder is then taken from a parallel resonance tank in order to secure the required high-impedance match (essentially high-resistance match) instead of the series resonance feed at the input side of the line. We have then the respective arrangements of Fig. 36 with an effective length

$$l_e = \frac{1}{I} \left| I \frac{\lambda}{2\pi} \sin \frac{2\pi}{\lambda} x \right|_{x=-\lambda/4}^{x=+\lambda/4=l/2} = \frac{\lambda}{2\pi} [\sin 90° - \sin(-90°)]$$

$$= \frac{\lambda}{2\pi} \underbrace{(\sin 90 + \sin 90)}_{2} = \frac{\lambda}{\pi} = \frac{2l}{\pi}$$

Hence, l_e is again 0.636 times the actual length l of the feeder, which would be expected, since the feeder consists of two ¼λ sections. We find also for

$$s = 0.05 = 5 \times 10^{-2}\,\mathrm{m}; \quad s^3 = 1.25 \times 10^{-4}, \quad \lambda_0 = {}^{300}\!/_{200} = 1.5\,\mathrm{m}, \quad \lambda \cong 0.95 \times 1.5 = 1.425\,\mathrm{m}$$

$$s/\lambda = 5 \times 10^{-2}/1.425 = 3.52 \times 10^{-2}; \quad 6.28s/\lambda = 2.2 \times 10^{-1}; \quad (6.28s/\lambda)^2 = 4.84 \times 10^{-2}$$

$$360s/\lambda = 12.6°$$

The value within brackets is then, as found for twice the spacing for the ¼λ feeder length, *i.e.*, { } = 6.5 × 10^{-3}, and

$$R_m = 9.56 \frac{\left(0.636 \frac{\lambda}{2}\right)^2 \lambda}{1.25 \times 10^{-4}} 6.5 \times 10^{-3} = 9.56 \frac{0.318^2 \times 1.425^3}{1.25 \times 10^{-4}} 6.5 \times 10^{-3}$$

$$= \frac{9.56 \times 1.012 \times 10^{-1} \times 2.893 \times 6.5 \times 10^{-3}}{1.25 \times 10^{-4}} = 145\Omega$$

Since the self-radiation resistances of each conductor are for $l = \lambda/2$, twice the values as for $l = \lambda/4$, we have $R_1 \cong 2 \times 20\Omega = R_2$, and the total radiation resistance $R_r = R_1 + R_2 - R_m = 4 \times 40 - 145 = 160 - 145 \cong 15$ ohms and 15 watts would be radiated for the effective loop current value of 1 amp on the Lecher feeder.

The above numerical evaluations assume unobstructed radiation, and the substitution of the elementary length dl by the effective length l_e is permissible only if the actual length does *not exceed one-half wavelength* distribution along the feeder. If $l > ½\lambda$, we have radiation interference from the in- and antiphase sections of one and the same conductor (go or return, respectively), and the method given on page 498 must be applied. In this reference, the case for traveling waves along the feeder is also treated.

40. Remarks on the Radiation Effects of Dipoles in the Neighborhood of Obstacles. If we deal with a plane surface that has a high conductivity, such as copper, we may say that the plane surface separating, for instance, a medium such as air from a copper medium behaves essentially as a perfect electric mirror. Hence, if an electric dipole is oriented *normal* to the surface, it has an image dipole just as far behind the surface as the actual dipole is in front of it. For VHF and UHF currents, a very thin copper skin acts as just as good an electric mirror as does a copper surface with a thickness to it. The electric image current is then just as large and in phase with the actual dipole current. For an electric linear radiator *parallel* with such an electric mirror and in front of it, the image linear radiator is also just as far behind the mirror as the actual radiator is in front of it but carries an equal *antiphase* image current distribution. It must be realized, however, that radiation energy can be diverted *only in front* of the electric mirror, just as a light flux from a candle, in front of an optical mirror, can act only in front of the reflecting mirror. If the reflecting surface is partially conductive and partially a dielectric, we obtain *blurred* (pp. 850, 882) electromagnetic image effects.

CHAPTER III

FUNDAMENTAL METHODS USED IN ELECTROMAGNETIC THEORY

So far, we have applied formulas that are the outcome of the electromagnetic theory and shall continue to do so. In some cases, a basic knowledge of the mathematical procedures is necessary, for which reason this chapter is included in order to bring out the assumptions made and the limitations of certain methods. Thus, for instance, the Poynting vector method, as generally applied, has held its place ever since it was derived in 1884, even though it is based on a volume integral that gives only the entire energy of a field region. We have, therefore, not an exact proof of how the magnetic and electric field energies are distributed. A method that applies to all radiation cases is based on the solution of Maxwell's equations, which satisfy all boundary conditions involved. We can then find the field in the radiation region as well as the current distribution along a radiator. This is not exactly an easy procedure when the shape of the radiator is not simple. We have also procedures that are easier. One such procedure requires the knowledge of the field distribution next to the radiator, while another procedure requires a knowledge of the potential and current distributions along the radiator. As far as the radiated power is concerned, we can compute it *without* the Poynting theorem. We then utilize the law of conservation of energy, which states that the radiated power must come from the radiator. Therefore, the current at points of a radiator must flow against an emf that has an inphase component. The counter emf is then proportional to the current with an inphase component, which is the current times the radiation resistance. We can also use the Hertz vector (a single vector) instead of the electric and magnetic field values when the current and potential distributions are known (or are assumed).

41. Review and Compilation of Circuit and Electromagnetic Relations in MKS Units.[1] In customary networks we are concerned with E volts, I amperes, the power in watts (time rate of change in energy), and gen-

[1] A detailed derivation based on laws by Gauss, Ampère, and Faraday, including Maxwell's conception of displacement current is given in Hund, "Phenomena in High-frequency Systems," Chap. IX, starting on p. 324, McGraw-Hill Book Company, Inc., New York, 1936. This review presents an abstract of essential relations in the more up-to-date mks units.

erally a load impedance in ohms with a real part, which is the ohmic resistance, and an imaginary part, which may be positive or negative and is known as the *"reactance in ohms," for harmonic excitations.*

If we deal with electric and magnetic fields in a medium of μ henrys per meter, κ farads per meter, and σ mhos per meter, we have at a space point the electric field intensity $\mathcal{E}$ volts per meter, the magnetic field intensity H amperes per meter and an electromagnetic power flow of p watts per square meter. The quantity p is known as the useful component of the complex Poynting vector.

In circuit analysis, we employ the magnetic flux density B, *i.e.*, magnetic lines of flux per square meter (webers per square meter) and electric flux density D coulombs per square meter, *i.e.*, electric lines of flux per unit area. The latter is also known as "electric displacement $D = \kappa\mathcal{E}$," while the magnetic flux density is also known as "magnetic induction $B = \mu H$," where κ and μ for a vacuum or empty space are $\kappa = \kappa_0 = 8.854 \times 10^{-12}$ farad/m and $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m.

For any medium we have, therefore, the electric and magnetic polarizations

$$P_e = D - \kappa_0\mathcal{E} = \overbrace{\kappa\mathcal{E} - \kappa_0\mathcal{E}}^{\text{contribution over vacuum}}; \qquad P_m = \overbrace{\frac{B}{\mu_0} - \frac{B}{\mu}}^{\text{contribution over vacuum}} = \frac{B}{\mu_0} - H \tag{189}$$

which are vectors of a medium (*due to matter*) and must, therefore, vanish for empty space. The quantities P_e and P_m are then the respective contributions to the induction due to the presence of a dielectric substance or the presence of a magnetic substance. Hence, for vacuum, $P_e = P_m = 0$ and $D = \kappa_0\mathcal{E}$ coulombs per square meter and $B = \mu_0 H$ webers per square meter. For a medium, *i.e.*, a *space with substance*, we have then the relations

$$D = P_e + \underbrace{\kappa_0\mathcal{E}}_{\text{for vacuum}} = \kappa\mathcal{E}; \qquad H = \underbrace{\frac{B}{\mu_0}}_{\text{for vacuum}} - P_m = \frac{B}{\mu} \tag{190}$$

where

$$\left.\begin{aligned} &\kappa = \frac{P_e}{\mathcal{E}} + 8.854 \times 10^{-12} \text{ farad/m}; \\ &\qquad \frac{1}{\mu} = \frac{10^6}{1.257} - \frac{P_m}{B} \quad \text{m/henry} \\ &1/\mu = \text{magnetic space reluctivity} \end{aligned}\right\} \tag{191}$$

In the classic Maxwellian theory[1] use is made of the Gauss theorem, Ampère's law, Faraday's law, and Maxwell's displacement current.

[1] *Ibid.*

Using Cartesian coordinates, the electric displacement has a resultant magnitude given by $D = \sqrt{D_x^2 + D_y^2 + D_z^2}$, where generally we have three components D_x, D_y, and D_z as in the case of the magnetic induction $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$ with respective components B_x, B_y, and B_z.

According to Gauss, the electric flux diverging from an electric charge is equal to the value of the charge. This means that the total normal induction over any closed surface is equal to the total electric charge surrounded by the surface. If ρ coulombs per cubic meter denotes the volume density of the electric charge, the classic Gauss relation reads

$$\rho = \text{div } D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{192}$$

since the divergence of $D = \kappa\mathcal{E}$ is defined by the total cross-surface flux with respect to the enclosed volume, or

$$\text{Average div } D = \frac{\int\int_s D\,dS}{\Delta x \cdot \Delta y \cdot \Delta z}$$

and divergence of D (written div D) at any space point holds for a very small volume $\Delta x\, \Delta y\, \Delta z$ (parallelepiped for which ρ is uniformly distributed) shrinking to a point ($\rightarrow 0$) with a surrounding surface $S \rightarrow 0$. In this expression, D acts *normal* to the surface element dS and *outward.* Relation (192) can also be written as a *scalar* or *dot* product[1] $\nabla \cdot D = \rho$, where strictly speaking the operator ∇ is a *directional* derivative of a scalar and given by $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\cdot$ Therefore, it also includes the unit vectors i, j, and k along the X, Y, and Z directions. For the sake

[1] For two vectors $\underset{\cdot}{A}$ and $\underset{\cdot}{B}$ of scalar values A and B making an angle θ with each other, we have the *interior* or *scalar* product $\underset{\cdot}{A} \cdot \underset{\cdot}{B} = AB \cos \theta$, which is also known as the "dot" product. The *exterior* or *vector* product is also known as the "cross" product $\underset{\cdot}{A} \times \underset{\cdot}{B} = AB \sin \theta$. Therefore, this product has a magnitude $AB \sin \theta$ and a direction perpendicular to $\underset{\cdot}{A}$ and $\underset{\cdot}{B}$ vectors. Assuming i, j, and k unit vectors along the X, Y, and Z directions, we have the scalar product $\underset{\cdot}{A} \cdot \underset{\cdot}{B} = A_xB_x + A_yB_y + A_zB_z$ and the vector product

$$\underset{\cdot}{A} \times \underset{\cdot}{B} = \underbrace{i(A_yB_z - A_zB_y)}_{(\underset{\cdot}{A} \times \underset{\cdot}{B})_x} + \underbrace{j(A_zB_x - A_xB_z)}_{(\underset{\cdot}{A} \times \underset{\cdot}{B})_y} + \underbrace{k(A_xB_y - A_yB_x)}_{(\underset{\cdot}{A} \times \underset{\cdot}{B})_z}$$

since we have the determinant

$$\underset{\cdot}{A} \times \underset{\cdot}{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

With the *cross* product, it is an *important* rule that $\underset{\cdot}{A} \times \underset{\cdot}{B} = -\underset{\cdot}{B} \times \underset{\cdot}{A}$, since the perpendicular vector $AB \sin \theta$ is in the direction of the *right*-hand screw penetrating the AB plane.

of simplicity, in many relations of this text these unit vectors are ignored, but nevertheless understood.

Inasmuch as the magnetic lines are closed, the net outward flux from a volume of 1 cu m must vanish, since just as many B lines return as leave the enveloping surface. Hence, $\nabla \cdot B = 0$ or div $B = 0$, and

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{193}$$

For no sources in a medium (*solenoidal* condition), both div $D = 0$ and div $B = 0$. The *divergence of a vector P is a scalar;* and according to the above relations,

$$\text{div } P \equiv \nabla \cdot P = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \tag{194}$$

where P may be the vector $\mathcal{E}$ or H as well as the more general vectors D and B, respectively.

Just as the divergence of a vector is defined by the cross-surface flux with respect to the *entirely* enclosed volume by the surrounding surface, we may define the *line integral* around a closed path with respect to the area enclosed by the path by a vector, which is known as a "curl" and called by some continental European writers a "rot," since it has to do with a "rotation" or a "whirl." This new vector is then a vector product $\nabla \times P \equiv \text{curl } P \equiv \text{rot } P$, since for a path element dl along the closed path l meters[1] of an enclosed area of ΔS square meters, we have for ΔS approaching zero,

$$\text{curl } P \equiv \text{rot } P = \frac{\oint_l P \cos \theta \, dl}{\Delta S \rightarrow 0}$$

if θ is the angle that P makes with the path element dl. Therefore, we have for the limiting conditions for the divergence and the curl of a vector P, the respective relations

$$\left.\begin{aligned} \text{div } P &\equiv \nabla \cdot P = \frac{\int\int_S P \, dS}{dx \, dy \, dz} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \\ \text{curl } P &\equiv \text{rot } P \equiv \nabla \times P = \frac{\oint_l P \cos \theta \, dl}{dS} \end{aligned}\right\} \tag{195}$$

where P is taken normal to the surface element dS.

[1] Generally, if a path l is closed and a force F makes an angle θ with the path, the line integral $\int_l F \cos \theta \, dl$ is known as the circulation, and the *circulation per unit area of an infinitely small loop is the curl of F* (written curl F, which is normal to the plane of the loop. The loop is in such a position that $\int_l F \cos \theta \, dl$ is a maximum. The operator ∇, used on pages 197–199 and throughout the text, is the Hamiltonian nabla.

For the current density J in amperes per square meter at any space point xyz, we have generally the component densities J_x, J_y, and J_z normal to respective YZ, XZ, and XY planes. We find then

$$\text{div } J \equiv \nabla \cdot J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

since div J is equal to the scalar product $\nabla \cdot J$.

For the curl of a vector P, we have the vector product $\nabla \times P$ and must, therefore, have three components: $\text{curl}_x P$, $\text{curl}_y P$, and $\text{curl}_z P$, which can also be written as $\text{rot}_x P$, $\text{rot}_y P$, and $\text{rot}_z P$, respectively. According to the footnote on page 197, for the "cross" or "vector" product, we have for a right-hand coordinate system for the X component

$$\text{curl}_x P = \text{rot}_x P = \frac{\partial P_z}{\partial y} - \frac{\partial P_y}{\partial z} \tag{196}$$

and by advancing in the right-hand rotation, we find the remaining components

$$\left.\begin{aligned} \text{curl}_y P &= \text{rot}_y P = \frac{\partial P_x}{\partial z} - \frac{\partial P_z}{\partial x} \\ \text{curl}\ P &\equiv \text{rot}_z P = \frac{\partial P_y}{\partial x} - \frac{\partial P_x}{\partial y} \end{aligned}\right\} \tag{196a}$$

Such curl, or rot, relations occur then when applying the Ampère-Maxwell current concepts as well as the Faraday induction law, which read generally

$$\text{curl } H = J + \frac{\partial(\kappa \mathcal{E})}{\partial t} = \text{rot } H; \qquad \text{curl } \mathcal{E} = -\frac{\partial(\mu H)}{\partial t} = \text{rot } \mathcal{E} \tag{197}$$

where we deal either in amperes per square meter or volts per square meter, respectively [as shown in connection with Eqs. (1)]. The velocity of wave propagation does not show up directly as in classic systems of units. For isotropic mediums, μ and κ can be taken out of the differential operator, since they are constant.

The component relations along the X, Y, and Z axes are found from (196) and (196a), by means of the curl relation (195). For the magnetic field vector H, we then have in (195) the identity $P \equiv H$; and for the $\text{curl}_x P$ of (196), we have in (195) the identity $dS \equiv dy\, dz$ as the surface element. We find then

$$\text{curl}_x H = \frac{\oint_l H \cos\theta\, dl}{dy \cdot dx} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \text{rot}_x H = \text{total current density across area } dy\, dx$$

For the total current densities in amperes per square meter, we have

$$\left.\begin{aligned}\text{curl}_x\, H \equiv \text{rot}_x\, H &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ &= J_x + \frac{\partial(\kappa \mathcal{E}_x)}{\partial t} \\ \text{curl}_y\, H \equiv \text{rot}_y\, H &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ &= J_y + \frac{\partial(\kappa \mathcal{E}_y)}{\partial t} \\ \text{curl}_z\, H \equiv \text{rot}_z\, H &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \\ &= J_z + \frac{\partial(\kappa \mathcal{E}_z)}{\partial t}\end{aligned}\right\}\text{Ampère-Maxwell laws} \qquad (198)$$

For the electric field vector $\mathcal{E}$ in the curl equation (195), we have the identity $P \equiv \mathcal{E}$; and for the curl component in the X direction according to (195) to (197),

$$\text{curl}_x\, \mathcal{E} \equiv \text{rot}_x\, \mathcal{E} = \frac{\oint_l \mathcal{E} \cos \theta \, dl}{dy\, dx} = \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = \underbrace{-\frac{\partial B_x}{\partial t}}_{\text{Faraday}} \qquad (199)$$

giving the three Cartesian components for the curl $\mathcal{E}$ relation (197a)

$$\left.\begin{aligned}\text{curl}_x\, \mathcal{E} \equiv \text{rot}_x\, \mathcal{E} &= \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\frac{\partial(\mu H_x)}{\partial t} \\ \text{curl}_y\, \mathcal{E} \equiv \text{rot}_y\, \mathcal{E} &= \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\frac{\partial(\mu H_y)}{\partial t} \\ \text{curl}_z\, \mathcal{E} \equiv \text{rot}_z\, \mathcal{E} &= \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\frac{\partial(\mu H_z)}{\partial t}\end{aligned}\right\}\text{Faraday law} \qquad (200)$$

The component equations (196) and (196a) and the corresponding general equations (197) can be expressed in the general differential and integral forms as

$$\left.\begin{aligned}&\text{curl}\, \mathcal{E} + \frac{\partial(\mu H)}{\partial t} = 0; \qquad \text{div}\,(\mu H) = 0; \\ &\quad \text{curl}\, H - \frac{\partial(\kappa \mathcal{E})}{\partial t} = J = \sigma \mathcal{E}; \\ &\qquad\qquad \text{div}\,(\kappa \mathcal{E}) = \rho \qquad \text{differential forms} \\ &\int \mathcal{E} \cos \theta \, dl + \frac{\partial}{\partial t} \int\!\!\int (\mu H)_n \, dS = 0; \\ &\quad \int\!\!\int (\mu H)_n \, dS = 0; \\ &\quad \int H \cos \theta \, dl - \frac{\partial}{\partial t} \int\!\!\int (\kappa \mathcal{E})_n \, dS = J = \sigma \mathcal{E}; \\ &\qquad\qquad \int\!\!\int (\kappa \mathcal{E})_n \, dS = \rho \qquad \text{integral forms}\end{aligned}\right\}\text{Maxwell's equations} \qquad (201)$$

where the subscript n designates that $\mu H = B$ and $\kappa \mathcal{E} = D$ are *normal to* respective *surface elements dS* and $\mathcal{E} \cos \theta$ as well as $H \cos \theta$ express these field components *along* respective line elements dl.

42. Notes on the Poynting Vector. The energy flow associated with an electromagnetic wave is found by developing Poynting's vector $p = \mathcal{E} \times H$, which is perpendicular to both $\mathcal{E}$ and H vectors and whose magnitude $\frac{1}{2}\mathcal{E} \cdot H \sin (\mathcal{E}, H)$ depends upon the product of $\mathcal{E}$ and H and the angle that vector $\mathcal{E}$ makes with vector H. Hence, the integral over a certain surface gives the rate at which energy flows through this surface. The details are as follows:

For harmonic fields $\mathcal{E} = \mathcal{E}_0 \epsilon^{j\omega t}$ and $H = H_0 \epsilon^{j\omega t}$, we have for each cubic-meter space the electric field energy $\frac{1}{2}\kappa \mathcal{E}^2$ and the magnetic field energy $\frac{1}{2}\mu H^2$ in joules per cubic meter, if $\mathcal{E}$ and H are in volts per meter and amperes per meter, respectively; κ in farads per meter; and μ in henrys per meter. The power is the time rate of change of energy (time rate of work in joules per second or watts) We deal then with a power density of energy crossing 1 sq m area during 1 sec (watts per square meter). The relation

$$\underbrace{\text{div } p \equiv \nabla \cdot p}_{\substack{\text{energy}\\ \text{diverging}\\ \text{from 1}\\ \text{cu m}}} = \underbrace{- \frac{\partial(\frac{1}{2}\kappa \mathcal{E}^2 + \frac{1}{2}\mu H^2)}{\partial t}}_{\substack{\text{time rate of decrease}\\ \text{of electromagnetic field}\\ \text{energy in 1 cu m}}} \quad \text{watts/m}^2 \tag{202}$$

or

$$\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z} = - \frac{\partial(\frac{1}{2}\kappa \mathcal{E}^2 + \frac{1}{2}\mu H^2)}{\partial t} \tag{202a}$$

according to (194). Here $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ and p_x, p_y, and p_z are the Cartesian components at a space point xyz delivering each second p_x watts per square meter across surface YZ, p_y watts per square meter across surface XZ, and p_z watts per square meter across surface XY of the unit cube of medium with a dielectric constant κ farads per meter and permeability μ henrys per meter The resulting Poynting vector p is along the direction of electromagnetic field energy flow at a field point xyz with a magnitude p watts per square meter per unit time, where the unit area of 1 sq m lies in the wave front of the advancing wave.

An expression for the time rate of the field energy can also be obtained by using the differential forms

$$\text{curl } \mathcal{E} = - \frac{\partial(\mu H)}{\partial t}; \quad \text{curl } H = J + \frac{\partial(\kappa \mathcal{E})}{\partial t} = \overbrace{\sigma \mathcal{E}}^{\text{conduction}} + \overbrace{\frac{\partial(\kappa \mathcal{E})}{\partial t}}^{\text{displacement}} \text{ current densities}$$

of Maxwell's equations (201) in the vector identity for the average divergence

$$\text{div } (\mathcal{E} \times H) \equiv H \text{ curl } \mathcal{E} - \mathcal{E} \text{ curl } H$$

yielding

$$\operatorname{div}(\mathcal{E} \times H) = -H\frac{\partial(\mu H)}{\partial t} - \mathcal{E}\left[J + \frac{\partial(\kappa\mathcal{E})}{\partial t}\right]$$
$$= -\mathcal{E}\frac{\partial(\kappa\mathcal{E})}{\partial t} - H\frac{\partial(\mu H)}{\partial t} - \mathcal{E}\cdot J \qquad (203)$$

In electromagnetic wave propagation, $\mathcal{E}$ and H are variables of the form $\mathcal{E} = \mathcal{E}_0\epsilon^p$ and $H = H_0\epsilon^p$ where $p = j(\omega t - \beta r)$, if r is the distance through which the wave has passed. Hence, we find for H

$$H = H_0\epsilon^p; \qquad H^2 = H_0^2\epsilon^{2p}$$
$$\frac{\partial H}{\partial t} = j\omega H; \qquad \frac{\partial H^2}{\partial t} = j2\omega H^2$$
$$H\frac{\partial H}{\partial t} = \omega H^2 = \tfrac{1}{2}\frac{\partial H^2}{\partial t}$$

Hence

$$H\frac{\partial(\mu H)}{\partial t} = \frac{\partial(\frac{1}{2}\mu H^2)}{\partial t}; \qquad \mathcal{E}\frac{\partial(\kappa\mathcal{E})}{\partial t} = \frac{\partial(\frac{1}{2}\kappa\mathcal{E}^2)}{\partial t}$$

Equation (203) can, therefore, be written in the form

$$\underbrace{\operatorname{div}(\mathcal{E} \times H)} = -\frac{\partial(\frac{1}{2}\kappa\mathcal{E}^2 + \frac{1}{2}\mu H^2)}{\partial t} - \sigma\mathcal{E}^2 \quad \text{watts/m}^3 \qquad (203a)$$

div p (energy flowing out over surface enclosing 1 cu m of a dissipative medium of conductivity σ mhos/m, permeability μ henrys/m, dielectric constant κ farads/m	rate of decrease of electromagnetic field energy	joulean heat loss in 1 cu m medium due to resistance $1/\sigma$ ohms across 1 m cube

since $J = \sigma\mathcal{E}$ and $\mathcal{E}J$ watts per cubic meter $= \sigma\mathcal{E}^2$ for the conductivity σ mhos across opposite faces of a cubical medium of 1 meter dimensions.

Therefore, for the *general* medium σ, μ, and κ, which is dissipative since it is not a perfect insulator ($\sigma \neq 0$), we have then div $(\mathcal{E} \times H) =$ div p, or the Poynting vector p equal to the vector product of $\mathcal{E}$ and H. This means that p is along the direction of wave propagation and normal to the wave front, since $\mathcal{E}$ and H are tangent to this front at points xyz of the radiation region.

Quite often we are interested in wave propagation only along a desired direction, for instance, the positive X direction of a Cartesian coordinate system. We can then use the component relation (202a) for the divergence of the Poynting vector p, since we are now no longer interested in the average divergence

$$\operatorname{div} p = \frac{\iint_S p\,dS}{dx\,dy\,dz} \qquad \text{watts/m}^3 \qquad (204)$$

i.e., the electromagnetic flux $dx\,dy\,dz$ div p through surrounding boundaries of space elements $dx\,dy\,dz$ or $\int\int_S p_n\,dS = \int\int\int_{xyz} \text{div}\, p\, dx\,dy\,dz$ if p_n is normal to dS surface elements but in *surface* divergences such as

$$\text{div}\, p_x = \frac{\int_x p_x\,dx}{dy\,dz}; \qquad \text{div}\, p_y = \frac{\int_y p_y\,dy}{dx\,dz};$$

$$\text{div}\, p_z = \frac{\int_z p_z\,dz}{dx\,dy} \qquad \text{watts/m}^2 \quad (205)$$

where div p_x is the power flow in watts across 1 sq m in planes parallel with the YZ plane, div p_y is the power density in planes parallel with the XZ plane, and p_z the power density in watts per square meter in planes parallel to the XY plane passing through a space point xyz. We recall that $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$, where the resultant p and the components p_x, p_y, p_z stand in this case for the absolute amount of these vectors.

For sharp beam radiation, the average volume divergence simplifies to a linear divergence, which is mathematically an ordinary derivative and called a "gradient."

The component vectors p_x, p_y, p_z can be readily found when substituting in footnote on page 197 in the determinant for the vector product $\underset{\cdot}{A} \times \underset{\cdot}{B}$, the electric field vector $\underset{\cdot}{\mathcal{E}}$ for $\underset{\cdot}{A}$ and the magnetic field vector $\underset{\cdot}{H}$ for $\underset{\cdot}{B}$. For simplicity's sake, the dot under $\mathcal{E}$ and H is omitted and both quantities refer to instantaneous values in the case of harmonic excitations for which $\mathcal{E}_0\epsilon^{j\omega t}$ and $H_0\epsilon^{j\omega t}$ are the time functions. The divergence for the vector product $\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H}$ then satisfies (203) and the result of (203*a*); and according to the footnote, we have for (205)

$$\left.\begin{aligned} \text{div}\, p_x &= \text{div}\,(\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H})_x = \text{div}\,(\mathcal{E}_y H_z - \mathcal{E}_z H_y)_x \\ \text{div}\, p_y &= \text{div}\,(\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H})_y = \text{div}\,(\mathcal{E}_z H_x - \mathcal{E}_x H\ )_y \\ \text{div}\, p_z &= \text{div}\,(\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H})_z = \text{div}\,(\mathcal{E}_x H_y - \mathcal{E}_y H_x)_z \end{aligned}\right\} \text{watts/m}^2 \quad (206)$$

From these expressions, we note that for the component p_x of the Poynting vector the power flow depends upon the product $\mathcal{E}_y H_z$ and $\mathcal{E}_z H_y$, since for the general orientation of $\mathcal{E}$ and H we must have $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$, *i.e.*, $\mathcal{E}_y$ and $\mathcal{E}_z$ components in a plane wave front parallel with the YZ plane and similarly H_y and H_z components. The same also holds true for the p_y and p_z Cartesian space components. From (206) we can obtain the values along X and find

$$\text{div}\, p_x = \frac{\partial(\mathcal{E}_y H_z - \mathcal{E}_z H_y)}{\partial x} = \mathcal{E}_y \frac{\partial H_z}{\partial x} + H_z \frac{\partial \mathcal{E}_y}{\partial x} - \mathcal{E}_z \frac{\partial H_y}{\partial x} - H_y \frac{\partial \mathcal{E}_z}{\partial x} \quad (207)$$

and similarly

$$\left.\begin{aligned}\text{div } p_y &= \frac{\partial p_y}{\partial y} = \mathcal{E}_z \frac{\partial H_x}{\partial y} + H_x \frac{\partial \mathcal{E}_z}{\partial y} - \mathcal{E}_x \frac{\partial H_z}{\partial y} - H_z \frac{\partial \mathcal{E}_x}{\partial y} \\ \text{div } p_z &= \frac{\partial p_z}{\partial z} = \mathcal{E}_x \frac{\partial H_y}{\partial z} + H_y \frac{\partial \mathcal{E}_x}{\partial z} - \mathcal{E}_y \frac{\partial H_x}{\partial z} - H_x \frac{\partial \mathcal{E}_y}{\partial z}\end{aligned}\right\} \qquad (208)$$

For each space element $dx\,dy\,dz$ we have the *outward* flux over its boundary

$$\begin{aligned}(\text{div } p_x + \text{div } p_y + \text{div } p_z)\,dx\,dy\,dz &= \left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z}\right) dx\,dy\,dz \\ = \Bigg(\mathcal{E}_y \frac{\partial H_z}{\partial x} + H_z \frac{\partial \mathcal{E}_y}{\partial x} - \mathcal{E}_z \frac{\partial H_y}{\partial x} - H_y \frac{\partial \mathcal{E}_z}{\partial x} &+ \mathcal{E}_z \frac{\partial H_x}{\partial y} + H_x \frac{\partial \mathcal{E}_z}{\partial y} \\ - \mathcal{E}_x \frac{\partial H_z}{\partial y} - H_z \frac{\partial \mathcal{E}_x}{\partial y} + \mathcal{E}_x \frac{\partial H_y}{\partial z} + H_y \frac{\partial \mathcal{E}_x}{\partial z} - \mathcal{E}_y \frac{\partial H_x}{\partial z} \\ - H_x \frac{\partial \mathcal{E}_y}{\partial z}\Bigg)\,dx\,dy\,dz \qquad \text{watts} \qquad (208a)\end{aligned}$$

These relations must satisfy Maxwell's equations (198) and (200), which can be changed into watts per cubic meter by multiplying the $\text{curl}_x H$ relation by $\mathcal{E}_x$, the $\text{curl}_y H$ relation by $\mathcal{E}_y$, the $\text{curl}_z H$ relation by $\mathcal{E}_z$, the $\text{curl}_x \mathcal{E}$ relation by $-H_x$, the $\text{curl}_y \mathcal{E}$ relation by $-H_y$, and the $\text{curl}_z \mathcal{E}$ relation by $-H_z$, yielding

$$\left.\begin{aligned}\mathcal{E}_x \text{ curl}_x H &= \mathcal{E}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = \mathcal{E}_x J_x + \mathcal{E}_x \frac{\partial(\kappa \mathcal{E}_x)}{\partial t} \\ &= \mathcal{E}_x J_x + \frac{1}{2}\frac{\partial(\kappa \mathcal{E}_x^2)}{\partial t} \\ \mathcal{E}_y \text{ curl}_y H &= \mathcal{E}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial y}\right) = \mathcal{E}_y J_y + \mathcal{E}_y \frac{\partial(\kappa \mathcal{E}_y)}{\partial t} \\ &= \mathcal{E}_y J_y + \frac{1}{2}\frac{\partial(\kappa \mathcal{E}_y^2)}{\partial t} \\ \mathcal{E}_z \text{ curl}_z H &= \mathcal{E}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \mathcal{E}_z J_z + \mathcal{E}_z \frac{\partial(\kappa \mathcal{E}_z)}{\partial t} \\ &= \mathcal{E}_z J_z + \frac{1}{2}\frac{\partial(\kappa \mathcal{E}_z^2)}{\partial t} \\ -H_x \text{ curl}_x \mathcal{E} &= H_x\left(\frac{\partial \mathcal{E}_y}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial y}\right) = H_x \frac{\partial(\mu H_x)}{\partial t} \\ &= \frac{1}{2}\frac{\partial(\mu H_x^2)}{\partial t} \\ -H_y \text{ curl}_y \mathcal{E} &= H_y\left(\frac{\partial \mathcal{E}_z}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial z}\right) = \frac{1}{2}\frac{\partial(\mu H_y^2)}{\partial t} \\ -H_z \text{ curl}_z \mathcal{E} &= H_z\left(\frac{\partial \mathcal{E}_x}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial x}\right) = \frac{1}{2}\frac{\partial(\mu H_z^2)}{\partial t}\end{aligned}\right\} \text{ watts/m}^3 \qquad (209)$$

Adding the first and the fourth relations, second and fifth relations, and third and sixth relations, respectively, yields

$$\left.\begin{aligned}
\mathcal{E}_x \operatorname{curl}_x H - H_x \operatorname{curl}_x \mathcal{E} &= \mathcal{E}_x J_x + \frac{\partial(\frac{1}{2}\kappa\mathcal{E}_x^2 + \frac{1}{2}\mu H_x^2)}{\partial t} \\
\mathcal{E}_y \operatorname{curl}_y H - H_y \operatorname{curl}_y \mathcal{E} &= \mathcal{E}_y J_y + \frac{\partial(\frac{1}{2}\kappa\mathcal{E}_y^2 + \frac{1}{2}\mu H_y^2)}{\partial t} \\
\mathcal{E}_z \operatorname{curl}_z H - H_z \operatorname{curl}_z \mathcal{E} &= \underbrace{\mathcal{E}_z J_z}_{\substack{\text{watt} \\ \text{component}}} + \underbrace{\frac{\partial(\frac{1}{2}\kappa\mathcal{E}_z^2 + \frac{1}{2}\mu H_z^2)}{\partial t}}_{\substack{\text{wattless field} \\ \text{component (available watts)}}}
\end{aligned}\right\} \text{watts/m}^3 \qquad (210)$$

We note that the left side is of the form $\mathcal{E}_q \operatorname{curl}_q H - H_q \operatorname{curl}_q \mathcal{E}$ with q giving the principal coordinate direction, resembling the form

$$\mathcal{E} \operatorname{curl} H - H \operatorname{curl} \mathcal{E} \equiv -\operatorname{div}(\mathcal{E} \times H)$$

used for obtaining the result of (203) and (203*a*). Since $\mathcal{E}_x J_x$ is in watts per cubic meter, it must represent the joulean heat loss in the medium through which the wave passes. This loss is due to any medium conductivity σ mhos per meter cube while electromagnetic field energy passes along the X direction. In the same way, $\mathcal{E}_y J_y = \sigma\mathcal{E}_y^2$ denotes the medium loss per cubic meter for the electromagnetic radiation along the Y direction, and $\mathcal{E}_z J_z = \sigma\mathcal{E}_z^2$ watts are lost per cubic meter owing to the radiation component along the Z axis. Hence, $\mathcal{E}_x J_x + \mathcal{E}_y J_y + \mathcal{E}_z J_z$ represents the total loss in watts within 1 cu m of the medium, and $\iiint_{xyz} (\mathcal{E}_x J_x + \mathcal{E}_y J_y + \mathcal{E}_z J_z)\, dx\, dy\, dz$ watts denotes the joulean heat loss for xyz cubic meters of the medium in case σ is not zero as would be essentially so in customary troposphere. Adding up the wattless electric and magnetic field components for all three relations (210), we find, for the xyz cubic-meter medium,

$$\frac{1}{2}\iiint_{xyz} \frac{\partial[\kappa(\mathcal{E}_x^2 + \mathcal{E}_y^2 + \mathcal{E}_z^2) + \mu(H_x^2 + H_y^2 + H_z^2)]\, dx\, dy\, dz}{\partial t}$$
$$= \iiint_{xyz} \frac{\partial(\kappa\mathcal{E}^2 + \mu H^2)}{\partial t}\, dx\, dy\, dz \qquad \text{watts}$$

since $\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2 + \mathcal{E}_z^2}$ and $H = \sqrt{H_x^2 + H_y^2 + H_z^2}$. We now have a means of solving (208*a*) when comparing it with relations (209). The comparison is to be made with respect to the parenthesis values of (208*a*), since these values are in watts per cubic meter as are all terms in (209). Putting $\mathcal{E}_x J_x + \mathcal{E}_y J_y + \mathcal{E}_z J_z = W$, which corresponds to watts required for the space dissipation, and $\frac{1}{2}\frac{\partial(\kappa\mathcal{E}^2 + \mu H^2)}{\partial t} = W_f$, which represents the time rate of the electromagnetic field energy, we find, when adding

up all terms of relations (209) by using the differential expressions for respective curls after rearranging the various differentials, that

$$W + W_f = -\underbrace{\left(\mathcal{E}_y \frac{\partial H_z}{\partial y} + H_z \frac{\partial \mathcal{E}_y}{\partial x} - \mathcal{E}_z \frac{\partial H_y}{\partial x} - H_y \frac{\partial \mathcal{E}_z}{\partial x}\right)}_{=\frac{\partial p_x}{\partial x}}$$

$$-\underbrace{\left(\mathcal{E}_z \frac{\partial H_x}{\partial y} + H_x \frac{\partial \mathcal{E}_z}{\partial y} - \mathcal{E}_x \frac{\partial H_z}{\partial y} - H_z \frac{\partial \mathcal{E}_x}{\partial y}\right)}_{=\frac{\partial p_y}{\partial y}}$$

$$-\underbrace{\left(\mathcal{E}_x \frac{\partial H_y}{\partial z} + H_y \frac{\partial \mathcal{E}_x}{\partial z} - \mathcal{E}_y \frac{\partial H_x}{\partial z} - H_x \frac{\partial \mathcal{E}_y}{\partial z}\right)}_{=\frac{\partial p}{\partial z}}$$

$$= -(\text{div } p_x + \text{div } p_y + \text{div } p_z)$$

Hence, for a medium of volume abc cubic meters, where a, b, and c are the dimensions in meters along the X, Y, and Z directions, we have

$$-\underbrace{\iiint_{abc} (\text{div } p_x + \text{div } p_y + \text{div } p_z)\, dx\, dy\, dz}_{\substack{-\text{div } p \\ \text{electromagnetic field power in watts escaping across surrounding} \\ \text{boundary of } abc \text{ cu m}}} - \underbrace{\sigma \iiint_{abc} \mathcal{E}^2\, dx\, dy\, dz}_{\substack{\text{medium loss in watts in} \\ abc \text{ cu m}}}$$

$$+ \tfrac{1}{2} \underbrace{\iiint_{abc} (\kappa\mathcal{E}^2 + \mu H^2)\, dx\, dy\, dz}_{\text{rate of decrease in electromagnetic field energy}} \quad (211)$$

This confirms the simpler solution with (203a) as the result. From (203a), (211), and (206) we see that we have for the Poynting vector p for its real components components

$$p = \mathcal{E} \times H \text{ (resultant)}; \qquad p_x = (\mathcal{E} \times H)_x = \mathcal{E}_y H_z - \mathcal{E}_z H_y;$$
$$p_y = (\mathcal{E} \times H)_y = \mathcal{E}_z H_x - \mathcal{E}_x H_z; \qquad p_z = (\mathcal{E} \times H)_z = \mathcal{E}_x H_y - \mathcal{E}_y H_x$$
$$\text{watts/m}^2 \quad (212)$$

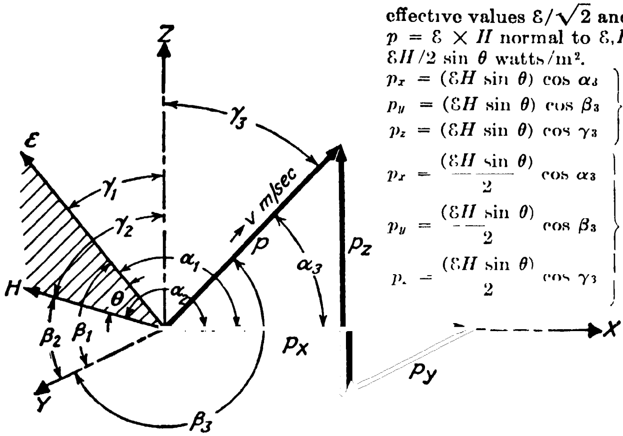
If $\mathcal{E}$ and H make an angle θ with each other and we take $\mathcal{E}$ and H as the absolute values, then $\mathcal{E}H \sin \theta$ denotes the power in watts transferred normally across a square-meter area. In the true radiation region, $\theta = 90$ deg and $Ap = \frac{1}{2}\mathcal{E}HA$ watts power flows across A square meters in the wave front if $\mathcal{E}$ and H are taken as the maximum field values. The component powers p_xA, p_yA, and p_zA watts would flow normal and across the same area of A square meters along the X, Y, and Z directions, respectively, and the absolute amounts of the component radiation contributions satisfies $p^2 = p_x^2 + p_y^2 + p_z^2$.

It is necessary to deal with direction cosines cos α_1, cos β_1, and cos γ_1 for the $\mathcal{E}$ vector with respect to the respective X, Y, and Z directions. For the resultant H vector, we have the corresponding direction cosines

$\mathcal{E}$ and H are maximum field values with the effective values $\mathcal{E}/\sqrt{2}$ and $H/\sqrt{2}$.
$p = \mathcal{E} \times H$ normal to $\mathcal{E},H$ plane of magnitude $\mathcal{E}H/2 \sin\theta$ watts/m².

$$\left.\begin{aligned} p_x &= (\mathcal{E}H \sin\theta)\cos\alpha_3 \\ p_y &= (\mathcal{E}H \sin\theta)\cos\beta_3 \\ p_z &= (\mathcal{E}H \sin\theta)\cos\gamma_3 \end{aligned}\right\} \text{peak components of } p$$

$$\left.\begin{aligned} p_x &= \frac{(\mathcal{E}H \sin\theta)}{2}\cos\alpha_3 \\ p_y &= \frac{(\mathcal{E}H \sin\theta)}{2}\cos\beta_3 \\ p_z &= \frac{(\mathcal{E}H \sin\theta)}{2}\cos\gamma_3 \end{aligned}\right\} \text{average watts/m}^2$$



$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\left.\begin{aligned} \cos(\mathcal{E},X) &= \cos\alpha_1 = \frac{\mathcal{E}_x}{\mathcal{E}} \\ \cos(\mathcal{E},Y) &= \cos\beta_1 = \frac{\mathcal{E}_y}{\mathcal{E}} \\ \cos(\mathcal{E},Z) &= \cos\gamma_1 = \frac{\mathcal{E}_z}{\mathcal{E}} \end{aligned}\right\} \mathcal{E} = \mathcal{E}_x\cos\alpha_1 + \mathcal{E}_y\cos\beta_1 + \mathcal{E}_z\cos\gamma_1$$

$$\left.\begin{aligned} \cos(H,X) &= \cos\alpha_2 = \frac{H_x}{H} \\ \cos(H,Y) &= \cos\beta_2 = \frac{H_y}{H} \\ \cos(H,Z) &= \cos\gamma_2 = \frac{H_z}{H} \end{aligned}\right\} H = H_x\cos\alpha_2 + H_y\cos\beta_2 + H_z\cos\gamma_2$$

$$\left.\begin{aligned} \cos(p,X) &= \cos\alpha_3 = \frac{p_x}{p} \\ \cos(p,Y) &= \cos\beta_3 = \frac{p_y}{p} \\ \cos(p,Z) &= \cos\gamma_3 = \frac{p_z}{p} \end{aligned}\right\} p = p_x\cos\alpha_3 + p_y\cos\beta_3 + p_z\cos\gamma_3$$

FIG. 37. Poynting vector p, electric $\mathcal{E}$ and magnetic H vectors, and their Cartesian components.

cos α_2, cos β_2, and cos γ_2, while the resultant Poynting vector is defined in space by means of the direction cosines cos α_3, cos β_3, and cos γ_3, respectively. For the general case where $\mathcal{E}$ and H make an angle θ, other than 90 space degrees, we then have the situation shown in Fig. 37. From this figure, $\mathcal{E}_x = \mathcal{E}\cos\alpha_1$, $\mathcal{E}_y = \mathcal{E}\cos\beta_1$, $\mathcal{E}_z = \mathcal{E}\cos\gamma_1$, $H_x =$

$H \cos \alpha_2$, $H_y = H \cos \beta_2$, $H_z = H \cos \gamma_2$; and for the peak values, we find

$$\begin{aligned}
p_x &= \mathcal{E}_y H_z - \mathcal{E}_z H_y = \mathcal{E}H(\cos \beta_1 \cos \gamma_2 - \cos \gamma_1 \cos \beta_2) = \mathcal{E}HA \\
p_y &= \mathcal{E}_z H_x - \mathcal{E}_x H_z = \mathcal{E}H(\cos \gamma_1 \cos \alpha_2 - \cos \alpha_1 \cos \gamma_2) = \mathcal{E}HB \\
p_z &= \mathcal{E}_x H_y - \mathcal{E}_y H_x = \mathcal{E}H(\cos \alpha_1 \cos \beta_2 - \cos \beta_1 \cos \alpha_2) = \mathcal{E}HC \\
p &= \sqrt{p_x^2 + p_y^2 + p_z^2} = \mathcal{E}H \sqrt{A^2 + B^2 + C^2} \\
&= \mathcal{E}H \sqrt{\sin^2 \theta \underbrace{(\cos^2 \alpha_3 + \cos^2 \beta_3 + \cos^2 \gamma_3)}_{1}} = \mathcal{E}H \sin \theta \qquad (213)
\end{aligned}$$

since from the peak values

$$\left.\begin{aligned}
p_x &= (\mathcal{E}H \sin \theta) \cos \alpha_3 = \mathcal{E}HA \\
p_y &= (\mathcal{E}H \sin \theta) \cos \beta_3 = \mathcal{E}HB \\
p_z &= (\mathcal{E}H \sin \theta) \cos \gamma_3 = \mathcal{E}HC
\end{aligned}\right\} \text{we obtain} \left\{\begin{aligned}
A &= \sin \theta \cos \alpha_3 \\
B &= \sin \theta \cos \beta_3 \\
C &= \sin \theta \cos \gamma_3
\end{aligned}\right.$$

where, of course, A has a value as given here and has nothing to do with

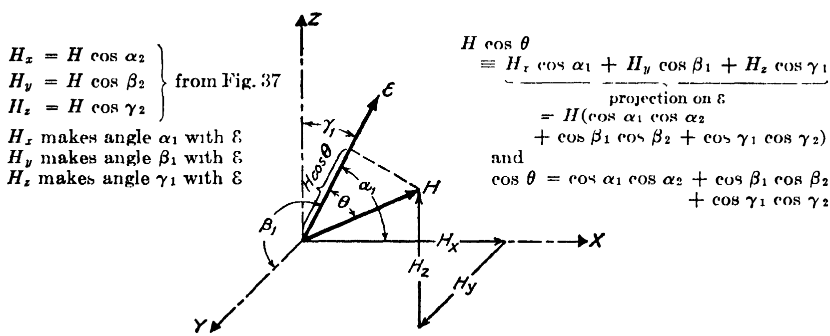


FIG. 38. Electric $\mathcal{E}$ and magnetic H vectors in terms of the direction cosines.

the area in square meters as used before. These results can also be obtained by means of

$$\begin{aligned}
p = p_x \cos \alpha_3 + p_y \cos \beta_3 + p_z \cos \gamma_3 \\
&= \mathcal{E}H(A \cos \alpha_3 + B \cos \beta_3 + C \cos \gamma_3) \\
&= \mathcal{E}H \sin \theta(\cos^2 \alpha_3 + \cos^2 \beta_3 + \cos^2 \gamma_3) \\
&= \mathcal{E}H \sin \theta
\end{aligned}$$

These results are based upon the peak-value expressions of Fig. 37 for p_x, p_y, and p_z, which contain also the factor $\mathcal{E}H \sin \theta$. Therefore, we still have to prove that the angle θ between $\mathcal{E}$ and H plays a part if it has other values than 90 or 270 deg, for which conditions we have ± 1 values or $\pm \frac{1}{2}\mathcal{E}H$ watts per square meter for the resultant power flux, where $\mathcal{E}$ and H again denote the maximum field values. It is now neces-

sary to express the angle θ between $\mathcal{E}$ and H vectors in terms of the direction cosines of these vectors with respect to the XYZ reference directions as in Fig. 38, where, for clearness, the $\mathcal{E}$ and H vectors of Fig. 37 are drawn into the positive XYZ quadrant. According to Fig. 37 and the result $p = \mathcal{E}H\sqrt{A^2 + B^2 + C^2}$ of Eqs. (20), the peak value p yields

$$p = \mathcal{E}H\sqrt{1 - (\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2)^2}$$
$$= \mathcal{E}H\sqrt{(1 - \cos^2\theta)} = \mathcal{E}H\sin\theta$$

confirming for the effective values $\mathcal{E}_e = \mathcal{E}/\sqrt{2}$ and $H_e = H/\sqrt{2}$ the well-known relation for the average power density

$$\bar{p} = \tfrac{1}{2}\mathcal{E}H\sin\theta = \mathcal{E}_e H_e \sin\theta \qquad \text{watts/m}^2$$

For plane waves, respective field associations $\mathcal{E}$ and H are normal to each other and in time phase and

$$p = \mathcal{E}_e H_e = \frac{\mathcal{E}_e^2}{376.7} = 2.65 \times 10^{-3}\mathcal{E}_e^2 \qquad \text{watts/m}^2$$

if the area of 1 sq m lies in the wave front, *i.e.*, perpendicular to the direction of wave propagation. Since vectors p_x, p_y, and p_z add under 90 deg, we have $p = ip_x + jp_y + kp_z$ for unit vectors i, j, and k. We have, therefore, for the absolute peak values of Eq. (212) for the power flux along the X, Y, and Z axes, the average power densities

$$\left.\begin{aligned}\bar{p}_x &= \tfrac{1}{2}(\mathcal{E}_y H_z - \mathcal{E}_z H_y) = 1.325 \times 10^{-3}(\mathcal{E}_y^2 - \mathcal{E}_z^2)\\ \bar{p}_y &= \tfrac{1}{2}(\mathcal{E}_z H_x - \mathcal{E}_x H_z) = 1.325 \times 10^{-3}(\mathcal{E}_z^2 - \mathcal{E}_x^2)\\ \bar{p}_z &= \tfrac{1}{2}(\mathcal{E}_x H_y - \mathcal{E}_y H_x) = 1.325 \times 10^{-3}(\mathcal{E}_x^2 - \mathcal{E}_y^2)\end{aligned}\right\} \text{watts/m}^2 \qquad (214)$$

since $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$ are peak values.

43. Notes on the Complex Poynting Vector. Inasmuch as this vector refers to a power density due to the product of two vectors, which are $\underset{\cdot}{\mathcal{E}}$ and $\underset{\cdot}{H}$ in the form $\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H}$, we must distinguish between variable time terms and fixed terms, since the latter account only for the average power in watts per square meter. We then have a similar case as in alternating-current engineering, where $\underset{\cdot}{E} = E\epsilon^{j\omega t}$ is the impressed voltage and $\underset{\cdot}{I} = I\epsilon^{j(\omega t+\varphi)}$ is the corresponding current, which is generally φ time degrees out of phase (leading or lagging, whichever the case). We have then

$$\epsilon^{j\omega t} = \cos\omega t + j\sin\omega t; \qquad \epsilon^{j(\omega t+\varphi)} = \cos(\omega t + \varphi) + j\sin(\omega t + \varphi)$$

It is the real part that is important, and the energy expended to force a current through an impedance $Z = R \pm X$ is

$$\int_0^t \underset{\cdot}{E} \cdot \underset{\cdot}{I}\, dt = EI \int_0^t \cos \omega t \cos (\omega t + \varphi)\, dt$$

$$= \tfrac{1}{2}EI \int_0^t [\underbrace{\cos (\omega t - \omega t - \varphi)}_{\cos \varphi} + \underbrace{\cos (\omega t + \omega t + \varphi)}_{\cos (2\omega t + \varphi)}]\, dt$$

$$= \left| \underbrace{\tfrac{1}{2}EIt \cos \varphi}_{\text{fixed}} + \underbrace{\frac{EI}{4\omega} \sin (2\omega t + \varphi)}_{\text{fluctuating}} \right|_0^t$$

Therefore, the average power $\bar{P} = \frac{1}{2}EI \cos \varphi$ watts if t is sufficiently long, since it is equal to $[(\frac{1}{2}EIt \cos \varphi)/t] + 0$.

If P^* denotes the *complex* power defined by

$$\left.\begin{aligned} P^* &= \tfrac{1}{2}\underset{\cdot}{E}\underset{\cdot}{I}^* = \tfrac{1}{2}E\epsilon^{j\omega t} \cdot I\epsilon^{-j(\omega t+\varphi)} \\ P^* &= \tfrac{1}{2}Z\underset{\cdot}{I}\underset{\cdot}{I}^* = \tfrac{1}{2}ZI\epsilon^{j(\omega t+\varphi)}I\epsilon^{-j(\omega t+\varphi)} = \tfrac{1}{2}ZI^2 \\ P^* &= \frac{1}{2}\frac{\underset{\cdot}{E}\underset{\cdot}{E}^*}{Z} = \frac{1}{2}\frac{E\epsilon^{j\omega t}E\epsilon^{-j\omega t}}{Z} = \frac{1}{2}\frac{E^2}{Z} \end{aligned}\right\} \tag{215}$$

and

$$\bar{P} = \tfrac{1}{2}EI \cos \varphi = \text{real part of } P^* \equiv \text{Re}\,(P^*) \tag{216}$$

In the above expressions, $\underset{\cdot}{E}^*$ is the *conjugate* vector of $\underset{\cdot}{E}$ and $\underset{\cdot}{I}^*$ the conjugate of $\underset{\cdot}{I}$. Hence, if $\underset{\cdot}{E} = a \pm jb$, we have $\underset{\cdot}{E}^* = a \mp jb$, since $E = \sqrt{a^2 + b^2}\, \epsilon^{\pm j\theta}$ and $E^* = \sqrt{a^2 + b^2}\, \epsilon^{\mp j\theta}$ for $\theta = \tan^{-1} (b/a)$.

In the electromagnetic field, we are concerned with the complex power density p^* in the general case; and therefore, $\bar{p} = \text{Re}\,(p^*)$ and $\mathcal{E}$ and H play the role of E and I in (215), the space or medium impedance Z_0 corresponds to the circuit impedance, and

$$p^* = \tfrac{1}{2}\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H}^*; \qquad \bar{p} = \text{Re}\,(\tfrac{1}{2}\underset{\cdot}{\mathcal{E}} \times \underset{\cdot}{H}^*) \tag{217}$$

since the average power density $\bar{p}$ is equal to the average value of the vector product $\text{Re}\,(\underset{\cdot}{\mathcal{E}}) \times \text{Re}\,(\underset{\cdot}{H})$.

The advantage of using the conjugate vector as one factor can be seen from the following, where the general case $\underset{\cdot}{\mathcal{E}} = \mathcal{E}\epsilon^{j\varphi_1}$ and $\underset{\cdot}{H} = H\epsilon^{j\varphi_2}$, *i.e.*, out-of-phase condition, is analyzed. We find then

$$\underset{\cdot}{\mathcal{E}} \cdot \underset{\cdot}{H} = \mathcal{E} \cdot H\epsilon^{j(\varphi_1+\varphi_2)} = \mathcal{E}H \cos (\varphi_1 + \varphi_2) + j\mathcal{E}H \sin (\varphi_1 + \varphi_2)$$

which has *no* particular bearing on the above case. But if we take the conjugate vector $\underset{\cdot}{H}^* = H\epsilon^{-j\varphi_2}$, we find

$$\underset{\cdot}{\mathcal{E}}\underset{\cdot}{H}^* = \mathcal{E}H\epsilon^{j(\varphi_1-\varphi_2)} = \underbrace{\mathcal{E}H \cos (\varphi_1 - \varphi_2)}_{\substack{\text{scalar}\\ \text{product}}} - \underbrace{j\mathcal{E}H \sin (\varphi_1 - \varphi_2)}_{\substack{\text{vector}\\ \text{product}}}$$

again showing that the *real* part yields the power $\frac{1}{2}\mathcal{E}H \cos (\varphi_1 - \varphi_2)$

watts per square meter, since the effective values are $\mathcal{E}/\sqrt{2}$ and $H/\sqrt{2}$. If we now give the magnetic vector the general form $\underset{\cdot}{H} = (A + jB)\epsilon^{j\varphi}$, we have

$$\underset{\cdot}{H} = (A + jB)(\cos\varphi + j\sin\varphi) = \underbrace{A\cos\varphi - B\sin\varphi}_{\text{Re}\,(\underset{\cdot}{H})} + j(B\cos\varphi + A\sin\varphi) \quad (218)$$

For the conjugate value, we have

$$\underset{\cdot}{H}^* = (A - jB)\epsilon^{-j\varphi} = (A - jB)(\cos\varphi - j\sin\varphi)$$
$$= \underbrace{A\cos\varphi - B\sin\varphi}_{\text{Re}\,(\underset{\cdot}{H})^*} - j(A\sin\varphi - B\cos\varphi) \quad (219)$$

Adding (218) and (219), we find

$$\underset{\cdot}{H} + \underset{\cdot}{H}^* = 2\,\text{Re}\,(\underset{\cdot}{H}) \quad (220)$$

Hence, half the sum of two conjugate vectors is equal to the common real part of each vector. For the product, we find

$$\underset{\cdot}{H} \cdot \underset{\cdot}{H}^* = (A + jB)(A - jB)\epsilon^{j(\varphi-\varphi)} = A^2 + B^2 \quad (221)$$

which is the square of the complex amplitude of each vector. Hence, for complex vectors $\underset{\cdot}{\mathcal{E}} = (C + jD)\epsilon^{j\varphi}$ and $\underset{\cdot}{H} = (A + jB)\epsilon^{j\varphi}$, we have for the real parts, according to (220), the product

$$p = \text{Re}\,(\underset{\cdot}{\mathcal{E}}) \cdot \text{Re}\,(\underset{\cdot}{H}) = \tfrac{1}{2}(\underset{\cdot}{\mathcal{E}} + \underset{\cdot}{\mathcal{E}}^*) \cdot \tfrac{1}{2}(\underset{\cdot}{H} + \underset{\cdot}{H}^*)$$
$$= 0.25(\underset{\cdot}{\mathcal{E}}\underset{\cdot}{H} + \underset{\cdot}{\mathcal{E}}\underset{\cdot}{H}^* + \underset{\cdot}{\mathcal{E}}^*\underset{\cdot}{H} + \underset{\cdot}{\mathcal{E}}^*\underset{\cdot}{H}^*)$$

with an average value

$$\bar{p} = 0.25(\underset{\cdot}{\mathcal{E}}\underset{\cdot}{H}^* + \underset{\cdot}{\mathcal{E}}^*\underset{\cdot}{H}) = 0.25(2AC + 2BD) = \tfrac{1}{2}(AC + BD) \quad (222)$$

since

$$\underset{\cdot}{\mathcal{E}} \cdot \underset{\cdot}{H}^* = (C + jD)\epsilon^{j\varphi}(A - jB)\epsilon^{-j\varphi} = CA - jBC + jAD + BD$$
$$\underset{\cdot}{\mathcal{E}}^* \cdot \underset{\cdot}{H} = (C - jD)\epsilon^{-j\varphi}(A + jB)\epsilon^{j\varphi} = CA + jBC - jAD + BD$$
$$\underset{\cdot}{\mathcal{E}} \cdot \underset{\cdot}{H}^* + \underset{\cdot}{\mathcal{E}}^* \cdot \underset{\cdot}{H} = 2AC + 2BD$$

and the remaining terms in the expression for p

$$\underset{\cdot}{\mathcal{E}} \cdot \underset{\cdot}{H} = (C + jD)\epsilon^{j\varphi}(A + jB)\epsilon^{j\varphi}$$
$$= [CA - BD + j(BC + AD)]\,[\cos(2\varphi) + j\sin(2\varphi)]$$
$$\underset{\cdot}{\mathcal{E}}^* \cdot \underset{\cdot}{H}^* = (C - jD)\epsilon^{-j\varphi}(A - jB)\epsilon^{-j\varphi}$$
$$= [CA - BD - j(BC + AD)]\,[\cos(2\varphi) - j\sin(2\varphi)]$$

All these terms are $2\omega t$ trigonometric functions and vanish when integrated for the average value.

From the gradients of $\mathcal{E}_x$ along the Z direction and H_y along the same direction for the resultant vectors $\mathcal{E}$ and H, we have the relations $\nabla \times \mathcal{E} = j\omega\mu H$ and $\nabla \times H = (\sigma + j\omega\kappa)\mathcal{E}$ with a conjugate vector relation $\nabla \times H^* = (\sigma - j\omega\kappa)\mathcal{E}^*$, where $\nabla \times \mathcal{E}$ and $\nabla \times H$ stand for curl $\mathcal{E}$ and curl H, respectively. For a dissipative medium (σ present), we then have a complex Poynting vector p^* and deal with div $(\mathcal{E} \times H^*)$ and find, corresponding to the divergence equation (203*a*) on page 202,

$$\text{div } p^* \equiv \nabla \cdot p^* = \underbrace{-\tfrac{1}{2}\sigma\mathcal{E} \cdot \mathcal{E}^*}_{\substack{\text{real part is the}\\\text{divergence of}\\\text{Re }(p^*)\text{ and}\\\text{equal to the}\\\text{joulean heat loss}\\\text{in 1 cu m of the}\\\text{medium}}} + \underbrace{j\omega(\tfrac{1}{2}\mu H \cdot H^* - \tfrac{1}{2}\kappa\mathcal{E} \cdot \mathcal{E}^*)}_{\substack{\text{divergence of imaginary part of}\\\text{complex Poynting vector } p^* \text{ (pro-}\\\text{portional to the difference between}\\\text{magnetic and electric field energy}\\\text{per m}^3}} \tag{223}$$

In a hollow wave guide for which there is only one conductor, we have a characteristic impedance of the form

$$Z_0 = \underbrace{\bar{p}}_{\text{Re }(Z_0)} + \underbrace{j2\omega(\bar{p}_m - \bar{p}_e)}_{j(Z_0)} \tag{224}$$

where $\bar{p}$ is the mean power passing through the guide and $\bar{p}_m$ and $\bar{p}_e$ are the average stored magnetic and electric field energies corresponding to 1 amp. The characteristic impedance may then be defined as the transverse complex Poynting vector p^* integrated over the cross section of the guide divided by the mean square current, since

$$p^* = \int_S \mathcal{E} \times H \, dS = \bar{p} + j2\omega(\bar{p}_m - \bar{p}_e) \tag{225}$$

since $\bar{p} = \text{Re }(p^*)$, $Z_0 = \text{Re }(Z_0) + j(Z_0)$, $\bar{p} = \tfrac{1}{2}\text{ Re }(Z_0)I^2$, and $2\omega(\bar{p}_m - \bar{p}_e) = \tfrac{1}{2}(Z_0)I^2$, where I denotes the total maximum current of form $I\epsilon^{j\omega t}$. Hence, in a circular wave guide of radius a meters, we have the average Poynting vector

$$\bar{p} = \tfrac{1}{2}\text{ Re }(Z_0)I^2 = \text{Re}\left[\int_0^{2\pi}\int_0^a r(\mathcal{E}_r H_\theta^* - \mathcal{E}_\theta H_r^*)\,dr\,d\theta\right] \tag{225a}$$

where the maximum values are such that $\mathcal{E}^2 = \mathcal{E}_\theta^2 + \mathcal{E}_r^2$ and $H^2 = H_\theta^2 + H_r^2$. This relation holds for both $\mathcal{E}$ and H waves and

$$\left.\begin{aligned} \mathcal{E}_r H_\theta^* - \mathcal{E}_\theta H_r^* &= \sqrt{\frac{\mu}{\kappa}}\frac{c}{c'}H^2 \qquad \text{for } \mathcal{E} \text{ waves} \\ \mathcal{E}_r H_\theta^* - \mathcal{E}_\theta H_r^* &= \sqrt{\frac{\mu}{\kappa}}\frac{c'}{c}H^2 \qquad \text{for } H \text{ waves} \end{aligned}\right\} \tag{226}$$

where $c = 3 \times 10^8$ m/sec and c' is the phase velocity in the guide.

The complex energy, *i.e.*, the power flow across S square meters of surface, is, according to the above-mentioned relations,

$$P^* = \tfrac{1}{2} \int \int_S (\mathcal{E} \times H^*) \, dS \qquad \text{watts} \tag{226a}$$

where the surface element dS is taken normal to $\mathcal{E} \times H^*$ and $p^* = \tfrac{1}{2}\mathcal{E} \times H^*$ is the complex Poynting vector of average value $\bar{p} = \text{Re}\,(p^*)$ watts per square meter. The complex power flow across the surface of a sphere of radius r meters is therefore

$$P^* = \tfrac{1}{2} r^2 \int\int (\mathcal{E}_\theta H_\varphi^* - \mathcal{E}_\varphi H_\theta^*) \sin\theta \, d\theta \, d\varphi \tag{227}$$

where θ is the angle with the Z axis and φ the angle with the X axis for a spherical coordinate system r, θ, φ so that the Laplacian transformations $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, and $z = r\cos\theta$ apply. The θ and φ components are then normal to each other. and the relations $\mathcal{E}_\theta = Z_0 H_\varphi$ and $\mathcal{E}_\varphi = -Z_0 H_\theta$ hold.

If the surface S is the boundary of a very good conductor ($\sigma \cong \infty$ mhos/m,) the power flow P given by (227) must be tangential, since no power across S is possible because tangential $\mathcal{E}$ components along a very highly conducting surface must vanish, which explains the screening effect of a closed metal surface. The same would be true if we had a medium of $Z_0 = \infty$ ohms, since $\mathcal{E}/H = \infty$ ohms, and therefore no tangential H components could exist across a boundary separating a medium of $Z_0 = \infty$ ohms from a medium that is, for instance, $z_0 = 377$ ohms (air). Hence, for such a hypothetical case, no electromagnetic power flow is possible across the boundary into the high-impedance medium. Since for the general medium

$$Z_0 = \sqrt{\frac{Y}{Z}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\kappa}} = \sqrt{\frac{\mu}{\kappa - j\dfrac{\sigma}{\omega}}} = \sqrt{\frac{\mu}{\kappa_e}} \qquad \text{ohms} \tag{228}$$

where, as mentioned already, κ_e denotes the equivalent dielectric constant for a dissipative medium with σ mhos per meter present in addition to μ henrys per meter and κ farads per meter, we can realize a value of only $Z_0 = \infty$ for a medium at very low frequencies. Also, on account of (227), we have the characteristic impedances

$$Z_{\theta\varphi} = \frac{\mathcal{E}_\theta}{H_\varphi} \qquad \text{and} \qquad Z_{\varphi\theta} = -\frac{\mathcal{E}_\varphi}{H_\theta} \qquad \text{ohms}$$

just as in any curvilinear system with coordinates ξ and η normal to

each other and

$$\left.\begin{aligned} P^* &= \tfrac{1}{2} \iint_S (\mathcal{E}_\xi H_\eta^* - \mathcal{E}_\eta H_\xi^*)\, dS \qquad \text{watts} \\ Z_{\xi\eta} &= \frac{\mathcal{E}_\xi}{H_\eta} \quad \text{ohms} \quad \text{and} \quad Z_{\eta\xi} = -\frac{\mathcal{E}_\eta}{H_\xi} \quad \text{ohms} \end{aligned}\right\} \tag{229}$$

We find

$$\left.\begin{aligned} P^* &= \tfrac{1}{2} \iint_S Z_{\eta\xi} H_\xi H_\xi^* + Z_{\xi\eta} H_\eta H_\eta^*)\, dS \\ P^* &= \frac{1}{2} \iint_S \left(\frac{\mathcal{E}_\xi \mathcal{E}_\xi^*}{Z_{\xi\eta}} + \frac{\mathcal{E}_\eta \mathcal{E}_\eta^*}{Z_{\eta\xi}} \right) dS \\ P^* &= \tfrac{1}{2} \iint (Z_{\varphi\theta} H_\theta H_\theta^* + Z_{\theta\varphi} H_\varphi H_\varphi^*) \sin\theta\, d\theta\, d\varphi \end{aligned}\right\} \tag{230}$$

which for an isotropic impedance condition, *i.e.*,

$$Z_{\xi\eta} = Z_{\eta\xi} = Z_{\theta\varphi} = Z_{\varphi\theta} = \sqrt{\frac{\mu}{\kappa_e}} = Z_0 \qquad \text{ohms}$$

yields

$$P^* = \tfrac{1}{2} \iint Z_0 (H_\xi H_\xi^* + H_\eta H_\eta^*)\, dS$$
$$P^* = \tfrac{1}{2} \iint Z_0 (H_\theta H_\theta^* + H_\varphi H_\varphi^*) \sin\theta\, d\theta\, d\varphi$$

where Z_0 then acts normal everywhere to the surfaces across which energy contributions making up P^* flow.

43A. Relationship between Electric Scalar and Magnetic Vector Potentials and the Electromagnetic Field Vectors and Meaning of the Retarded Potentials. As mentioned in the introductionary remarks of this chapter, we can find the electromagnetic fields if we know the current and charge distributions on a radiator. The analysis can then often be made easier by using potentials that are auxiliary functions. We then postulate a scalar potential V volts in terms of the electric field intensity of $\mathcal{E}$ volts per meter and a vector potential Λ volt · seconds per meter, in terms of the magnetic flux density B in webers per square meter, and such that they satisfy the relations

$$\mathcal{E} = -\operatorname{grad} V - \frac{\partial \Lambda}{\partial t}; \qquad B = \operatorname{curl} \Lambda = \mu H \tag{231}$$

The quantity V is then a *scalar* point function, making the *negative* gradient a *vector* point function. We know from ordinary field theory that a surface on which V is fixed (is constant) is an equipotential surface and along the normal between two different equipotential surfaces we have a potential gradient. Suppose we have a conductor of volume τ as

indicated in the Cartesian coordinate system of Fig. 39. Suppose we know the current density J at any point P_0 of coordinates x, y, and z of the solid conductor. Such a conductor point P_0 refers then to a conductor element $d\tau = dx\,dy\,dz$ in the XYZ system with indicated components yielding $J = \sqrt{J_x^2 + J_y^2 + J_z^2}$ if we have the scalar values of the com-

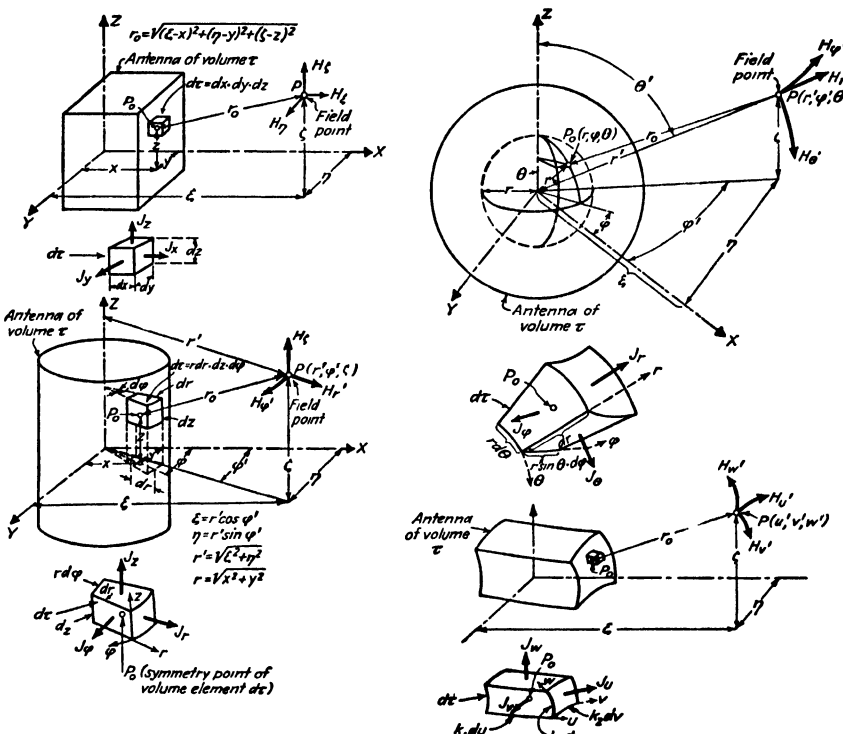


FIG. 39. Antennas and antenna elements and the $\mathcal{E}$ and H components at a distant field point P. All current density components J are space functions. P is field point, P_0 is symmetry center of volume element $d\tau$ of radiator, J_x is a function of x, J_y a function of y, J_z a function of z.

ponent densities in mind. Now, if the charge density ρ in coulombs per unit volume is used, we have, for an elementary volume of $d\tau$, an electric charge of $q = \rho\,d\tau$ coulombs and, according to Coulomb's law, an electric field force $d\mathcal{E} = q(4\pi\kappa r_0^2)$ volts per meter at the field point P of Fig. 39. For air as a medium, we have $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m. The resultant field vector $\mathcal{E}$ due to the elementary charge q in the volume element $d\tau$ is then along the r_0 direction. Since the field point is located at ξ, η, ζ, we find the electric field components

$$\left.\begin{aligned}
d\mathcal{E}_\xi &= -\operatorname{grad}_\xi V \equiv -\frac{\partial V}{\partial \xi} = -\frac{1}{4\pi}\frac{\partial\left(\frac{q}{\kappa r_0}\right)}{\partial \xi}\\
&= +\frac{q}{4\pi\kappa r_0^2}\frac{\partial r_0}{\partial \xi}\\
&= \frac{q}{4\pi\kappa r_0^2}\frac{\partial[(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2]^{1/2}}{\partial \xi}\\
&= \frac{q}{4\pi\kappa r_0^2}\frac{\xi - x}{r_0} = \frac{\xi - x}{4\pi\kappa r_0^3}\rho\, d\tau \qquad \text{volts/m}\\
d\mathcal{E}_\eta &= -\frac{\partial V}{\partial \eta} = \frac{\eta - y}{4\pi\kappa r_0^3}\rho\, d\tau \qquad \text{volts/m}\\
d\mathcal{E}_\zeta &= -\frac{\partial V}{\partial \zeta} = \frac{\zeta - z}{4\pi\kappa r_0^3}\rho\, d\tau \qquad \text{volts/m}
\end{aligned}\right\}$$

correct if no *fast* time variation of elementary charges $\rho\, d\tau$ coulombs occur

Therefore, for static charges or slow $\rho\epsilon^{j\omega t}$ variations, we have then in (231) the simplified expression $\mathcal{E} = -\operatorname{grad} V$ and find, for the total field contribution at field point P,

$$V^{(\text{volts})} = \frac{1}{4\pi}\iiint_\tau \frac{\rho}{\kappa r_0}\, d\tau$$

$$\mathcal{E}_\xi^{(\text{volts/m})} = \iiint_\tau d\mathcal{E}_\xi = -\frac{\partial\left(\frac{1}{4\pi}\iiint_\tau \frac{\rho}{\kappa r_0}\, d\tau\right)}{\partial \xi} = -\frac{\partial V}{\partial \xi}$$

$$= \frac{1}{4\pi}\iiint_\tau \frac{\xi - x}{\kappa r_0^3}\rho\, d\tau$$

$$\mathcal{E}_\eta = \frac{1}{4\pi}\iiint_\tau \frac{\eta - y}{\kappa r_0^3}\rho\, d\tau; \qquad \mathcal{E}_\zeta = \frac{1}{4\pi}\iiint_\tau \frac{\zeta - z}{\kappa r_0^3}\rho\, d\tau$$

Relations of this type then apply, for instance, to vacuum tubes when the frequency is not too high, and we speak of customary scalar potentials of V volts and the corresponding electric field components. As a matter of fact, we can go up to quite high frequencies before the second part[1] of the $\mathcal{E}$ expression of (231) plays a part, since the cathode surface, where the charges are initiated, is only a short distance from the plate at which the output effects take place. But when we come to frequencies above 100 Mc and especially above 500 Mc/sec, we can no longer ignore the transit time effect in an electron tube. Now if we had, instead of the large volume (finite volume) τ and even instead of the elementary volume $d\tau$ of the Cartesian system XYZ of Fig. 39, only a vertical current filament of length dz, we should deal only with an elementary current radiator

[1] For details consult A. Hund, "High-frequency Measurements," 2d ed., Sec. 182, Mcgraw-Hill Book Company, Inc., New York, 1951.

dz oriented along the Z direction. Since this radiator is infinitesimal in length, we have along its entire length a fixed current distribution of I amperes, with a current moment of $I\,dz$ meter · amperes, which is along the $\pm dz$ directions. Hence, in any r_0 direction, we have only the $I\,dz \cos\theta$ effect, since θ is the angle that r_0 makes with the dz direction. If we take again the near-field condition and for slow time variations, then we have for the resultant magnetic field intensity dH_φ in amperes per meter, at the field point P

$$dH_\varphi = \frac{I\,dz}{4\pi r_0^2} \sin\theta \qquad \text{Biot-Savart law}$$

for dz and r_0 in meters, since the circular magnetic field is around the dz current element. We have then only two magnetic components

$$dH_x = -\frac{I\,dz}{4\pi r_0^3}(\eta - y); \qquad dH_y = \frac{I\,dz}{4\pi r_0^3}(\xi - x); \qquad \text{since } dH_z = 0$$

In these component relations, the characteristic term is still $I\,dz/4\pi r_0^2$, since it is multiplied by $(\eta - y)/r_0$ and $(\xi - x)/r_0$, respectively. We can then compute the magnetic field values in the distant radiation region also from the auxiliary function $d\Lambda_\zeta = \mu I\,dz/(4\pi r_0)$ volt · seconds per meter, since, according to Fig. 17*a*, we have no Λ components along the X and Y directions with a current filament only along the Z direction. The quantity Λ is then called the "magnetic vector" potential at the field point P, just as V is the "electric scalar" potential at this point. For electromagnetic fields, we then assume such auxiliary potentials in the distant radiation field. Because it takes the scalar and the vector potential a certain time to reach such distant field points with a time function $\epsilon^{j(\omega t - \beta r_0)}$ instead of $\epsilon^{j\omega t}$, we use the *retarded* electric *scalar* and *retarded* magnetic *vector* potentials. We then find that *only inverse distance terms* ($1/r_0$ terms) play a part instead of inverse $1/r_0^2$ terms. The total field contribution at field point P is then found again by integration. If we deal with a three-dimensional body such as in Fig. 39, we have for each volume element $d\tau$ a magnitude of the resultant current density of the form $J = \sqrt{J_x^2 + J_y^2 + J_z^2}$ amperes per square meter. The currents along the three Cartesian directions then are $I_x = J_x\,dy\,dz$, $I_y = J_y\,dx\,dz$, and $I_z = J_z\,dx\,dy$ amperes. We then treat each component like a linear current element and find, at field point $P(\xi, \eta, \zeta)$ in the distant radiation region,

$$d\Lambda_\xi = \frac{\mu I_x\,dx}{4\pi r_0} = \frac{\mu J_x \overbrace{dx\,dy\,dz}^{d\tau}}{4\pi r_0}; \qquad d\Lambda_\eta = 0; \qquad d\Lambda_\zeta = 0 \qquad \text{for the } I_x\,dx \text{ current element}$$

$$d\Lambda_\eta = \frac{\mu I_y\,dy}{4\pi r_0} = \frac{\mu J_y\,d\tau}{4\pi r_0}; \qquad d\Lambda_\xi = 0; \qquad d\Lambda_\zeta = 0 \qquad \text{for the } I_y\,dy \text{ current element}$$

$$d\Lambda_\zeta = \frac{\mu I_z\, dz}{4\pi r_0} = \frac{\mu J_z\, d\tau}{4\pi r_0}; \qquad d\Lambda_\xi = 0; \qquad d\Lambda_\eta = 0 \quad \text{for the } I_z\, dz \text{ current element}$$

in volt · seconds per meter. The total vector potential contributions at field point P are then

$$\Lambda_\xi = \frac{\mu}{4\pi}\int\int\int_\tau \frac{[J_x]\, d\tau}{r_0}; \qquad \Lambda_\eta = \frac{\mu}{4\pi}\int\int\int_\tau \frac{[J_y]\, d\tau}{r_0};$$
$$\Lambda_\zeta = \frac{\mu}{4\pi}\int\int\int_\tau \frac{[J_z]\, d\tau}{r_0} \qquad \text{volt} \cdot \text{sec/m} \qquad (232)$$

where $[J_x]$, $[J_y]$, and $[J_z]$ mean that we deal with *retarded* effects of time function $\epsilon^{j(\omega t - \beta r_0)}$. Since, according to (231), we have $\mu H = \text{curl}\ \Lambda$, we find from the Cartesian components of Eq. (137),

$$\mu H_\xi = \frac{\partial \Lambda_\zeta}{\partial \eta} - \frac{\partial \Lambda_\eta}{\partial \zeta}; \qquad \mu H_\eta = \frac{\partial \Lambda_\xi}{\partial \zeta} - \frac{\partial \Lambda_\zeta}{\partial \xi};$$
$$\mu H_\zeta = \frac{\partial \Lambda_\eta}{\partial \xi} - \frac{\partial \Lambda_\xi}{\partial \eta} \qquad \text{webers/m}^2 \qquad (233)$$

This is followed up in Sec. 45 for all three coordinate systems.

Since we have $\mu H = B = \text{curl}\ \Lambda$ and, according to Maxwell, curl $\mathcal{E} = -\partial(B)/\partial t$, we have also

$$\text{curl}\ \mathcal{E} = -\frac{\partial B}{\partial t} = -\frac{\partial(\text{curl}\ \Lambda)}{\partial t}; \qquad \text{curl}\ \overbrace{\left(\mathcal{E} + \frac{\partial \Lambda}{\partial t}\right)}^{-\ \text{grad}\ V} = 0$$

44. Notes on the Complex Field Vector, Scalar Potential, Vector Potential, and Hertz Vector in Connection with Maxwell's Equations. Generally, wave potentials, as used in the Lorentz equations, are retarded as well as attenuated, since we deal with mediums that are *not* necessarily *free from dissipation*. We have then to distinguish between scalar and vector potentials. The Hertzian solutions are based on such potentials. The integral $K\int(A\, d\tau)/r$ is called a "scalar potential," where A denotes any scalar quantity that is arbitrarily distributed in a region with volume elements $d\tau$, the quantity r is the distance from $d\tau$ to a fixed point, and K is a factor depending on the system of units used and the characteristics of the wave potential of interest. If A denotes a vector function of position, the integral gives the vector potential. Thus the electric field due to stationary charges is a vector $\mathcal{E}$, which can be derived from a scalar potential V according to $\mathcal{E} = -\text{grad}\ V$. A vector potential Λ can also be considered as a vector such that the line integral of Λ around any closed path is equal to the particular flux through the path, or $\oint_l \Lambda \cos \alpha\, dl = \int\int_S F \cos \beta\, dS$, where dl is the line element of the path

that completely encloses the area S and F is the particular flux density. If this density represents $B = \mu H$, we deal with the magnetic vector potential $\Lambda = \Lambda_m$, since the so-called magnetic current is operative. Instead of respective scalar and vector potentials, the Hertzian vector (*polarization* potential) and the Fitzgerald vector (*magnetization* potential) can also be used for field solutions. This requires, however, that space charge densities be taken care of by space polarization and magnetization densities. Inasmuch as we can have a closed-line integral due to an electric field force of $\mathcal{E}$ volts per meter and to a magnetic field force of H amperes per meter, where respective forces are vectors (magnitude and direction), we have also in electromagnetic relations two distinct *scalar* quantities defined by

$$V = \oint_l \mathcal{E}\, dl \quad \text{volts} \qquad \text{and} \qquad U = \oint_l H\, dl \quad \text{amp} \tag{234}$$

where l denotes the closed path, expressed in meters, enclosing an area of S square meters and both forces, *i.e.*, $\mathcal{E}$ and H, act along respective path elements dl expressed in meters. Based on such symmetric relations, we find the electromotive force of V volts equal to the negative surface integral $-\int\int_S J_m\, dS$ if J_m is taken as the so-called magnetic current density in volts per square meter. For the magnetomotive force U, we have the customary surface integral $\int\int_S J\, dS$ for the electric current density J in amperes per square meter. Also here J_m and J act normal to the respective surface or area element dS; otherwise we have to multiply by the cosine of the angle that respective current densities make with dS. We have then $J_m = \partial(\mu H)/\partial t$ volts per square meter, since according to Faraday's induction law $V = -S\partial(\mu H)/\partial t$ volts. Since in one case, we have a *negative* surface integral (for V volts) and, in the other, a *positive* surface integral (for U amperes), the electric-current case follows a corkscrew law while the magnetic current case follows an anticorkscrew law. Hence, when the electric-current density J amperes per square meter flows along the positive X direction of a Cartesian system, the magnetomotive force of U amperes circles, or acts around, in a counterclockwise direction in the YZ plane and encloses an area of 1 sq m. On the other hand, if the magnetic current density J_m volts per square meter acts along the positive X direction, the electromotive force V volts acts around in a clockwise direction.

To these scalar wave potentials of V volts and U amperes correspond the vector potentials Λ_m and Λ_e, the first one, namely, Λ_m, being called the "magnetic" vector potential, since it is affected by the scalar potential V, which is due to the negative time rate of the magnetic flux (μHS).

The electric vector potential Λ_e is affected by the scalar potential U in amperes which is due to the time rate of the electric flux ($\kappa\mathcal{E}s$). The quantity μHS is in (henrys/m)(amp/m)(m^2) or volt · seconds, since according to $e^{(\text{volts})} = L^{(\text{henrys})}[di^{(\text{amp})}]/[dt^{(\text{sec})}]$, we have henrys = volt · seconds per ampere. The electric flux ($\kappa\mathcal{E}S$) is in the same way in (farads/m) (volts/m)(m^2), *i.e.*, in ampere · seconds.

The electric field intensity $\mathcal{E}$ and magnetic field intensity H can be expressed in terms of wave potentials: the electric scalar potential V, the magnetic scalar potential U, the magnetic vector potential Λ_m, and the electric vector potential Λ_e. It is the magnetic vector potential Λ_m that is dealt with in this section and is, for the sake of simplicity, just called the vector potential and written as Λ.

By using a complex field vector C with a real magnetic part μH and an imaginary electric part $\sqrt{\mu\kappa}\,\mathcal{E}$, containing magnetic and electric field vectors, electromagnetic field relations can be written in a more compact form. Similarly, when using scalar and vector potentials, in many cases a single Hertz vector Π can be employed. The integration of electromagnetic relations is also simplified thereby. The general Maxwellian equations read

$$\left.\begin{aligned} &\text{curl}\ \mathcal{E} + \frac{\partial(\mu H)}{\partial t} = 0; \qquad \text{div}\ (\mu H) = 0\ \text{(always solenoidal)} \\ &\text{curl}\ H - \frac{\partial(\kappa\mathcal{E})}{\partial t} = J; \qquad \text{div}\ (\kappa\mathcal{E}) = \rho \end{aligned}\right\} \tag{235}$$

For $B = \mu H$ and $D = \kappa\mathcal{E}$ and the Laplace operator ∇, we have the forms

$$\left.\begin{aligned} \nabla \times \mathcal{E} &= -\frac{\partial B}{\partial t}; \qquad \nabla \cdot B = 0 \\ \nabla \times H &= J + \frac{\partial D}{\partial t}; \qquad \nabla \cdot D = \rho \end{aligned}\right\} . \tag{235a}$$

Using a *single* but complex field vector

$$C = \mu H + j\sqrt{\mu\kappa}\,\mathcal{E}$$

the above relations then reduce to two equations

$$\nabla \times C + j\sqrt{\mu\kappa}\,\frac{\partial C}{\partial t} = \mu J = \mu\sigma\mathcal{E}; \qquad \nabla \cdot C = j\rho\sqrt{\frac{\mu}{\kappa}} \tag{236}$$

or

$$\text{curl}\ C = -j\sqrt{\mu\kappa}\,\frac{\partial C}{\partial t} + \mu\sigma\mathcal{E}; \qquad \text{div}\ C = j\rho\sqrt{\frac{\mu}{\kappa}} \tag{236a}$$

The use of a complex vector of the type as C was suggested by L. Silber-

stein[1] and followed up by others. In the above relations, we have again $\mathcal{E}$ in volts per meter, H in amperes per meter, C in volt · seconds per square meter, since C must have the same dimension as the real quantity, or the j quantity. The latter is $\sqrt{\mu\kappa}$ times $\mathcal{E}$. The first factor is the reciprocal of a velocity, *i.e.*, in seconds per meter, while the second factor is in volts per meter, giving the above dimension of volt · seconds per square meter. The other dimensions are, as usual, μ in henrys per meter, κ in farads per meter, σ in mhos per meter, J in amperes per square meter, and ρ in coulombs per cubic meter. For the customary radiation field (atmosphere), both the charge ρ and the conduction current J are equal to zero, since we deal with solenoidal conditions and have neither sources nor sinks in the medium and have, because of $\sigma = 0$ mhos/m, also $J = \sigma\mathcal{E} = 0$, hence no conduction current density. This means that for the phase velocity $c' = 1/\sqrt{\mu\kappa}$ meters per second,

$$\text{curl } C = j\frac{1}{c'}\frac{\partial C}{\partial t} \qquad \text{and} \qquad \text{div } C = 0 \tag{237}$$

The velocity of propagation is then $c' = c = 3 \times 10^8$ m/sec unless the radiation field is a perfect dielectric other than air.

The magnetic vector potential Λ in volt · seconds per meter is then proportional to the time rate of the Hertz vector Π, with a constant of proportionality equal to the reciprocal of the square of the velocity of propagation. The Hertz vector is then in meter · volts, and we have

$$\Lambda = \mu\kappa\frac{\partial\Pi}{\partial t} \quad \text{volt} \cdot \text{sec/m}; \qquad B = \mu\kappa\nabla \times \frac{\partial\Pi}{\partial t}$$

$$\text{webers/m}^2 \text{ (or volt} \cdot \text{sec/m}^2) \tag{238}$$

$$\mathcal{E} = -\nabla V - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} = -\text{ grad } V - \frac{\partial\Lambda}{\partial t} \qquad \text{volts/m}$$

where V denotes the *electric* scalar potential (a position function) and Π is the *electric* polarization potential. The source as well as the electro-

[1] SILBERSTEIN, L., *Ann. phys.*, **22, 24** (1907); *Phil. Mag.*, **23** (6), 790 (1912). BATEMAN, H., "Electrical and Optical Wave Motion," p. 4, Cambridge University Press, London, 1915. STRATTON, J. A., "Electromagnetic Theory," p. 32, McGraw-Hill Book Company, Inc., New York, 1941. In these original papers, we find $C = H \pm j\mathcal{E}$ with the simpler solutions curl $C \doteq \mp j\frac{1}{c}\frac{\partial C}{\partial t}$ and div $C = 0$, since solenoidal or sourceless conditions were assumed and above simplification stands for the Heaviside symmetrical forms curl $H = \frac{1}{c}\frac{\partial\mathcal{E}}{\partial t}$; curl $\mathcal{E} = -\frac{1}{c}\frac{\partial H}{\partial t}$; div $\mathcal{E} = 0 =$ div H. We note that in Heaviside units, $C =$

magnetic field are distributions of an *electric polarization*.[1] By means of the Hertz vector Π, it is often possible to express an electromagnetic field by a single vector function.

If a scalar function V volts, as in relations (238), is distributed in a certain volume τ with volume elements $d\tau$, expressed in cubic meters, for the retarded scalar V and vector Λ potentials, at a field point ξ, η, and ζ a distance r_0 meters from a volume element $d\tau$ located at xyz in Fig. 39, we have the relations[2]

$$\left.\begin{aligned} dV &= \frac{[\rho]}{4\pi r_0\kappa}\,d\tau \quad \text{volts}; \qquad d\Lambda = \frac{\mu[J]}{4\pi r_0}\,d\tau \quad \text{volt}\cdot\text{sec/m} \\ &\text{or} \\ V &= \frac{1}{4\pi\kappa}\int\!\!\int\!\!\int_\tau \frac{[\rho]\,dx\,dy\,dz}{r_0} \quad \text{volts}; \\ \Lambda &= \frac{\mu}{4\pi}\int\!\!\int\!\!\int_\tau \frac{[J]\,dx\,dy\,dz}{r_0} \quad \text{volt}\cdot\text{sec/m} \end{aligned}\right\} \qquad (239)$$

if [ ] shows that the respective enclosed scalar (ρ coulombs per cubic meter) and vector (J amperes per square meter) are at the field point P at ξ, η, ζ at the retarded time $t - r/v$ seconds, where $v = 1/\sqrt{\mu\kappa}$ meters per second in a perfect dielectric medium ($\sigma = 0$ mhos/m) and κ_e is to be used instead of the actual dielectric κ in farads per meter when σ mhos per meter becomes noticeable ($\kappa_e = \kappa - j\sigma/\omega$ farads per meter with $\omega/6.28$ in cycles per second). For the retarded quantities, we then have $[\rho] = \rho_{(t-r_0/v)} = \rho\epsilon^{-j\beta r_0}$, since $\beta = \omega/v$, $[J] = J\epsilon^{-j\beta r_0}$, $J = \rho v$ amperes per square meter, for $\sigma = 0$ mhos/m. The quantities ρ and J are $\epsilon^{j\omega t}$ variations. These expressions are based upon the concept that a space distribution of electric currents can be divided into elementary currents, each causing elementary fields. For an elementary volume of $d\tau$ cubic meters with a total current flow of I amperes normal to a cross section of dS square meters, for the current density J in amperes per square meter, we have the relation $J = I/dS$. For an

[1] Consult Secs. 245 and 246. A good discussion is also given in Stratton, *op cit.*, pp. 28–32. It is shown that for $\rho = 0 = J$, the electromagnetic field can be resolved in two separate fields, one derived from a vector Π and the other from a vector Π^*. The field associated with Π^* is due to magnetic polarization distribution. A good discussion on polarization and magnetization potentials of this type is also given in R. W. P. King, "Electromagnetic Engineering," Vol. I, pp. 174–177, McGraw-Hill Book Company, Inc., New York, 1945.

[2] In order to be consistent with the nomenclature used for cylindrical (r, φ, z) and spherical (r, φ, θ) coordinates at other places in this text, the letter r_0 is used for the distance of the volume element $d\tau$ to the field point P. The character ρ is used for the elementary electric charge density in coulombs per cubic meter.

elementary radiator of length dl, we have an electric moment $m_e = I\,dl$ meter · amperes; and for an elementary volume $d\tau$, we have a space moment $J\,d\tau$.

Equations (239) hold for nondissipative regions (zero-space conductivity). We note that for the magnetic vector potential (Λ is Λ_m in this case), the current density J amperes per square meter enters. It is still the result of the electric volume density ρ coulombs per cubic meter utilized in the expression for the scalar potential of V volts. It refers now to moving electric charges that constitute a current flow. Generally, this flow is a *convection* current, since an electric volume density ρ multiplied by the average velocity acting on it causes such a current flow. Therefore, if all charges making up the elementary space-charge concentration per cubic meter and in coulombs are drifting with the mean velocity v of so many meters per second, then ρv denotes the current density J in amperes per square meter. This means that the volume density of a convection current refers to charge per 1 sec and per square meter area. Since a current, *i.e.*, a charge motion, has a direction, the elementary convection current J amperes per square meter represents a vector acting at the central symmetry point of the elementary volume $d\tau$ expressed in cubic meters. The convection current density J amperes per square meter changes to a *conduction* current density J if only free electrons are moving *while* the *positive* charges *remain at rest.* This happens in a *perfect conductor.*

For the general medium (dissipative), we have, in addition, the effect of the specific conductivity σ. This means that in the propagation constant $\Gamma = \alpha + j\beta$, we can no longer use the exponent $j\beta$ only, where $\beta = \omega/v$ or $2\pi/\lambda$ and λ meters $= 300/f$, where f is in megacycles per second. We have then for the propagation constant Γ, the expression

$$\left.\begin{aligned} \Gamma &= j\omega\sqrt{\mu\kappa_e} = \alpha + j\beta \qquad \text{generalized hyperbolic radians/m} \\ \kappa_e &= \kappa - j\frac{\sigma}{\omega} \qquad \text{farads/m} \end{aligned}\right\} \tag{240}$$

where $\omega/6.28$ is in cycles per second and $v = \omega/\beta$ is in meters per second, since the attenuation constant α is in pure hyperbolic radians per meter, *i.e.*, in nepers per meter, and the phase or wavelength constant β is in circular radians per meter. We then find for the dissipative medium

$$\begin{aligned} V &= -\iiint_\tau \frac{\operatorname{div} J\epsilon^{-r_0\Gamma}}{4\pi(\sigma + j\omega\kappa)r_0}\,d\tau \qquad \text{volts} \\ \Lambda &= \frac{\mu}{4\pi}\iiint_\tau \frac{J\epsilon^{-r_0\Gamma}}{r_0}\,d\tau \qquad \text{volt}\cdot\text{sec/m} \end{aligned} \tag{241}$$

In these relations, J stands again for the retarded value $J_{(t-r_0/c')}$; and in

the scalar V relation, the corresponding current density is directly used. The vector potential stands also here for the magnetic vector potential Λ_m. For the other scalar potential, namely, the *magnetic* potential U in amperes and the associated *electric* vector potential Λ_e in ampere · seconds per meter, we find, for a dissipative medium,

$$U = -\iiint_\tau \frac{\text{div } J_m \epsilon^{-r_0\Gamma}}{4\pi j\omega\mu r_0}\,d\tau \qquad \text{amp}$$

$$\Lambda_e = \frac{\kappa_e}{4\pi}\iiint_\tau \frac{J_m \epsilon^{-r_0\Gamma}}{r_0}\,d\tau \qquad \text{amp} \cdot \text{sec/m} \qquad (242)$$

where the magnetic current density is again retarded and in volts per square meter and the equivalent dielectric constant $\kappa_e = \kappa - j\sigma/\omega$ in farads per meter appears instead of the actual dielectric constant κ, which would hold for the electric vector potential in the absence of field dissipation ($\sigma = 0$ mhos/m). The dimension of this vector potential can be readily checked. We know that the charge q of a condenser C is all the greater the larger the value of C farads and the greater the voltage E volts is applied, since $q = CE$ with a corresponding unit equation, coulombs ≡ ampere · seconds = farad · volts. Hence, we find farads = ampere · seconds/volt. Now κ_e is in farads per meter = ampere · seconds per meter · volts, J_m is in volts per square meter, r_0 is in meters, $d\tau$ is in cubic meters, and the unit of Λ_e is

$$\left|\frac{\kappa_e J_m\,d\tau}{r_0}\right| = \frac{\dfrac{\text{amp} \cdot \text{sec}}{\text{m} \cdot \text{volts}} \cdot \dfrac{\text{volts}}{\text{m}^2} \cdot \text{m}^3}{\text{m}} = \text{amp} \cdot \text{sec/m}$$

As mentioned before, it is the *magnetic* vector potential relation of Eq. (241) that is made use of in this section. It is the customary vector potential used in most texts, since by means of it and the *electric* scalar potential V, we can find the electric field intensity $\mathcal{E}$ in volts per meter and the magnetic field intensity H in amperes per meter, since for $B = \mu H$, we have

$$B = \text{curl } \Lambda \quad \text{webers/m}^2; \qquad \mathcal{E} = -\text{grad } V - \frac{\partial \Lambda}{\partial t} \quad \text{volts/m} \qquad (243)$$

These are important relations and are often written in the form

$$B = \nabla \times \Lambda; \qquad \mathcal{E} = -\nabla V - \frac{\partial \Lambda}{\partial t} \qquad (243a)$$

Now the magnetic vector potential $\Lambda = \Lambda_m$ has, according to Eq. (241), the dimension of volt · sec/m. This is also verified by the second relation in Eq. (243), since the time rate of Λ has the same dimension as $\mathcal{E}$, *i.e.*, volts/m = $|\Lambda|$/sec, or Λ is in volt · sec/m. However, according to the first of Eqs. (243*a*), it is in webers/m; since ∇ is

per meter and $|\nabla \times \Lambda| = |1/m \times \Lambda|$ webers/m², or $|\Lambda| = |(\text{webers/m}^2) \cdot (\text{m})|$, which is webers per meter. We can also convert volt · sec/m to volt · amp · sec/(m · amp), which is watt · sec/(m · amp), or joules/amp/m. This means that the magnetic vector potential $\Lambda = \Lambda_m$ is an energy per ampere per meter. *Therefore, when we multiply a current with a magnetic vector potential, we deal with energy.* In the same way, we can convert the amp · sec/m unit for the electric vector potential Λ_e and find amp · sec/m = volt · amp · sec/(m · volt) = joules/volt/m. Hence, when we *multiply* the *electric vector potential* Λ_e *by a voltage, we likewise obtain energy.* Hence, Λ_m and Λ_e can be referred to as energy when multiplied by a suitable corresponding factor.

Figure 39 gives an application of such vector potentials with respect to the curvilinear reference, Cartesian, cylindrical, and spherical coordinate systems. The conditions for Cartesian coordinates have already been dealt with. For cylinder coordinates r, φ, and z, the volume element of which a radiator is composed of is $d\tau = r\,dr\,dz\,d\varphi$, expressed in cubic meters. We then have $r = \sqrt{x^2 + y^2}$, $x = r\cos\varphi$, and $y = r\sin\varphi$, where r, when in planes parallel with the XY plane, makes the angle φ with the $+X$ direction. For spherical coordinates of the space occupied by the radiator, the volume element of the radiator is again expressed in cubic meters and given by $d\tau = r^2 \sin\theta\,dr\,d\varphi\,d\theta$, since we have the coordinates r, φ, and θ, and $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. The direction of r with the Z axis makes an angle θ, and the projection of r into the XY plane makes an angle φ with the X axis. For orthogonal curvilinear coordinates, $u = f_1(x, y, z)$, $v = f_2(x, y, z)$, and $w = f_3(x, y, z)$, the respective differential curved coordinates producing the elementary volume $d\tau$, are proportional to the differentials of the curved coordinates u, v, and w, and

$$d\tau = (k_1\,du)(k_2\,dv)(k_3\,dw)$$

where k_1, k_2, and k_3 are the respective constants of proportionality. We found already that the vector potential Λ, whether magnetic or electric, is a vector in which the line integral of Λ around a closed path is equal to the flux across the area enclosed by the path. For an isotropic medium, the electromagnetic constants of the medium do not depend upon $\mathcal{E}$ and H and

$$\left.\begin{aligned} \mu H &= B = \nabla \times \Lambda \equiv \text{curl } \Lambda \\ \kappa\mathcal{E} &= D = -\kappa\nabla V - \kappa\frac{\partial\Lambda}{\partial t} \equiv -\kappa\left[\text{div } V + \frac{\partial\Lambda}{\partial t}\right] \\ &\text{or} \\ H^{(\text{amp/m})} &= \frac{\text{curl } \Lambda}{\mu}; \qquad \mathcal{E}^{(\text{volts/m})} = -\text{ div } V - \frac{\partial\Lambda}{\partial t} \end{aligned}\right\} \tag{244}$$

Also

$$\nabla^2\Lambda + \mu J = \mu\kappa\frac{\partial^2\Lambda}{\partial t^2} \tag{245}$$

Relations (244) hold, since curl $\mathcal{E} = -\partial B/\partial t$ and div $B = 0$. From $B = \text{curl}\,\Lambda$ and $\mathcal{E} = -\text{grad}\,V - \partial\Lambda/\partial t$, we can compute the H and $\mathcal{E}$ vectors when Λ and V are known. We find then first the H vector, because it is the curl $\Lambda = \mu H$ expression that is used to compute the radiation effects[1] in the distant radiation region for different antenna shapes, since the magnetic H_ξ, H_η, and H_ζ components at space points ξ, η, and ζ are defined by the resultant magnetic field intensity $H = \sqrt{H_\xi^2 + H_\eta^2 + H_\zeta^2}$. We have also for the resultant electric field intensity $\mathcal{E} = \sqrt{\mathcal{E}_\xi^2 + \mathcal{E}_\eta^2 + \mathcal{E}_\zeta^2}$ and for plane waves in the distant radiation field, $\mathcal{E}/H = Z_0 = \sqrt{\mu/\kappa_e}$ ohms, where as above $\kappa_e = \kappa - j\sigma/\omega$ farads per meter with $\omega/6.28$ cycles per second, κ in farads per meter, σ in mhos per meter, and μ in henrys per meter. For atmosphere, $\sigma = 0$ and $Z_0 \cong 377$ ohms. Since Λ is found from (239) for the three Cartesian components Λ_ξ, Λ_η, and Λ_ζ and the abbreviations $K = \mu/12.56$ and $d\tau = dx\,dy\,dz$ expressed in cubic meters, we have

$$\Lambda_\xi = K\int\int\int_\tau \frac{[J_x]}{r_0}\,d\tau; \qquad \Lambda_\eta = K\int\int\int_\tau \frac{[J_y]}{r_0}\,d\tau;$$
$$\Lambda_\zeta = K\int\int\int_\tau \frac{[J_z]}{r_0}\,d\tau \qquad \text{volt}\cdot\text{sec/m} \qquad (246)$$

Inasmuch as the resultant current density J as well as its components along the XYZ directions in $J = \sqrt{J_x^2 + J_y^2 + J_z^2}$, if expressed in absolute values, act across 1-sq-m surfaces, J_x refers in these expressions to surface element $dy\,dz$, J_y to $dx\,dz$, and J_z to $dx\,dy$. *Nevertheless*, we still refer to the *volume element* $d\tau$, except as far as the current flows are concerned; but for each component, the magnetic vector potential, for instance, for the Λ_ξ component, current flows only across YZ planes play a part. Having found such components, from $\mu H = \text{curl}\,\Lambda$, we obtain the magnetic field components

$$H_\xi = \frac{\text{curl}_\xi\,\Lambda}{\mu} = \frac{(\partial\Lambda_\zeta/\partial\eta) - (\partial\Lambda_\eta/\partial_\zeta)}{\mu}; \qquad H_\eta = \frac{\text{curl}_\eta\,\Lambda}{\mu};$$
$$H_\eta = \frac{(\partial\Lambda_\xi/\partial_\zeta) - (\partial\Lambda_\zeta/\partial_\xi)}{\mu}; \qquad H_\zeta = \frac{\text{curl}_\zeta\,\Lambda}{\mu} = \frac{(\partial\Lambda_\eta/\partial\xi) - (\partial\Lambda_\xi/\partial\eta)}{\mu}$$
$$\text{amp/m} \qquad (247)$$

A check on the dimensions shows that μ is in henrys per meter. Because $L = e\,dt/di$ in volt · seconds per ampere, this is the dimension for the henry unit. The vector potential Λ is in volt · seconds per meter. The curl Λ is, for its components, composed of Λ gradients in volt · seconds per square meter. Hence, the dimensions of μ^{-1} curl Λ become

[1] HUND, A., *op. cit.*, pp. 342–347.

$$\underbrace{\frac{\text{amp}\cdot\text{m}}{\text{volt}\cdot\text{sec}}}_{\text{for }1/\mu}\ \underbrace{\frac{\text{volt}\cdot\text{sec}}{\text{m}^2}}_{\text{for curl }\Lambda} = \underbrace{\frac{\text{amp}}{\text{m}}}_{\text{for }H} \qquad \text{proving the correctness of (247)}$$

The dimensions[1] of Λ used here are found most easily by the last expression of (238), since the term $\partial\Lambda/\partial t$ is the dimension desired divided by the second on account of the ∂t operator and must be equal to the dimension of $\mathcal{E}$, which is volts per meter. Hence, Λ is in volt · seconds per meter.

If cylindrical coordinates must be used because of a certain shape of the radiator of volume τ expressed in cubic meters, we have

$$\left.\begin{aligned} H_{r'} &= \frac{\text{curl}_{r'}\,\Lambda}{\mu} = \frac{\dfrac{1}{r'}\dfrac{\partial\Lambda_\zeta}{\partial\varphi'} - \dfrac{\partial\Lambda_{\varphi'}}{\partial\zeta}}{\mu} \\ H_{\varphi'} &= \frac{\text{curl}_{\varphi'}\,\Lambda}{\mu} = \frac{\dfrac{\partial\Lambda_{r'}}{\partial\zeta} - \dfrac{\partial\Lambda_\zeta}{r'}}{\mu} \\ H_\zeta &= \frac{\text{curl}_\zeta\,\Lambda}{\mu} = \frac{\dfrac{1}{r'}\left[\dfrac{\partial(r'\Lambda_{\varphi'})}{\partial r'} - \dfrac{\partial\Lambda_{r'}}{\partial\varphi'}\right]}{\mu} \end{aligned}\right\}\ \text{amp/m} \qquad (248)$$

where for $d\tau = r\,dr\,dz\,d\varphi$ in cubic meters and for $K = \mu/12.56$,

$$\Lambda_{r'} = K\iiint_\tau \frac{[J_r]}{r_0}\,d\tau; \qquad \Lambda_{\varphi'} = K\iiint_\tau \frac{[J_\varphi]}{r_0}\,d\tau;$$

$$\Lambda_\zeta = K\iiint_\tau \frac{[J_z]}{r_0}\,d\tau \qquad \text{volt}\cdot\text{sec/m} \quad (249)$$

For Cartesian coordinates, we have the vectorial relation $J = iJ_x + jJ_y + kJ_z$ for the resultant current density in amperes per square meter, where the unit vectors i, j, and k are along the orthogonal X, Y, and Z directions, respectively. For 1 cu m of a medium, the vector J is thought of as acting at the central symmetry point of this reference volume. For Cartesian coordinates, this point is the intersection point of the three diagonal planes of the reference unit volume. Regardless of whether we deal with the resultant vector J or any one of its three components, such as J_y, for instance, the current flow is in each case in amperes per square meter. For the resultant J it is the flow across a square-meter area through said symmetry point and normal to the J direction. For the

[1] A check on dimensions is a good way of finding whether or not a mathematical result is correct. A term such as μ might be overlooked, and the dimension would show that a factor is missing. A simple numerical application is a further check on formulas.

component J_y, it is a square-meter area parallel to the XZ plane and through the symmetry point or any other parallel plane occupied by 1 cu m. For the current density J, we have simply the total current effect belonging to 1 cu m of isotropic space. For any of the three components of J, we have only the current contribution along the particular component direction and across a square meter normal to the particular direction. If the resultant vector J is along any one of the edges of the unit reference volume, there is only one component current density, which then gives the total current effect per cubic meter. Generally, the volume density of a current has the dimension of a charge (coulombs) per second and per square meter in the mks system of units.

For cylindrical coordinates, we also have orthogonal conditions for the differential edges of the differential volume element $d\tau$. The differential distance dr corresponds to the differential distance dx, $R\,d\varphi$ to dy, and dz to dz according to Fig. 39. The edges dr, $r\,d\varphi$, and dz of the cylindrical differential volume element are, therefore, perpendicular to each other. We have the unit vectors i, j, and k along r, φ, and z coordinate directions; and for the vectors, $J = iJ_r + jJ_\varphi + kJ_z$ and, when expressed in absolute amplitude value, $J = \sqrt{J_r^2 + J_\varphi^2 + J_z^2}$. Since for J_x, the elementary areas are $dy\,dz$ square meters, we have for the total current flow across such faces $I_x = J_x\,dy\,dz$ amperes; and for the same reason, $J_y = I_y/(dx\,dz)$, $J_z = I_z/(dx\,dy)$, where I_y and I_z are the total current flows in the Y and Z directions, respectively. These are then the current flows aross the respective faces of the elementary space element $d\tau = dx\,dy\,dz$ expressed in cubic meters. For cylindrical coordinates, we have, similarly, $I_r = J_r r\,d\varphi\,dz$, $I_\varphi = J_\varphi\,dr\,dz$, and $I_z = J_z r\,d\varphi\,dr$ amperes. For antenna shapes whose volume τ requires differential volume elements $d\tau$, which build up a sphere, we require spherical coordinates r, φ, and θ. For the space point P, where the respective electromagnetic fields act, we have the coordinates r', φ', and θ', which are again orthogonal to each other. We obtain the magnetic field intensity components at the space point $P(r', \varphi', \theta')$ as

$$\left.\begin{aligned} H_{r'} &= \frac{\text{curl}_{r'}\,\Lambda}{\mu} = \frac{\dfrac{\partial(\sin\theta'\,\Lambda\varphi')}{\partial\theta'} - \dfrac{\partial(\Lambda_{\theta'})}{\partial\varphi'}}{\mu r'\sin\theta'} \\ H_{\varphi'} &= \frac{\text{curl}_{\varphi'}\,\Lambda}{\mu} = \frac{\dfrac{\partial(r'\Lambda_{\theta'})}{\partial r'} - \dfrac{\partial(\Lambda_{r'})}{\partial\theta'}}{\mu r'} \\ H_{\theta'} &= \frac{\text{curl}_{\theta'}\,\Lambda}{\mu} = \frac{\dfrac{\partial(\Lambda_{r'})}{\partial\varphi'} - \sin\theta'\dfrac{\partial(r'\Lambda\varphi')}{\partial r'}}{\mu r'\sin\theta'} \end{aligned}\right\}\ \text{amp/m} \qquad (250)$$

where, for $d\tau = r^2 \sin\theta \, dr \, d\varphi \, d\theta$ and $K = \mu/12.56$,

$$\Lambda_{r'} = K \int\int\int_\tau \frac{[J_r]}{r_0} d\tau; \qquad \Lambda_{\varphi'} = K \int\int\int_\tau \frac{[J_\varphi]}{r_0} d\tau;$$

$$\Lambda_{\theta'} = K \int\int\int_\tau \frac{[J_\theta]}{r_0} d\tau \qquad \text{volt} \cdot \text{sec/m} \qquad (251)$$

Then the unit vectors i, j, and k are along the orthogonal line r and orthogonal arcs φ and θ, and $J = iJ_r + jJ_\varphi + kJ_\theta$, where $J_r = I_r/(r^2 \sin\theta \, d\theta \, d\varphi)$, $J_\varphi = I_\varphi/(r \, dr \, d\theta)$, and $J_\theta = I_\theta/(r \sin\theta \, dr \, d\varphi)$ in amperes per square meter.

For antenna shapes that make use of orthogonal coordinates u, v, w, each a particular function of x, y, z, for respective constants k_1, k_2, and k_3, we have $d\tau = k_1 \, du k_2 \, dv k_3 \, dw$, expressed in cubic meters; and for $K = \mu/12.56$, we have the vector potential components

$$\Lambda_u = K \int\int\int_\tau \frac{[J_u]}{r_0} d\tau; \qquad \Lambda_v = K \int\int\int_\tau \frac{[J_v]}{r_0} d\tau;$$

$$\Lambda_w = K \int\int\int_\tau \frac{[J_w]}{r_0} d\tau \qquad \text{volt} \cdot \text{sec/m} \qquad (252)$$

yielding the magnetic field components

$$\left.\begin{aligned} H_{u'} &= \frac{\text{curl}_{u'} \Lambda}{\mu} = \frac{\dfrac{\partial(k_3\Lambda_{w'})}{\partial v'} - \dfrac{\partial(k_2\Lambda_{v'})}{\partial w'}}{\mu k_2 k_3} \\ H_{v'} &= \frac{\text{curl}_{v'} \Lambda}{\mu} = \frac{\dfrac{\partial(k_1\Lambda_{u'})}{\partial w'} - \dfrac{\partial(k_3\Lambda_{u'})}{\partial u'}}{\mu k_1 k_3} \\ H_{w'} &= \frac{\text{curl}_{w'} \Lambda}{\mu} = \frac{\dfrac{\partial(k_2\Lambda_{v'})}{\partial w'} - \dfrac{\partial(k_1\Lambda_{u'})}{\partial v'}}{\mu k_1 k_2} \end{aligned}\right\} \text{amp/m} \qquad (253)$$

since u', v', and w' are the coordinates of the distant field point P of the radiation region.

45. Applications for Finding the Electric and Magnetic Field Intensity by Means of the Electric Scalar Potential and the Magnetic Vector Potential. For a thin wire (filament) as an antenna along the Z axis, we find for a sinusoidal current distribution at any place z along the vertical radiator

$$I_z = I \sin \frac{q\pi z}{l} \epsilon^{j\omega t} \qquad \text{amp}$$

where l is the length of the linear radiator in meters and q is an integer.

Equations (239) then reduce to a one-dimensional value for $d\tau$, since for a filament antenna the cross section is negligible compared with the length l and also with respect to the operating wavelength determining the phase distribution exhibited along radiator dimensions.[1] We then

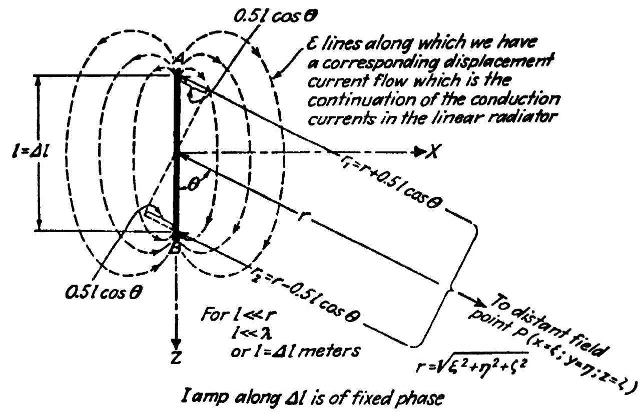


FIG. 40. Short linear radiator.

find for the electric scalar potential V volts and the magnetic vector potential Λ volt · seconds per meter

$$V = \frac{1}{4\pi\kappa}\int_0^l \frac{[\rho]\,dz}{r} \quad \text{volts;} \qquad \Lambda_\zeta = \frac{\mu}{4\pi}\int_0^l \frac{[I_z]\,dz}{r} \quad \text{volt}\cdot\text{sec/m} \qquad (254)$$

where r_0 is taken as r meters as is shown in Fig. 40. The vector potential components Λ_ξ and Λ_η are both equal to zero, since for a filament radiator there can be no current flow in such directions. Since we know the current distribution I_z, we have, for the time rate of the charge, $\partial\rho/\partial t = -\partial I_z/\partial z$ (conservation of electricity). The distribution of charge and the distribution of a current are interdependent like light and shadow. We find for the scalar

$$\rho = -\int_0^t \frac{\partial I_z}{\partial z}\,dt = \frac{j}{\omega}\frac{\partial I_z}{\partial z}\,\epsilon^{j\omega t} = j\,\frac{q\pi I}{\omega l}\cos\frac{q\pi z}{l}\,\epsilon^{j\omega t}$$

The retarded values are then, for $\beta = \omega/v$ and $v = 1/\sqrt{\mu\kappa} \cong 3 \times 10^8$ m/sec $= c$ holding for atmosphere,

[1] In many antenna solutions, we assume sinusoidal current and voltage distributions that for a thin wire, occur along the length dimension. These distributions are, in either case, phase distributions and depend on the phase velocity c' along the antenna or any other direction if the antenna has a shape other than mentioned above. A wavelength distribution along an aerial corresponds to 360 electrical degrees and is developed along the wire somewhat shorter than in free space.

$$\left.\begin{aligned} [\rho] &= \rho\epsilon^{-j\beta r} = j\,\frac{q\pi I\epsilon^{j\omega(t-r/c)}}{\omega l}\cos\frac{q\pi z}{l} \\ [I_z] &= I_z\epsilon^{-j\beta r} = I\epsilon^{j\omega(t-r/c)}\sin\frac{q\pi z}{l} \end{aligned}\right\} \tag{255}$$

which values are used for computing the scalar potential V and the only vector potential component Λ_ζ of (254) and for finding the electric field intensity $\mathcal{E}$ at the space point ξ, η, ζ from (238) by means of

$$\mathcal{E}^{(\text{volts/m})} = -\operatorname{grad} V - \frac{\partial \Lambda_\zeta}{\partial t} \tag{256}$$

However, it is easier to use the curl relation (247), since terms with Λ_η and Λ_ξ vanish in this particular example, because the fine wire linear antenna is parallel with the ζ coordinate of space point P. Hence,

$$H_\xi = \frac{\partial\Lambda_\zeta/\partial\eta}{\mu}; \qquad H_\eta = \frac{-\partial\Lambda_\zeta/\partial\xi}{\mu}; \qquad H = \sqrt{H_\xi^2 + H_\eta^2} \qquad \text{amp/m} \tag{257}$$

where H is the amplitude value of the resultant magnetic field intensity $H\epsilon^{j\omega t}$ at field point P with an electric field association

$$\mathcal{E} = H\sqrt{\frac{\mu}{\kappa}} \cong 377H \qquad \text{volts/m} \tag{258}$$

if it is assumed that field propagation occurs in unobstructed atmosphere, so that only a direct electromagnetic ray is possible.

In Fig. 40, where Δl is taken very short compared with the wavelength, the current distribution I_z along Δl is practically constant. If the field point P is very far away compared with the antenna length Δl, the distances r_1, r, and r_2 essentially become parallel, and r_1 and r_2 are by the same amount $\frac{1}{2}\Delta l\cos\theta$ longer and shorter, respectively, than the reference distance r (distance from field point P to the radiation center). The only vector potential left is again

$$\Lambda_\zeta = \frac{\mu[I_z]}{4\pi r}\,\Delta l = \frac{\mu\,\Delta l I}{4\pi r}\,\epsilon^{j\omega(t-r/v)} \qquad \text{volt}\cdot\text{sec/m}$$

where $\epsilon^{ja} = \cos a + j\sin a$, and the real part $\cos a$ or the real portion $\sin a$ of the imaginary term is a solution representing the physical state. Using $\sin a = \sin(\omega t - \beta r)$ for $\beta = \omega/v$ and $v = 1/\sqrt{\mu\kappa}$ for any perfect dielectric medium and equal to 3×10^8 m/sec for atmosphere, we have the physical condition

$$\Lambda_\zeta = \frac{\mu\,\Delta l I}{4\pi r}\sin(\omega t - \beta r) \qquad \text{volt}\cdot\text{sec/m} \tag{259}$$

and find from (247)

$$\left.\begin{aligned}
H_\xi &= \frac{\partial\Lambda_\zeta/\partial\eta}{\mu} = \frac{-\eta}{4\pi r}\left[\frac{I\,\Delta l}{r^2}\sin(\omega t - \beta r) + \frac{\beta I\,\Delta l}{r}\cos(\omega t - \beta r)\right] \\
H_\eta &= \frac{-\partial\Lambda_\zeta/\partial\xi}{\mu} = \frac{\xi}{4\pi r}\left[\frac{I\,\Delta l}{r^2}\sin(\omega t - \beta r) + \frac{\beta I\,\Delta l}{r}\cos(\omega t - \beta r)\right] \\
H_{\varphi'} &= H = \sqrt{H_\xi^2 + H_\eta^2} \\
&= \frac{I\,\Delta l\,\sqrt{\xi^2+\eta^2}}{4\pi r^2}\left[\frac{\sin(\omega t - \beta r)}{r} + \beta\cos(\omega t - \beta r)\right]
\end{aligned}\right\}\ \text{amp/m} \qquad (260)$$

because the resultant horizontal component H is the $H_{\varphi'}$ component in the spherical r', φ', θ' system of Fig. 39, except that with our case of a fine wire antenna of length Δl (Fig. 40), we have $r = r_0$, where r_0 is the value of Fig. 39. The factor $\sqrt{\xi^2 + \eta^2}$ is the result of the differentiations of Λ_ζ with respect to ξ and η, respectively, and value $r = \sqrt{\xi^2 + \eta^2 + \zeta^2}$ in formula (259) for the vector potential Λ_ζ. Putting $\omega t - \beta r = a$, we have for the amplitude values of the magnetic field components

$$\left.\begin{aligned}
&H_\xi = -\frac{\eta I\,\Delta l}{4\pi r^2}F; \qquad H_\eta = \frac{\xi I\,\Delta l}{4\pi r^2}F; \qquad H = \frac{I\,\Delta l\,\sqrt{\xi^2+\eta^2}}{4\pi r^2}F \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{amp/m} \\
&F = \frac{\sin a}{r} + \beta\cos a; \qquad \beta = \frac{\omega}{v} = \omega\sqrt{\mu\kappa}
\end{aligned}\right\} \qquad (261)$$

for a nondissipative ($\sigma = 0$) medium of μ henrys per meter and κ farads per meter.

At first glance, these relations give the impression that we are dealing with a wattless term that is proportional to $1/r^3$ and a radiation term (power term) that is proportional to $1/r^2$. This is not so, since for H_ξ we must also multiply by the η coordinate of the space point and for H_η by the ξ coordinate, which are, according to Fig. 39, the respective distances to planes through the field point P parallel with the ZX and ZY planes. Hence, for P in the equatorial plane of the vertical radiator of Fig. 40, the numerical values of r and η are identical for the H_ξ formula, and $r = \xi$ for the H_η expression. Therefore,

$$\left.\begin{aligned} H_\xi &= -\frac{I\,\Delta l}{4\pi\eta}F = -\Bigg(\underbrace{\frac{I\,\Delta l}{4\pi\eta^2}\sin a}_{\substack{\text{Induction field}\\\text{(always inter-}\\\text{linked with}\\\text{the radiator)}}} + \underbrace{\frac{\beta I\,\Delta l}{4\pi\eta}\cos a}_{\substack{\text{radiation field}\\\text{(escapes from}\\\text{the radiator)}}}\Bigg) \\ H_\eta &= \frac{I\,\Delta l}{4\pi\xi}F = \overbrace{\frac{I\,\Delta l}{4\pi\xi^2}\sin a} + \overbrace{\frac{\beta I\,\Delta l}{4\pi\xi}\cos a} \end{aligned}\right\}\ \text{amp/m} \qquad (262)$$

For an infinitesimally short radiation length $l = dl = dz$, we have an elementary radiator $I_z\,dz = I_z\,dl = I\sin\omega t\,dl$, where I is again the constant-current distribution along dl. This is, in effect, also equivalent to the Hertzian doublet radiator consisting of two lumped charges of $+q$ and $-q$ (at a certain instant) coulombs connected by an infinitely short fine wire so that for oscillating charges, a constant-current distribution I_z, again exists along the connection dl. The charges are shown in Fig. 40 at A and B, respectively, if Δl denotes the connection dl. Because of the conservation of energy, we have for the volume element $d\tau$,

$$d\tau\,\operatorname{div} I_z = \underbrace{d\tau\frac{\partial I_z}{\partial z} = -\frac{\partial\rho}{\partial t}\,d\tau}_{\substack{\text{conservation of}\\\text{electric energy}}} = -\frac{\partial q}{\partial t}$$

and Eq. (239) yields

$$\left.\begin{aligned} V^{(\text{volts})} &= \frac{1}{4\pi\kappa}\iiint_\tau \frac{[\rho]}{r}\,d\tau = \frac{[q]}{4\pi\kappa r_1} + \frac{[-q]}{4\pi\kappa r_2} \\ &= \frac{[I_z\,dt]}{4\pi\kappa r_1} - \frac{[I_z\,dt]}{4\pi\kappa r_2} \\ &= \frac{I\cos(\omega t-\beta r_1)}{4\pi\kappa\omega r_1} - \frac{I\cos(\omega t-\beta r_2)}{4\pi\kappa\omega r_2} \\ A_\zeta = A_z &= \frac{\mu}{12.56}\iiint_\tau \frac{[I_z]}{r}\,d\tau = \frac{\mu I\,dl}{4\pi r}\sin(\omega t-\beta r) \\ r_1 &= r + \tfrac{1}{2}\,dl\cos\theta; \qquad r_2 = r - \tfrac{1}{2}\,dl\cos\theta \end{aligned}\right\} \qquad (263)$$

But

$$\left.\begin{aligned} \omega t - \beta r_1 &= \omega t - \beta(r + 0.5dl\cos\theta) = \underbrace{\omega t - \beta r}_{a} \\ &\qquad\qquad\qquad\qquad - \underbrace{0.5\beta\,dl\cos\theta}_{b} \\ \omega t - \beta r_2 &= \omega t - \beta(r - 0.5dl\cos\theta) = a + b \\ \cos(\omega t-\beta r_1) &= \cos(a-b) = \cos a\cos b - \sin a\sin b \\ \cos(\omega t-\beta r_2) &= \cos(a+b) = \cos a\cos b - \sin a\sin b \end{aligned}\right\} \qquad (264)$$

Putting $r_1 = r_2 = r$ for the inverse distance effect, for the scalar potential, Eq. (263) yields

$$V = \frac{I}{2\pi\kappa\omega r}\sin a \sin b = \frac{I}{2\pi\kappa\omega r}\sin(\omega t - \beta r)\overbrace{\sin(\tfrac{1}{2}\beta\, dl \cos\theta)}^{\cong 0.5\beta\, dl \cos\theta}$$
$$= \frac{\beta\, dlI \cos\theta}{4\pi\kappa\omega r}\sin(\omega t - \beta r) \qquad \text{volts} \qquad (265)$$

If no approximation for the inverse distance effect is made, before introducing the substitutions of (264), we find

$$\sin b = \sin(0.5\beta\, dl \cos\theta) \cong 0.5\beta\, dl \cos\theta$$
$$\cos b = \cos(0.5\beta\, dl \cos\theta) \cong 1$$

and

$$V = \frac{I(\cos a \cos b + \sin a \sin b)}{4\pi\kappa\omega r_1} - \frac{I(\cos a \cos b - \sin a \sin b)}{4\pi\kappa\omega r_2}$$
$$= \frac{I}{4\pi\kappa\omega}\left(\frac{\overbrace{\cos a + 0.5\beta\, dl \cos\theta \sin a}^{m}}{r_1} - \frac{\overbrace{\cos a - 0.5\beta\, dl \cos\theta \sin a}^{n}}{r_2}\right)$$
$$= \frac{I}{4\pi\kappa\omega}\left(\frac{m}{r + \underbrace{0.5\, dl \cos\theta}_{\Delta}} - \frac{n}{r - \Delta}\right)$$
$$= \frac{I}{4\pi\kappa\omega}\left[\frac{(r - \Delta)m - (r + \Delta)n}{r_2 - \Delta^2}\right] \cong \frac{I[r(m - n) - \Delta(m + n)]}{\kappa\omega r^2} \qquad \text{volts}$$

where

$$m - n = \beta\, dl \cos\theta \sin a = \beta\, dl \cos\theta \sin(\omega t - \beta r);$$
$$m + n = 2\cos a = 2\cos(\omega t - \beta r)$$

Hence, for the maximum value of the scalar potential of an elementary radiator in *unobstructed* space, forming a perfect dielectric ($\sigma = 0$), we have

$$V = \frac{I}{4\pi\kappa\omega}\left[\frac{m - n}{r} - \frac{\Delta(m + n)}{r^2}\right]$$
$$= \frac{I\beta\, dl \cos\theta \sin(\omega t - \beta r)}{4\pi\kappa\omega r} - \frac{I\, dl \cos\theta \cos(\omega t - \beta r)}{4\pi\kappa\omega r^2} \qquad \text{volts} \qquad (266)$$

where $M_e = I\, dl/\sqrt{2}$ denotes the *effective* electric moment in meter · amperes of the elementary electromagnetic radiator. This formula discloses an inverse r^2 and an inverse r term, while the approximation formula (265) misses the inverse r^2 term altogether.

According to (259), for an infinitesimally short linear radiator, we have

$$A_\zeta = \frac{\mu\, dlI}{4\pi r} \sin (\omega t - \beta r) \qquad \text{volt} \cdot \text{sec/m} \tag{267}$$

which result together with $-$ grad V, according to (256), would yield a very large value for the term $(\epsilon^{-j\beta r})/r$ next to the surface of the radiator, since r is almost zero at the surface[1] of a very thin wire. For the limiting case where, r approaches zero value, L. Brillouin replaces the $(1/r)\epsilon^{-j\omega(r/c')}$ term by a function $\psi_\omega(r)$, which behaves like the $1/r$ function with its epsilon factor for large values of r (field point a distance away from the surface of the radiator) but takes a very large value for $r = 0$ (right on the thin filament radiator) in such a way as to yield *finite* integrals. The result is a *longitudinal* electric field along the radiator. This still conflicts, since no tangential

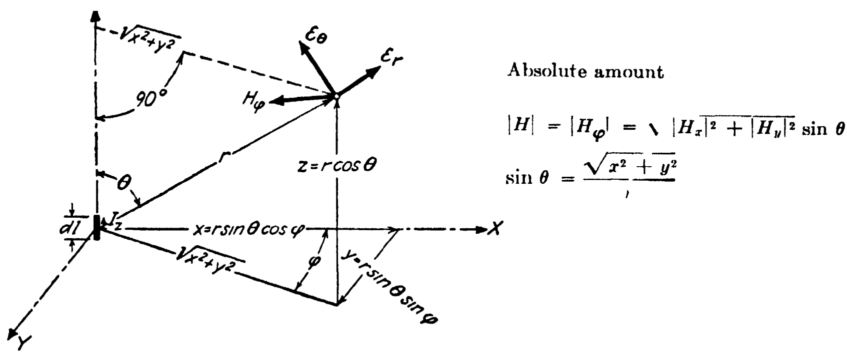


FIG. 41. Elementary linear radiator of length dl. At P, elementary spherical coordinates are dr, $r\,d\theta$, and $r \sin\theta\, d\varphi$.

electric field can exist along a perfect conductor. But under resonance conditions, sustained oscillations can exist when an additional external field just balances out the longitudinal field. This external field supplies every second to the antenna as many watts as is radiated from the antenna into the radiation region. In other words, when the radiation field escapes from a radiator (detaches itself from the radiator), it must still come from the source that causes the induction as well as the radiation field about the radiator. We have then a component of the voltage that is parallel to the antenna wire (at its surface) but in antiphase with the current flowing in the antenna. Hence, the phase between antenna current and total emf then gives rise to a wattless field (induction field, always interlinked with the antenna, and the escaping watt or radiation field). J. A. Stratton[2] gives also an exact calculation of the radiation field of a linear radiator by using the Hertz vector Π.

[1] For other details consult L. Brillouin, *Radioélectricité*, April, 1922; *Elec. Commun.*, 14, 15 (1944). R. Bechmann, *Proc. IRE*, **19**, 461, March, 1931; 682, April, 1931; 1471, August, 1931. A. A. Pistolkors, *Proc. IRE*, **17**, 562, March, 1929. J. A. Stratton, *op. cit.*, pp. 454–460.

[2] *Loc. cit.*

Formulas as given by (266) for the scalar potential V in volts and the vector potential Λ_ξ in volts · seconds per meter by (267), in the case of an elementary linear radiator, can be used in accounting for radiation effects of customary antennas by integration over the total length l of the antenna. It is then customary to express the field in terms of spherical coordinates as in Fig. 39, where the elementary radiator is at the origin of the XYZ system and $r' = r$. Figure 39 then simplifies to Fig. 41. Since a current flow only along the length element dl is of concern, we have a vector potential Λ_z only, and $\Lambda_x = \Lambda_y = 0$; hence, $\Lambda_\varphi = 0$. We require the evaluation of Λ_θ, Λ_r and of gradients $\text{grad}_\theta V$ and $\text{grad}_r V$ for the last expression of (238), yielding the components

$$\mathcal{E}_\theta = -\text{grad}_\theta V - \frac{\partial \Lambda_\theta}{\partial t}; \qquad \mathcal{E}_r = -\text{grad}_r V - \frac{\partial \Lambda_r}{\partial t} \qquad \text{volts/m} \qquad (268)$$

The vector potential component Λ_r is the resolution of Λ_z at field point P along the r direction; and therefore, $\Lambda_r = +\Lambda_z \cos\theta$. The component Λ_θ of the resultant vector potential Λ_z is tangent to the meridian through P; and therefore, $\Lambda_\theta = -\Lambda_z \sin\theta$, since the angle is $90 - \theta$. Hence, for $\omega t - \beta r = a$, we find

$$\Lambda_\theta = -\frac{\mu I\, dl \sin a}{4\pi r}\sin\theta; \qquad \Lambda_r = \frac{\mu I\, dl \sin a}{4\pi r}\cos\theta \qquad \text{volt} \cdot \text{sec/m}$$

$$\begin{aligned}
\text{grad}_\theta V &= \frac{\partial V}{r\,\partial\theta} = \frac{I\beta\, dl \sin a}{4\pi\kappa\omega r^2}\frac{\partial(\cos\theta)}{\partial\theta} - \frac{I\, dl \cos a}{4\pi\kappa\omega r^3}\frac{\partial(\cos\theta)}{\partial\theta} \\
&= -\frac{I\beta\, dl \sin a}{4\pi\kappa\omega r^2}\sin\theta + \frac{I\, dl\cos a}{4\pi\kappa\omega r^3}\sin\theta \\
&= \frac{I\, dl \sin\theta}{4\pi\kappa\omega r}\left(\frac{\cos a}{r^2} - \frac{\beta \sin a}{r}\right) \qquad \text{volts/m}
\end{aligned}$$

$$\begin{aligned}
\text{grad}_r V &= \frac{\partial V}{\partial r} = \frac{I\beta\, dl\cos\theta}{4\pi\kappa\omega}\frac{\partial\left[\dfrac{\sin(\omega t - \beta r)}{r}\right]}{\partial r} - \frac{I\, dl\cos\theta}{4\pi\kappa\omega}\frac{\partial\left[\dfrac{\cos(\omega t - \beta r)}{r^2}\right]}{\partial r} \\
&= \frac{I\beta\, dl\cos\theta}{4\pi\kappa\omega}\frac{(-\beta r\cos a - \sin a)}{r^2} - \frac{I\, dl\cos\theta}{4\pi\kappa\omega}\frac{(\beta r^2\sin a - 2r\cos a)}{r^4} \\
&= \frac{I\, dl\cos\theta}{4\pi\kappa\omega r^2}\left(-\beta^2 r\cos a - \beta\sin a - \beta\sin a + \frac{2\cos a}{r}\right) \\
&= \frac{I\, dl\cos\theta}{4\pi\kappa\omega r^2}\left[\left(\frac{2}{r} - \beta^2 r\right)\cos a - 2\beta\sin a\right] \qquad \text{volts/m}
\end{aligned}$$

$$\left.\begin{aligned}
\frac{\partial\Lambda_\theta}{\partial t} &= \frac{\mu I\, dl\sin\theta}{4\pi r}\frac{\partial(\sin\omega t - \beta r)}{\partial t} = -\frac{\omega\mu I\, dl\sin\theta}{4\pi r}\cos a \\
\frac{\partial\Lambda_r}{\partial t} &= \frac{\mu I\, dl\cos\theta}{4\pi r}\frac{\partial(\sin\omega t - \beta r)}{\partial t} = +\frac{\omega\mu I\, dl\cos\theta}{4\pi r}\cos a
\end{aligned}\right\} \text{volts/m}$$

giving for Eqs. (268) the expressions

$$\left.\begin{aligned}
\mathcal{E}_\theta &= -\operatorname{grad}_\theta V - \frac{\partial \Lambda_\theta}{\partial t} = \frac{I\,dl \sin\theta}{4\pi\kappa\omega r}\left(\frac{\beta \sin a}{r} - \frac{\cos a}{r^2} + \underbrace{\omega^2\kappa\mu}_{\beta^2} \cos a\right) \\
&= \frac{I\,dl\sin\theta}{4\pi\kappa\omega r}\left[\frac{\beta}{r}\sin(\omega t - \beta r) \right. \\
&\qquad \left. - \left(\frac{1}{r^2} - \beta^2\right)\cos(\omega t - \beta r)\right] \\
\mathcal{E}_r &= -\operatorname{grad}_r V - \frac{\partial \Lambda_r}{\partial t} = \frac{I\,dl\cos\theta}{4\pi\kappa\omega r^2}\left[\left(\beta^2 r - \frac{2}{r}\right)\cos a + 2\beta\sin a \right. \\
&\qquad \left. - \underbrace{\omega^2\kappa\mu}_{\beta^2}\, r\cos a\right] \\
&= \frac{I\,dl\cos\theta}{2\pi\kappa\omega r^2}\left[\beta\sin(\omega t - \beta r) - \frac{\cos(\omega t - \beta r)}{r}\right]
\end{aligned}\right\}\ \text{volts/m}$$

Therefore, we arrive at the simple expressions

$$\left.\begin{aligned}
\mathcal{E}_\theta &= \frac{I\,dl\sin\theta}{4\pi\kappa\omega r}\left[\frac{\beta}{r}\sin a - \left(\frac{1}{r^2} - \beta^2\right)\cos a\right] \\
\mathcal{E}_r &= \frac{I\,dl\cos\theta}{2\pi\kappa\omega r^2}\left(\beta\sin a - \frac{\cos a}{r}\right) \\
a &= \omega t - \beta r; \qquad \beta = \frac{\omega}{v}; \qquad v = \frac{1}{\sqrt{\mu\kappa}}
\end{aligned}\right\}\ \text{volts/m} \tag{269}$$

Since the effective current in amperes is $I/\sqrt{2}$ in these formulas, we have $M_e = (I\,dl)/\sqrt{2}$ as the effective electric moment of the elementary radiator in meter · amperes and along dl. The quantity θ is the angle of direction r meters toward the field point with the length direction dl of the elementary radiator, where dl is expressed in meters, μ in henrys per meter, κ in farads per meter, $\omega/6.28$ the frequency in cycles per second, $v = 1/\sqrt{\mu\kappa}$ the velocity of propagation in meters per second, and $\beta = \omega/v$ the field phase constant in circular radians per meter. Making these substitutions in (269), we find the alternate formulas

$$\left.\begin{aligned}
\mathcal{E}_\theta^{(\text{volts/m})} &= \frac{I\,dl\sin\theta}{4\pi\kappa}\left(\underbrace{\frac{\omega\kappa\mu\cos a}{r}}_{A} + \underbrace{\frac{\sqrt{\kappa\mu}\sin a}{r^2}}_{B} - \underbrace{\frac{\cos a}{\omega r^3}}_{C}\right) \\
&= \frac{I\,dl\sin\theta}{4\pi\kappa}(A + B - C) \\
\mathcal{E}_r^{(\text{volts/m})} &= \frac{I\,dl\cos\theta}{2\pi\kappa}\left(\frac{\sqrt{\kappa\mu}\sin a}{r^2} - \frac{\cos a}{\omega r^3}\right) \\
&= \frac{I\,dl\cos\theta}{4\pi\kappa}(B - C)
\end{aligned}\right\} \tag{270}$$

$$\left.\begin{aligned} H_\varphi^{(\mathrm{amp/m})} = H = \sqrt{H_x^2 + H_y^2} &= \frac{I\,dl\,\sqrt{x^2 + y^2}}{4\pi}\left(\frac{\sin a}{r^3} + \frac{\beta \cos a}{r^2}\right) \quad \text{according to (260)} \\ &= \frac{I\,dl}{4\pi}\frac{\sqrt{x^2+y^2}}{r}\left(\frac{\sin a}{r^2} + \frac{\omega\sqrt{\kappa\mu}\cos a}{r}\right) \\ &= \frac{I\,dl \sin\theta}{4\pi}\Bigl(\underbrace{\frac{\sin a}{r^2}}_{\substack{\text{induction}\\ \text{field}}} + \underbrace{\frac{\omega\sqrt{\kappa\mu}\cos a}{r}}_{\substack{\text{radiation}\\ \text{field}}}\Bigr) \quad \text{according to Fig. 41} \end{aligned}\right\} \tag{270}$$

These are the only components which can occur, and we note that at distances $r \gg 1$ and $r \gg \lambda$ meters, only terms with an inverse-distance law ($1/r$ law) can play a part. Hence, $\mathcal{E}_r$ is zero in the distant radiation region, since B and C are negligible as well as $(\sin a)/r^2$, leaving, for the instantaneous electric moment m_e, only the relations

$$\left.\begin{aligned} \mathcal{E}_\theta &= \frac{I\,dl\omega\mu \sin\theta}{4\pi r}\cos(\omega t - \beta r) \quad \text{volts/m} \\ H_\varphi &= \frac{I\,dl\omega\sqrt{\kappa\mu}\sin\theta}{4\pi r}\cos(\omega t - \beta r) \quad \text{amp/m} \\ \frac{\mathcal{E}_\theta}{H_\varphi} &= \sqrt{\frac{\mu}{\kappa}} \quad \text{ohms} \\ \omega I\,dl\cos(\omega t - \beta r) &= \frac{dm_e}{dt} \\ m_e &= I\,dl\sin(\omega t - \beta r) \end{aligned}\right\} \text{in the distant radiation region for a non-dissipative medium} \tag{271}$$

Here $\mathcal{E}_\theta$ and H_φ are in time phase[1] and space quadrature with each other. Since these fields follow an exponential ($j\omega t$) law, we deal with sinusoidal time variations and $\mathcal{E}_\theta$ and H_φ of these expressions can also be taken as instantaneous field values. The electric moment m_e is likewise an instantaneous current of a ($dlI\epsilon^{j\omega t}$) radiator and of its corresponding *retarded* $dlI\epsilon^{j(\omega t - \beta r)}$ effect. The effective moment M_e of the same radiator is equal to $0.707I\,dl$, since $dl(I/\sqrt{2})$ denotes the effective current times the length of the elementary radiator. We have, for the instantaneous radiator current, $i = I\epsilon^{j\omega t}$ and, for the retarded current effect, $[i] = I\epsilon^{j(\omega t - \beta r)}$ for $\beta = \omega/c'$ and $c' = 1/\sqrt{\mu\kappa}$ meters per second, for a perfectly insulating medium. For customary atmosphere, $\mu = \mu_0 = 1.257 \times 10^{-6}$

[1] In these expressions $\mathcal{E}_\theta$ and H_φ denote the absolute amounts (scalar values) in volts per meter and amperes per meter of respective field vectors $\mathcal{E}_\theta$ and H_φ.

henry/m, $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m, $\sqrt{\kappa_0\mu_0} = 1/(2.998 \times 10^8)$ sec/m, $\sqrt{\mu_0/\kappa_0} = 376.7$ ohms, and $1/(4\pi) = 0.0796$. If I_e denotes the fixed effective current $I/\sqrt{2}$ along the elementary radiator $I_e\,dl$ of effective electric moment $M_e = I_e\,dl$ meter · amperes, we have, for the effective field values for f in megacycles per second and r in meters (toward a distant unobstructed radiation region),

$$\mathcal{E}_\theta^{(\text{volts/m})} = \frac{623 \times 10^{-3} M_e f \sin\theta}{r}; \quad H_\varphi^{(\text{amp/m})} = \frac{166 \times 10^{-5} M_e f \sin\theta}{r} \tag{272}$$

Therefore, when we have a half-wave radiator $l = \frac{1}{2}\lambda$ with an effective current of I_e amperes at the current maximum (where a $\lambda/2$ dipole is fed), the effective length l_e is $0.637l$ meters, and

$$M_e = I_e l_e = 0.637 I_e l = 0.637 I_e \frac{\lambda}{2} = 0.637 I_e \frac{150}{f} = \frac{96 I_e}{f}$$

in meter · amperes, where the frequency f is in megacycles per second. The electric and magnetic field associations then read

$$\left.\begin{aligned} \mathcal{E}_\theta &= \frac{623 \times 10^{-3} \times 96 I_e D_\theta}{r} = \frac{60 I_e D_\theta}{r} \quad \text{volts/m} \\ H_\varphi &= \frac{166 \times 10^{-5} \times 96 I_e D_\theta}{r} = \frac{0.159 I_e D_\theta}{r} \quad \text{amp/m} \end{aligned}\right\} \begin{array}{l}\text{for a}\\ \text{half-wave}\\ \text{dipole}\end{array} \tag{273}$$

In these relations, we have to use the meridian direction factor

$$D_\theta = \frac{\cos(90\cos\theta)}{\sin\theta}$$

instead of the directivity, following a "$\sin\theta$ law" holding for an elementary radiator *only*. The distance r from the half-wave radiator to the field point is again in meters.

If we deal with an elementary linear radiator as shown in Fig. 41 or a radiator as in Fig. 40 of length Δl, also short enough with respect to the operating wavelength so that essentially fixed-current distribution prevails along Δl, then the antenna conduction current I leaves the radiator at A as a displacement current and returns through point B. The instantaneous electric current moment $m_e = I\,\Delta l \epsilon^{j\omega t}$ for such a short radiator or for $I\,dl\epsilon^{j\omega t}$, in the case of an elementary radiator, is then a maximum along the directions of the radiator. The quantity m_e holds for the $i = I\epsilon^{j\omega t}$ value of the fixed-current distribution along the radiator in this case.

Inasmuch as m_e acts along dl directions, it is $m_e \cos\theta$ in any other direction in the meridian planes of dl, where θ denotes the angle that the direction makes with dl, in the case of filament radiators. We can also

derive formulas for the amplitude value $J\,dS$ of a moment for an elementary surface radiator of dS square-meter area. The quantity J denotes the amplitude value of the surface current density normal to dS in amperes per square meter. If the surface radiator is in the XY plane or parallel to it, then $dS = dx\,dy$ and the amplitude value of the moment is $J_z\,dx\,dy$ amperes. For an elementary *space* volume radiator, we have a moment $m_e = J\epsilon^{j\omega t}\,d\tau$ meter · amperes, where J is the resultant current density in amperes per square meter and $d\tau$ the space element expressed in cubic meters. If the effective current density $J/\sqrt{2}$ instead of the amplitude value J is used, then we have the effective moment $M_e \doteq 0.707J\,d\tau$. For the space radiator, we can then resolve the crest value $J\,d\tau$ of the volume moment into surface moments $J_x\,dy\,dz$, $J_y\,dx\,dz$, and $J_z\,dx\,dy$.

46. Notes on Arbitrary Charge Distribution in Volume Elements with Respect to Electric Polarization. If we have unequal charges q_1, $-q_2$, q_3, · · · , arbitrarily distributed in a unit volume, we can speak of an average charge density. This would refer then to a total charge $q = q_1 - q_2 + q_3$, · · · , coulombs in a certain space. As a matter of fact, this is the same as the volume density expressed by ρ coulombs per cubic meter, since it likewise refers only to an average state of affairs within 1 cu m of a region. We therefore do not know how the respective negative and positive charges are relatively positioned and in which direction they act.

All matter consists of molecules, which generally contain different atoms. Even the atom is a complicated portion of matter, since it is made up of indivisible electrons, which represent negative elementary charges. The atom also contains positively charged protons made up of positrons (positively charged) of about the same mass as the electron and uncharged neutrons having about the mass of the protons. An electromagnetic field may distort the random motion of any of the charges in matter, since this is true also for bound charges (likewise subject to random motions). A metallic radiator such as an antenna has an abundance of freely moving electrons, while in perfect dielectric regions this is not the case, and we can expect only distortions of random motions of so-called "bound" charges but no transfer of them. If Δl represents the effect of the distortion, *i.e.*, the small distance at which the total negative charge $-q$ is displaced from the equal positive charge q, we have a doublet $q\,\Delta l$ effect (dipole effect), where Δl is a vector and q is a scalar. Therefore, $q\,\Delta l$ is another vector with respect to an origin near the random position where the charges $\pm q$ coulombs would be if an electromagnetic field had not caused distortion. Hence, if $\sum_{j=1}^{j=n} m_j$ denotes the algebraic sum of n different dipole effects $q_j\,\Delta l_j$ within a small volume

$\Delta\tau$, this summation divided by $\Delta\tau$ is the volume density of polarization expressed in meter · coulombs per cubic meter, *i.e.*, in ampere · seconds per square meter. This is a vector acting at the central symmetry point of each volume $\Delta\tau$.

47. Doublet and Elementary Linear Radiator Relations. Inasmuch as the Hertzian electric doublet consists of two opposite charges of equal magnitude q coulombs, which are connected by an infinitesimally short fine wire of length dl expressed in meters, and a corresponding elementary linear radiator $i\,dl$, carrying an instantaneous current $i = I\epsilon^{j\omega t}$ amperes—we have to distinguish between a doublet moment $m = q\,dl$ in meter · coulombs and the moment $m_e = i\,dl$ in meter · amperes for a linear current element. Both are instantaneous moments.

When the charge of the doublet oscillates harmonically, we have an instantaneous moment $m = q\epsilon^{j\omega t}\,dl$ meter · coulombs or meter · amperes · seconds and in consequence thereof an instantaneous current flow of i amperes along the linear length of dl meters. Since $q = \int i\,dt$ or $i = dq/dt$, we find

$$i = \frac{d(m/dl)}{dt} = \frac{1}{dl}\frac{dm}{dt}; \qquad i\,dl = \frac{dm}{dt} = m_e \quad \text{meter} \cdot \text{amp} \tag{274}$$

Another differentiation, with respect to time, yields

$$\frac{dm_e}{dt} = \frac{d^2m}{dt^2} = dl\frac{di}{dt} \qquad \text{meter} \cdot \text{amp/sec} \tag{275}$$

It is this derivative which appears in the field formulas (270) of an elementary radiator of length dl. We can, therefore, utilize the doublet moment $m = q\,dl$ or the current moment $m_e = i\,dl$ by merely integrating such effects over the entire length l of an antenna. That (275) is instrumental in the basic formulas (270) is seen if we consider first only terms that play a part in the distant radiation field and find from (271)

$$\left.\begin{aligned}
\mathcal{E}_0^{(\text{volts/m})} &= \frac{\omega\mu\sin\theta}{4\pi r} I\,dl\cos(\omega t - \beta r) \\
&= \frac{\mu\sin\theta}{4\pi r}\frac{d\overbrace{[I\,dl\sin(\omega t - \beta r)]}^{m_e}}{dt} = \frac{\mu\sin\theta}{4\pi r}\frac{dm_e}{dt} \\
H_\varphi^{(\text{amp/m})} &= \frac{\omega\sqrt{\mu\kappa}\sin\theta}{4\pi r} I\,dl\cos(\omega t - \beta r) = \frac{\sqrt{\mu\kappa}\sin\theta}{4\pi r}\frac{dm_e}{dt} \\
&= \frac{\mu}{\sqrt{\mu^2}}\frac{\sqrt{\mu\kappa}\sin\theta}{4\pi r}\frac{dm_e}{dt} = \underbrace{\frac{\mu\sin\theta}{4\pi r}\frac{dm_e}{dt}}_{=\,\mathcal{E}_\theta}\sqrt{\frac{\kappa}{\mu}} \\
Z_0 &= \sqrt{\frac{\mu}{\kappa}}; \qquad v = \frac{1}{\sqrt{\mu\kappa}}
\end{aligned}\right\} \tag{276}$$

Hence, we can find the instantaneous field effect of an electric current element by means of the time derivative of its moment m_e. Taking into consideration all terms of (270) that hold for any frequency and distance, we find, by also referring to the form of (269) and by using the abbreviations $a = \omega t - \beta r$ and $\beta = \omega\sqrt{\mu\kappa}$ (in circular radians per meter) and equivalences $I\,dl \sin a = m_e$, $\omega I\,dl \cos a = +dm_e/dt$,

$$\left.\begin{aligned}
\mathcal{E}_\theta^{(\text{volts/m})} &= \frac{\beta \sin\theta}{4\pi\kappa\omega r^2} m_e - \frac{\sin\theta}{4\pi\kappa\omega^2 r^3}\frac{dm_e}{dt} + \frac{\beta^2 \sin\theta}{4\pi\kappa\omega^2 r}\frac{dm_e}{dt} \\
&= \frac{\beta^3 \sin\theta}{4\pi\kappa}\left[\frac{dm_e/dt}{\omega^2\beta r} + \frac{m_e}{\omega(\beta r)^2} - \frac{dm_e/dt}{\omega^2(\beta r)^3}\right] \\
\mathcal{E}_r^{(\text{volts/m})} &= \frac{\beta \cos\theta}{2\pi\kappa\omega r^2} m_e - \frac{\cos\theta}{2\pi\kappa\omega^2 r^3}\frac{dm_e}{dt} \\
&= \frac{\beta^3 \cos\theta}{2\pi\kappa}\left[\frac{m_e}{\omega(\beta r)^2} - \frac{dm_e/dt}{\omega^2(\beta r)^3}\right] \\
H_\varphi^{(\text{amp/m})} &= \frac{\sin\theta}{4\pi r^2} m_e + \underbrace{\frac{\sqrt{\mu\kappa}\sin\theta}{4\pi r}}_{\frac{\beta \sin\theta}{4\pi r\omega}}\frac{dm_e}{dt} = \frac{\beta^2 \sin\theta}{4\pi}\left[\frac{dm_e/dt}{\omega\beta r} + \frac{m_e}{(\beta r)^2}\right]
\end{aligned}\right\} \quad (277)$$

This result is the differential form and can be simplified if we introduce complex terms holding for harmonic wave propagation. Then in the equatorial plane of the elementary radiator of maximum electric moment $m_e = I\,dl$ meter · amperes, we have

$$m_e = I\,dl\epsilon^{j(\omega t - \beta r)}; \qquad \frac{dm_e}{dt} = j\omega I\,dl\epsilon^{j(\omega t - \beta r)} = j\omega m_e$$

and the results of (277), for the scalar values of the field components, read

$$\left.\begin{aligned}
\mathcal{E}_\theta &= \frac{\beta^3 m_e \sin\theta}{4\pi\kappa}\left[\frac{j\omega}{\omega^2\beta r} + \frac{1}{\omega(\beta r)^2} - \frac{j\omega}{\omega^2(\beta r)^3}\right] \\
&= \frac{\beta^3 m_e \sin\theta}{4\pi\omega\kappa}\left[\frac{j}{\beta r} + \frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3}\right] \\
\mathcal{E}_r &= \frac{\beta^3 m_e \cos\theta}{2\pi\omega\kappa}\left[\frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3}\right] \quad \text{volts/m} \\
H_\varphi &= \frac{\beta^2 m_e \sin\theta}{4\pi}\left[\frac{j}{\beta r} + \frac{1}{(\beta r)^2}\right] \quad \text{amp/m}
\end{aligned}\right\} \text{at any distance of } r \text{ meters} \quad (278)$$

These are the relations also with respect to the maximum values of $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ if $I\,dl$ is used instead of m_e, since the moment of the radiator element is strictly $m_e\epsilon^{j(\omega t - \beta r)}$, because $I\,dl\epsilon^{j(\omega t-\beta r)}$ is the equivalent. Hence, if $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ are taken as the effective field values, $I\,dl/\sqrt{2} = M_e$ is to be used instead of m_e in these relations. These relations also hold for the half-wavelength radiator if the directivity $D_\theta = [\cos(90 \cos\theta)]/\sin\theta$ is used instead of $\sin\theta$ in the $\mathcal{E}_\theta$ and H_φ relations.

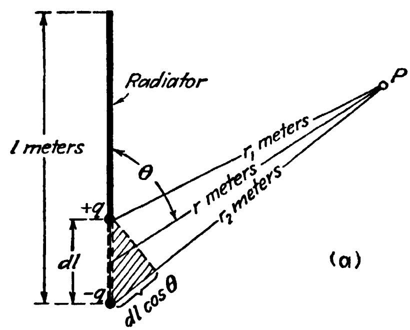
If we wish to express the above relations in terms of the Hertzian doublet moment m, according to (274), we have the relation $m_e = dm/dt$ and

$$m = q\,dl = \int m_e\,dt = dl\int I\epsilon^{j(\omega t-\beta r)}\,dt = \frac{dl}{j\omega}I\epsilon^{j(\omega t-\beta r)} = \frac{m_e}{j\omega}$$

Hence by putting $m_e = j\omega m$ in (278), for the doublet moment m, we find the absolute field values

$$\left.\begin{aligned} \mathcal{E}_\theta &= \frac{\beta^3 m \sin\theta}{4\pi\kappa}\left[-\frac{1}{\beta r}+\frac{j}{(\beta r)^2}+\frac{1}{(\beta r)^3}\right] \\ \mathcal{E}_r &= \frac{\beta^3 m \cos\theta}{2\pi\kappa}\left[\frac{j}{(\beta r)^2}+\frac{1}{(\beta r)^3}\right] \end{aligned}\right\} \text{volts/m} \qquad \left.\begin{aligned} & \\ & H_\varphi = \frac{\beta^2 \omega m \sin\theta}{4\pi}\left[-\frac{1}{\beta r}+\frac{j}{(\beta r)^2}\right] \quad \text{amp/m} \end{aligned}\right\} \text{at any distance of } r \text{ meters} \qquad (279)$$

The moment concept for m_e of (278) and m of (279) is then based upon the representations in Fig. 42*a* and *b*. In each case, the length dl is infinitesi-



Potential at P is

$$V_p^{(\text{volts})} = \frac{q/r_1 - q/r_2}{4\pi\kappa} = \frac{q}{4\pi\kappa}\,\frac{r_2 - r_1}{r_1 r_2} = \frac{q\,dl\cos\theta}{4\pi\kappa r^2} = \frac{m\cos\theta}{4\pi\kappa r^2}$$

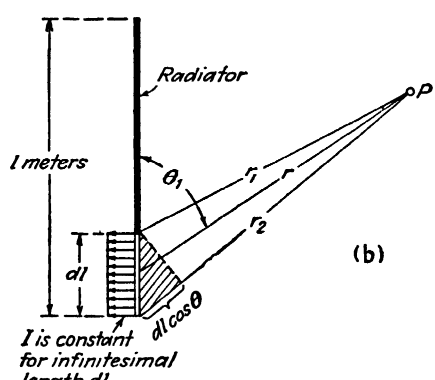
Hertzian Doublet

Instantaneous doublet moment

$$m = M\epsilon^{j(\omega t-\beta r)} = q\,dl\cdot\epsilon^{j(\omega t-\beta r)} \quad \text{meter coulombs}$$

$$r = \frac{r_1 + r_2}{2}; \qquad \beta = \omega\sqrt{\mu\kappa};$$

$\omega/6.28$ in cycles/sec



Potential at P is

$$V_p^{\text{volts}} = \frac{\beta m_e}{4\pi\kappa\omega r} - \frac{dm_e/dt}{4\pi\kappa\omega r^2}$$

For maximum values, we have

$$V_p = \frac{\beta m_e}{4\pi\kappa\omega r} - j\frac{m_e}{4\pi\kappa r^2} \quad \text{volts}$$

Elementary Linear Radiator

Instantaneous current element moment

$$m_e = M_e\epsilon^{j(\omega t-\beta r)} = I\,dl\,\epsilon^{j(\omega t-\beta r)}$$

meter-amp

FIG. 42. A comparison of a Hertzian doublet and a linear elementary radiator. q is in coulombs, κ in farads/m, μ in henrys/m, r in meters.

mally short. The expression for the electric potential V_P at field point P for the oscillating doublet charges $\pm q$ is based on the scalar potential just as is the V_P value for the $I\,dl\epsilon^{j(\omega t-\beta r)}$ radiator, since according to (263)

$$V_P^{(\text{volts})} = \frac{1}{\kappa}\int\int\int_\tau \frac{[\rho]}{4\pi r}\,d\tau = \frac{[q]}{4\pi\kappa r_1} + \frac{-[q]}{4\pi\kappa r_2} \quad \text{for doublet} \quad q\,dl\epsilon^{j(\omega t-\beta r)}$$

$$= \frac{[\int I_l\,dt]}{4\pi\kappa r_1} - \frac{[\int I_l\,dt]}{4\pi\kappa r_2} \quad \text{for current element} \quad I\,dl\epsilon^{j(\omega t-\beta r)}$$

where I_l denotes the I function along the total radiator length of l meters. Hence, *either moment can be used* to account for the effect of any length element dl. The expressions in terms of m_e seem more direct, since we know the current distribution I_l along the radiator length or can at least assume it.

Furthermore, the general formulas (278) show that for distances less than 1 m, the *higher* power inverse-distance terms *become* more *significant*, since the absolute field values are then essentially

$$\left.\begin{aligned}
\mathcal{E}_\theta^{(\text{volts/m})} &= A\sin\theta\underbrace{\left[\frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3}\right]}_{B} = AB\sin\theta \\
\mathcal{E}_r^{(\text{volts/m})} &= 2AB\cos\theta \\
\mathcal{E} &= \sqrt{\mathcal{E}_\theta^2 + \mathcal{E}_r^2} = AB\sqrt{\sin^2\theta + 4\cos^2\theta} \\
&= AB\sqrt{\underbrace{\sin^2\theta + \cos^2\theta}_{1} + 3\cos^2\theta} \\
&= AB\sqrt{1 + 3\cos^2\theta} \quad \text{volts/m} \\
H_\varphi^{(\text{amp/m})} &= \frac{m_e\sin\theta}{4\pi r^2} \\
A &= \frac{\beta^3 m_e}{4\pi\omega\kappa};\quad B = \frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3}; \\
&\qquad \beta = \omega\sqrt{\mu\kappa}
\end{aligned}\right\}\ \text{for } r < 1\text{ m} \quad (280)$$

Putting $\beta r = a$, we find, for the vector $\underset{\cdot}{B}$, its absolute amount B, and its phase ψ,

$$\left.\begin{aligned}
\underset{\cdot}{B} &= \frac{1}{a^2} - \frac{j}{a^3} \\
B &= \sqrt{\frac{1}{a^4} + \frac{1}{a^6}} = \sqrt{\frac{a^6 + a^4}{a^{10}}} = \sqrt{\frac{a^4(a^2+1)}{a^{10}}} \\
&= \frac{\sqrt{1+a^2}}{a^3} = \frac{\sqrt{1+(\beta r)^2}}{(\beta r)^3} \\
\psi &= \tan^{-1}\left(\frac{-1/a^3}{1/a^2}\right) = \tan^{-1}\left(-\frac{1}{\beta r}\right) \quad \text{deg}
\end{aligned}\right\} \quad (281)$$

For a region very close to the surface of the radiator

$$
\left.
\begin{aligned}
\mathcal{E}_\theta^{(\text{volts/m})} &= -j\frac{A \sin\theta}{a^3} = -jC\sin\theta \\
\mathcal{E}_r^{(\text{volts/m})} &= -j\frac{2A \cos\theta}{a^3} = -2jC\cos\theta \\
\mathcal{E} &= \sqrt{\mathcal{E}_\theta^2 + \mathcal{E}_r^2} = \frac{A\sqrt{1 + 3\cos^2\theta}}{(\beta r)^3} \\
&= \frac{m_e\sqrt{1 + 3\cos^2\theta}}{4\pi\omega\kappa r^3} \qquad \text{volts/m} \\
H_\varphi &= \frac{m_e \sin\theta}{4\pi r^2} \qquad \text{amp/m}
\end{aligned}
\right\}
\begin{array}{l}\text{field point next}\\ \text{to the radiator}\\ \text{surface (rather}\\ \text{close to it)}\end{array}
\qquad (282)
$$

Moreover, at any distance, we find for the ratio of the normal field associations $\mathcal{E}_\theta$ and H_φ from (278) for $a = \beta r$

$$
Z = \frac{\mathcal{E}_\theta}{H_\varphi} = \frac{\dfrac{\beta^3 m_e \sin\theta}{4\pi\omega\kappa}\left(\dfrac{j}{a} + \dfrac{1}{a^2} - \dfrac{j}{a^3}\right)}{\dfrac{\beta^2 m_e \sin\theta}{4\pi}\left(\dfrac{j}{a} + \dfrac{1}{a^2}\right)} = \frac{\beta\left(\dfrac{j}{a} + \dfrac{1}{a^2} - \dfrac{j}{a^3}\right)\left(\dfrac{1}{a^2} - \dfrac{j}{a}\right)}{\omega\kappa\left(\dfrac{1}{a^4} + \dfrac{1}{a^2}\right)}
$$

where

$$
\frac{\beta}{\omega\kappa} = \frac{\omega\sqrt{\mu\kappa}}{\omega\kappa} = \sqrt{\frac{\mu}{\kappa}} = Z_0; \qquad \frac{1}{a^4} + \frac{1}{a^2} = \frac{a^2 + a^4}{a^6} = \frac{1 + a^2}{a^4}
$$

$$
\left(\frac{j}{a} + \frac{1}{a^2} - \frac{j}{a^3}\right)\left(\frac{1}{a^2} - \frac{j}{a}\right) = \frac{1}{a^2}\left[\left(j + \frac{1}{a} - \frac{j}{a^2}\right)\left(\frac{1}{a} - j\right)\right]
$$

$$
= \frac{1}{a^2}\left(1 - \frac{j}{a^3}\right) = \frac{a^3 - j}{a^5}
$$

Hence,

$$
Z = \frac{\mathcal{E}_\theta}{H_\varphi} = \sqrt{\frac{\mu}{\kappa}}\,\frac{(a^3 - j)/a^5}{(1 + a^2)/a^4} = \sqrt{\frac{\mu}{\kappa}}\,\frac{a^3 - j}{a + a^3}
$$

We see, therefore, that this impedance, at any distance r meters, is

$$
Z = \frac{a^2}{1 + a^2}\sqrt{\frac{\mu}{\kappa}} - j\frac{\sqrt{\mu/\kappa}}{a + a^3} \qquad \text{ohms, compared with } Z_0 = \sqrt{\mu/\kappa} \text{ in the distant radiation field} \qquad (283)
$$

We have, therefore, a *real and an imaginary* component, each one a certain multiple of the customary distant field impedance to waves. If we expand the ratios $a^2/(1 + a^2)$

and $1/(a + a^3)$, we find

$$\frac{a^2}{1 + a^2} = a^2(1 + a^2)^{-1}; \qquad \frac{1}{a + a^3} = \frac{1}{a(1 + a^2)} = \frac{(1 + a^2)^{-1}}{a}$$

where for $a^2 = x$, we find by means of the binominal theorem

$$(1 + a^2)^{-1} = (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = 1 - a^2 + a^4 - a^6 + a^8 - a^{10} + \cdots$$

$$\frac{a^2}{1 + a^2} = a^2(1 + a^2)^{-1} = a^2 - a^4 + a^6 - a^8 + a^{10} - a^{12} + \cdots$$

$$\frac{1}{a + a^3} = \frac{(1 + a^2)^{-1}}{a} = \frac{1}{a} - a + a^3 - a^5 + a^7 - \cdots$$

Since $\beta = \omega\sqrt{\mu\kappa}$ and $Z_0 = \sqrt{\mu/\kappa}$ ohms, we have

$$Z = \frac{\mathcal{E}_\theta}{H_\varphi} = Z_0\Bigg[\underbrace{a^2 - a^4 + a^6 - a^8 + a^{10} - \cdots}_{\frac{a^2}{1 + a^2}} - j\bigg(\underbrace{\frac{1}{a} - a + a^3 - a^5 + a^7 - \cdots}_{\frac{1}{a + a^3}}\bigg)\Bigg]$$

$$= \sqrt{\frac{\mu}{\kappa}}\left[-\frac{j}{\beta r} + j\beta r + (\beta r)^2 - j(\beta r)^3 - (\beta r)^4 + j(\beta r)^5 + \quad\right]$$

where all j terms are *reactive* components (reactances) and all real terms represent *resistances.*

We find for the ratio of the orthogonal field associations $\mathcal{E}_\theta$ and H_φ,

$$Z = \frac{\mathcal{E}_\theta}{H_\varphi} = \underbrace{\frac{1}{j\omega\mu\kappa r}}_{\text{capacitive reactance}} + \underbrace{j\omega\mu r}_{\text{inductive reactance}} + \underbrace{\frac{\mu\kappa\omega^2 r^2 Z_0}{(\beta r)^2\sqrt{\frac{\mu}{\kappa}}}}_{\text{a resistance}}$$

$$- \underbrace{j[Z_0(\beta r)^3 - Z_0(\beta r)^5 + \cdots]}_{\text{remaining reactances}} - \underbrace{[Z_0(\beta r) - Z_0(\beta r)^6 + \cdots]}_{\text{remaining resistances}} \tag{284}$$

Inasmuch as both the real and the imaginary terms are affected by the series $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$, and this series is actually equal to $1 - (\beta r)^2 + (\beta r)^4 - (\beta r)^6 + \cdots$, we can expect convergence only for $(\beta r)^4 < 1$. But the real terms in (284) are all the outcome of the multiplication by the factor $1/(\beta r)$, the latter tending toward convergence if $\beta r > 1$. Therefore, it is of advantage to expand the real and imaginary terms of (283) by some other method. Using customary algebraic division for $1/(1 + a^2)$, we find

$$
\begin{array}{rl}
(a^2 + 1)\,\underline{|(0 + 1)} & = 0 + \dfrac{1}{a^2} - \dfrac{1}{a^4} + \dfrac{1}{a^6} - \\
\underline{-0 - 0} & \\
1 + 0 & \\
\underline{-1 - \dfrac{1}{a^2}} & \\
-\dfrac{1}{a^2} + 0 & \\
\underline{+\dfrac{1}{a^2} + \dfrac{1}{a^4}} & \\
+\dfrac{1}{a^4} + 0 & \text{remainder to be divided by } (a^2 + 1)
\end{array}
$$

and, therefore,

$$
\frac{a^2}{1 + a^2} = a^2\left(\frac{1}{a^2} - \frac{1}{a^4} + \frac{1}{a^6} - \frac{1}{a^8} + \cdots\right)
$$

$$
= 1 - \frac{1}{a^2} + \frac{1}{a^4} - \frac{1}{a^6} + \cdots + \overbrace{(-1)^{n+1}\frac{1}{a^{2n-2}}}^{n\text{th term}} + \overbrace{(-1)^n \frac{a^{1/(2n-4)}}{1 + a^2}}^{\text{remainder}} \tag{285}
$$

For the imaginary part, if divided by $\sqrt{\mu/\kappa}$, we find that

$$
\frac{1}{a + a^3} = \frac{1}{a}\left(\frac{1}{1 + a^2}\right) = \frac{1}{a^3} - \frac{1}{a^5} + \frac{1}{a^7} - \frac{1}{a^9} + \cdots + \underbrace{(-1)^{n+1}\frac{1}{a^{2n+1}}}_{n\text{th term}}
$$

$$
+ \underbrace{\frac{a^{1/(2n-4)}}{1 + a^2}}_{\text{remainder}} \tag{286}
$$

and

$$
\frac{a^3 - j}{a + a^3} = \underbrace{1 - \frac{1}{a^2} + \frac{1}{a^4} - \cdots}_{\text{equal to } \frac{a^2}{1 + a^2}} - j\underbrace{\left(\frac{1}{a^3} - \frac{1}{a^5} + \frac{1}{a^7} - \cdots\right)}_{\text{equal to } \frac{1}{a + a^3}}
$$

Therefore, for $a = \beta r$, we find

$$
Z = \sqrt{\frac{\mu}{\kappa}}\left[1 - \frac{1}{a^2} + \frac{1}{a^4} - \cdots - j\left(\frac{1}{a^3} + \frac{1}{a^5} + \frac{1}{a^7} + \cdots\right)\right]
$$

$$
= Z_0\left[1 - \frac{1}{(\beta r)^2} - \frac{j}{(\beta r)^3} + \frac{1}{(\beta r^4)} + \frac{j}{(\beta r)^5} - \frac{1}{(\beta r)^6} - \frac{j}{(\beta r)^7} + \cdots\right] \tag{287}
$$

This *expansion has* the advantage that both the real terms (resistance)

and the imaginary terms (reactance) follow inverse (βr) laws with ascending power with *rapid convergence* if $\beta r > 1$. Since

$$\beta = \omega\sqrt{\mu\kappa} = \frac{\omega}{v} \cong \frac{6.28 f^{(\text{cycles/sec})}}{3 \times 10^8}$$

we have

$$\beta r = 2.093 \times 10^{-2} f^{(\text{Mc})} r^{(\text{m})} \tag{288}$$

and $fr = 47.9$ makes βr equal to unity. For convergence the quantity βr must be larger than 47.9. For a 1-Mc wave excitation, r is larger than 47.9 m. Since for 1 Mc/sec, we have $\lambda = {}^{300}\!/_1 = 300$ m, it must be larger than $47.9/300 = 0.16\lambda$. This holds for any frequency, since we find, for instance, for $f = 3{,}000$ Mc/sec the free-space wavelength $\lambda = 300/3{,}000 = 0.1$ m or 10 cm, and $2.093 \times 10^{-2} \times 3{,}000r$ is larger than unity or $62.8r > 1$, *i.e.*, $r \geqq 1/62.8 \geqq 0.016$ m. This is again 0.16λ. At 1 m distance for 3,000-Mc waves, we have $\beta r = 62.8$, $(\beta r)^2 = 3{,}944$, $(\beta r)^3 = 247{,}673$, yielding, according to (287), in the case of atmosphere for which $Z_0 = \sqrt{\mu/\kappa} = 376.7$ ohms

$$Z = 376.7 - \frac{376.7}{3{,}944} - j\frac{376.7}{247{,}673} = \underbrace{376.7 - 0.096}_{376.604 \text{ resistive ohms}} - \underbrace{j0.00152}_{\text{capacitive reactance}} \quad \text{ohms}$$

showing that at a distance of 10 wavelengths from the radiator, we have essentially the characteristic impedance of 377 ohms, since any higher terms including $(\beta r)^4$, $(\beta r)^5$, etc., are entirely negligible. The first reactance term is likewise negligible. This holds even more for reactance (j) terms of still higher inverse (βr) power.

This discussion is not limited to 3,000-Mc waves, since the distance is expressed in wavelengths. Therefore, it may be said that for distances of the order of 10 wavelengths and more, the characteristic impedance of 377 ohms is essentially effective and that a distance of more than 0.16λ is required to make the expression of (287) for Z converge so that only a limited number of terms are needed. Applying (284) to our example and using the first few terms only in a 3,000-Mc wave propagation, we deal with consecutive terms that increase very rapidly and alternate in polarity, since βr is large ($= 6.28$), and

$$\begin{aligned} Z &= -j\frac{Z_0}{\beta r} - jZ_0\beta r + Z_0(\beta r)^2 - jZ_0(\beta r)^3 - Z_0(\beta r)^4 + \cdots \\ &= j\frac{376.7}{62.8} + j376.7 \times 62.8 + 376.7 \times 3{,}944 - j376.7 \times 247{,}673 - 376.7 \\ &\qquad \times 247{,}673 \times 62.8 + \cdots \\ &= -j60 + j23.5 \times 10^3 + \underbrace{14.8 \times 10^5}_{\text{resistive}} - j94 \times 10^6 - \underbrace{59 \times 10^8}_{\text{resistive}} + \cdots \end{aligned}$$

This result is difficult to explain. Therefore, it is not advisable to use the series of

(284) when βr is that large compared with unity. However, it is better for $\beta r < 1$, where the form (287) would give results of this type. Formula (283), for which (284) and (287) are expansions, is the relation for *all* cases, since all the terms are used. For the above case, we find then $a = \beta r = 62.8$, $a^2 = 3{,}944$, $a^3 = 247{,}673$, and

$$Z = \frac{3{,}944}{3{,}945} 376.7 - j \frac{376.7}{62.8 + 247{,}673} = 376.7 - j0.00152 \qquad \text{ohm}$$

checking the result found with expansion (287) and showing that the reactance term is negligible.

48. Poynting Vector, Hertz Vector, Vector Potential, and Formulas for Field Relations. For the *distant radiation region*, according to (271), the electric vector $\mathcal{E}_\theta$ and the magnetic vector H_φ are normal to each other and in time phase, and the power radiated through 1 sq m of the plane wave front is

$$\left.\begin{aligned} p &= \mathcal{E}_\theta \cdot H_\varphi = \frac{\mathcal{E}_\theta^2}{Z_0} = \frac{\mathcal{E}_\theta^2}{377} = 2.65 \times 10^{-3}\mathcal{E}_\theta^2 \\ p &= \mathcal{E}_\theta \cdot H_\varphi = Z_0 H_\varphi^2 = 377 H_\varphi^2 \end{aligned}\right\} \text{watts/m}^2 \qquad (289)$$

In these formulas, $\mathcal{E}_\theta$ in volts per meter and H_φ in amperes per meter are the *effective* values and p is known as the value of the Poynting vector, giving the amount of energy flow, in 1 sec across 1 sq m. The electromagnetic energy flow p is normal to the $\mathcal{E}_\theta$, H_φ plane and outward in unobstructed radiation space.

If $\mathcal{E}_\theta$ and H_φ were the peak values, we had $2.65 \times 10^{-3}(\mathcal{E}_\theta/\sqrt{2})^2 = (377H_\varphi/\sqrt{2})^2$ watts. These relations, therefore, require the determination of only one of the field values. Equations (271) give the instantaneous values for $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter at any point in the distant radiation field. The Poynting vector flow of energy across 1 sq m of area in 1 sec is $p = \mathcal{E}_\theta \times H_\varphi$ watts per square meter. In unobstructed radiation space, the electromagnetic field energy spreads perpendicularly through expanding spherical surfaces with elementary areas of $dS = r^2 \sin\theta\, d\theta\, d\varphi$ square meters, where r is the radius of the expanding wave front in meters from the radiation center where a radiator element $I\, dl \sin \omega t$ acts. Using the instantaneous field values of (271) gives the total power flow through the entire spherical surface of $4\pi r^2$ as

$$\begin{aligned} P &= \iint \mathcal{E}_\theta \cdot H_\varphi \cdot r^2 \sin\theta\, d\theta\, d\varphi \\ &= \underbrace{\frac{\mu\omega^2\sqrt{\mu\kappa}}{16\pi^2}}_{A} \underbrace{[I\, dl \cos(\omega t - \beta r)]^2}_{B} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta\, d\theta \\ &= AB \left|\varphi\right|_0^{2\pi} \cdot \left| -\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right|_0^{\pi} = AB\, 2\pi \frac{4}{3} = \frac{8\pi}{3} AB \end{aligned}$$

But

$$\frac{1}{T}\int_0^T \cos^2(\omega t - \beta r)\,dt = \frac{1}{2}$$

is the average value of $\cos^2(\omega t - \beta r)$. The average power flow across the spherical surface is then

$$\bar{P} = \frac{8\pi}{3}\frac{\mu\omega^2\sqrt{\mu\kappa}}{32\pi^2}(I\,dl)^2 = \frac{\mu\omega^2\sqrt{\mu\kappa}}{12\pi}(I\,dl)^2 = \frac{\mu\omega^2\sqrt{\mu\kappa}}{6\pi}(I_e\,dl)^2 \quad \text{watts} \qquad (290)$$

because the effective current $I_e = I/\sqrt{2}$ amperes. We have, therefore, for the total power flow $\bar{P}$ normal through the spherical surface of sufficient large radius of r meters, and the average[1] amount $\bar{p}$ of the Poynting vector p in any direction r, φ, θ

$$\left.\begin{aligned} \bar{P} &= 2.09 \times 10^{12}\mu f^2\sqrt{\mu\kappa}\,(I_e\,dl)^2 \quad \text{watts through } 4\pi r^2 \text{ sq m} \\ \bar{p} &= \frac{2.5 \times 10^{11}\mu f^2\sqrt{\mu\kappa}\,(I_e\,dl)^2}{r^2}\sin^2\theta \quad \text{watts/m}^2 \end{aligned}\right\} \qquad (291)$$

where f is in megacycles per second, r and dl are in meters, μ in henrys per meter, κ in farads per meter and $\sqrt{\mu\kappa}$ is the reciprocal of the velocity of propagation ($v = 1/\sqrt{\mu\kappa}$ meters/sec) since

$$p \doteq \frac{[I\,dl\cos(\omega t - \beta r)]^2\mu\omega^2\sqrt{\mu\kappa}}{16\pi^2 r^2}\sin^2\theta \quad \text{watts/m}^2$$

$$\text{Average}\,[\cos^2(\omega t - \beta r)] = \tfrac{1}{2}; \qquad I = \sqrt{2}\,I_e$$

The formula for $\bar{p}$ shows that most electromagnetic power flow occurs along the $\theta = 90$ deg direction, *i.e.*, in the equatorial plane of the $I_e\,dl$ radiator. For atmospheric wave propagation, $\mu \cong \mu_0 = 1.257 \times 10^{-6}$ henry/m, $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m, $\sqrt{\mu\kappa} \cong 10^{-8}/3$ sec/m, and $\mu\sqrt{\mu\kappa} \cong 0.419 \times 10^{-14}$ sec$^2\cdot$volts/(m$^2\cdot$amp). Hence,

$$\left.\begin{aligned} \bar{P} &= 8.78 \times 10^{-3}\underbrace{f^2 I_e^2(dl)^2}_{M_e^2} \quad \text{watts through } 4\pi r^2 \text{ sq m} \\ \bar{p} &= \frac{1.05 \times 10^{-3}M_e^2 f^2\sin^2\theta}{r^2} \quad \text{watts/m}^2 \end{aligned}\right\} \text{for atmospheric propagation} \qquad (292)$$

[1] According to (290), the average power $\bar{P}$ is independent of the radius of the expanding spherical wave front. This is obvious, since in nondissipative unobstructed radiation space no power is lost but only spread over more and more surface as the spherical wave front grows. By a radius r sufficiently large is meant that for the formulas to hold, we have to be first in the true radiation region, but so much farther out that the spherical wave front utilized by receiving antenna structures is a flat, *i.e.*, a plane, wave-front portion in effect as far as one square-meter wave front is concerned.

Since the radiation resistance R_r in ohms multiplied by the square of the effective elementary radiator current I_e in amperes is equal to P and R_θ is the radiation resistance effect in the r, φ, θ direction, the power flow with a square-meter wave front at a distance of r meters (r is to reach into the distant radiation region; $r \geqq 10$), we have, since

$$f^{(\text{Mc/sec})}\lambda^{(\text{m})} = 300$$

$$\left.\begin{aligned} R_r I_e^2 &= \bar{P}; \qquad R_\theta I_e^2 = \bar{p}; \qquad R_r = 8.78 \times 10^{-3}(dl)^2 f^2 \\ &= 789.5\left(\frac{dl}{\lambda}\right)^2 \qquad \text{ohms} \\ R_\theta &= \frac{1.05 \times 10^{-3}(dl)^2 f^2 \sin^2\theta}{r^2} = \frac{94.5\left(\frac{dl}{\lambda}\right)^2 \sin^2\theta}{r^2} \\ &\qquad \text{ohms/m}^2 \end{aligned}\right\} \qquad (293)$$

For a half-wave radiator in unobstructed space, we have

$$R_r = 789.5\left(\frac{l_e}{\lambda}\right)^2 \qquad \text{ohms}$$

if l_e denotes the effective length, which is $l_e = 0.637l = 0.637(\lambda/2)$, since l denotes the actual physical length of the half-wave radiator. We find then a radiation resistance $R_r = 789.5(0.637 \times 0.5)^2 \cong 80$ ohms. In the equatorial plane of the half-wave radiator, we find $R_\theta = 94.5(0.637 \times 0.5)^2/r^2 = 9.7/r^2$ ohms per square meter, since any linear radiator, in unobstructed radiation space, shows maximum circular directivity (in all directions) in the equatorial plane, and for a half-wave radiator, the space directivity factor $D_{\varphi\,\theta} = \dfrac{\cos\,(90\cos\theta)}{\sin\theta}$ gives $D_{\varphi,90} = 1$ for $\theta = 90$ deg (holding for all φ directions with the X axis of equatorial XY plane with the radiator along the Z axis). For an elementary radiator $I_e\,dl$, the R_θ value would be affected by a factor of $\sin^2\theta$, according to (293). Hence, for a 45 deg direction in any meridian plane through the radiator, $\sin^2\theta = \sin^2 45 = 0.7071^2 = 0.4998$. This means practically R_θ in this direction; *i.e.*, the R_{45} value would be only 50 per cent of the $R_\theta = R_{90}$ value holding in all φ directions in the equatorial plane. However, for a half-wave radiator, we have the above directivity $D_{\varphi,\theta}$ law, which is the same as the D_θ law, since the directivity is independent of the angle φ in unobstructed radiation space. For $\theta = 45$ deg, we have cos 45 deg = 0.7071, 90 deg cos 45 deg = 0.7071 × 90 = 63.64 deg, cos 63.64 deg = 0.4446. This result for the numerator of the $D_{\varphi,\theta}$ formula has to be divided by sin 45 deg = 0.7071, yielding $D_{\varphi,\theta} = D_\theta = 0.4446/0.7071 = 0.632$. In formula (293), the value of $D_\theta^2 = 0.632^2 = 0.3994$ has to be

used for the half-wave radiator instead of the $\sin^2 \theta = 0.4998$ value for the $I_e\,dl$ radiator. Hence, for a half-wave radiator, $R_\theta = R_{45}$ is only 39.94 per cent of the equatorial value. We found that the equatorial value is $R_\theta = R_{90} = 9.7/r^2$ ohms per square meter. Therefore, in the 45-deg direction with the half-wave radiator, we have $0.3994 \times 9.7/r^2 = 3.88/r^2$ ohms per square meter. Such R_θ values should be useful for VHF and UHF work if we choose the distance r to the field point in question properly (*i.e.*, not too close to the radiator). We expect, in such cases, a plane wave front, plane at least within 1-sq-m surface of the $4\pi r^2$ surface. These expressions for R_θ show that we have an inverse square distance law to contend with.

Equations (271), (272), and (292), holding for the distant radiation region, for which we have also $\mathcal{E}_\theta/H_\varphi = \sqrt{\mu/\kappa} = \sqrt{\mu_0/\kappa_0} \cong 377$ ohms, can, for f in megacycles per second, be written as

$$\left.\begin{aligned}
M_e &= I_e\,dl && \text{m}\cdot\text{amp}\\
\bar{P} &= 8.73 \times 10^{-3} M_e^2 f^2 && \text{watts through } 4\pi r^2 \text{ sq m}\\
M_e f &= 10.75\sqrt{\bar{P}} && \text{m}\cdot\text{amp/sec}\\
\mathcal{E}_\theta &= \frac{6.23 \times 10^{-1} M_e f \sin\theta}{r} = \frac{6.7\sqrt{\bar{P}}\sin\theta}{r} && \text{effective volts/m}\\
H_\varphi &= \frac{1.66 \times 10^{-3} M_e f \sin\theta}{r} = \frac{1.78 \times 10^{-2}\sqrt{\bar{P}}\sin\theta}{r} && \text{effective amp/m}
\end{aligned}\right\} \qquad (294)$$

Inasmuch as, according to (238) and (239), the vector potential

$$\Lambda = \mu\kappa\frac{\partial \Pi}{\partial t} = \frac{\mu}{4\pi}\iiint \frac{[J]}{r}\,d\tau \qquad \text{volt}\cdot\text{sec/m} \qquad (295)$$

is proportional to the time derivative of the Hertz vector Π, we can also relate the current density J acting at the symmetry center of an elementary space element $d\tau$ to such a vector and obtain results as given in (295).

With respect to (275) we also note that the retarded doublet moment m and the moment $m_e = i\,dl = dl I \epsilon^{j(\omega t - \beta r)}$ for harmonic current excitations are

$$m = q\,dl; \qquad m' = \frac{dm}{dt} = i\,dl = dl I \epsilon^{j(\omega t-\beta r)} = m_e$$

$$m_e' = \frac{dm_e}{dt} = \frac{d^2 m}{dt^2} = m'' = \omega\, dl I \epsilon^{j(\omega t - \beta r)} = j\omega m_e = j\omega m'$$

giving for the electric and magnetic field associations $\mathcal{E}_\theta$ and H_φ according to (276)

for the distant radiation region

$$\left.\begin{aligned}
&\mathcal{E}_\theta = j\omega m'A = \omega A\frac{dm}{dt} = j\omega m_e A \quad \text{volts/m;}\\
&\qquad H_\varphi = j\omega m'B = \omega B\frac{dm}{dt} = j\omega m_e B \quad \text{amp/m}\\
&A = \frac{\mu\sin\theta}{4\pi r}, \quad B = \frac{\sqrt{\mu\kappa}\sin\theta}{4\pi r}; \quad m = \frac{dlI\epsilon^{ja}}{j\omega} = \int m_e\,dt\\
&\qquad \text{meter}\cdot\text{coulombs}, \quad a = \omega t - \beta r\\
&m_e = dlI\epsilon^{j(\omega t-\beta r)} = \frac{dm}{dt} = m' \quad \text{meter}\cdot\text{amp/sec}\\
&j\omega m' = j\omega m_e = \frac{dm_e}{dt} = \frac{d^2m}{dt^2} = m'' \quad \text{meter}\cdot\text{amp/sec}
\end{aligned}\right\} \quad (296)$$

For an infinitesimal volume element $d\tau$, if referred to Cartesian coordinates, we have $d\tau = dx\,dy\,dz$ cubic meters, $J_x\,dy\,dz$ amperes, $J_y\,dx\,dz$ amperes, $J_z\,dx\,dy$ amperes. When the space element is enveloped by resultant lines of current flow and dS square meters is the normal cross section to the resultant current density J amperes per square meter, then we deal with a flow tube of total current $I = J\,dS$ amperes. The electric moment $m_e = I\,dl\,\epsilon^{j\omega t}$ is equal to $J\,d\tau\,\epsilon^{j\omega t}$, where $d\tau$ is the volume of the elementary flow tube. We have then for the retarded vector potential, for any coordinate system (Cartesian, cylindrical, spherical as well as curvilinear)

$$\mathrm{A} = \frac{\mu}{4\pi}\int\int\int_\tau \frac{J\epsilon^{j(\omega t-\beta r)}}{r}\,d\tau \qquad \text{volt}\cdot\text{sec/m} \tag{297}$$

The retarded scalar potential then becomes

$$V = \int\int\int_\tau \frac{\rho\epsilon^{j(\omega t-\beta r)}}{4\pi\kappa r}\,d\tau = \int\int\int_\tau \frac{\operatorname{div} J\epsilon^{j(\omega t-\beta r)}}{4\pi(\sigma + j\omega\kappa)r}\,d\tau \qquad \text{volts} \tag{298}$$

where σ in mhos per meter, κ in farads per meter, and μ in henrys per meter define the electromagnetic properties of the volume element $d\tau$ and ρ is the volume charge density in coulombs per cubic meter. We may also write $\nabla\cdot J\epsilon^{j(\omega t-\beta r)}$ instead of the divergence of the resultant current density function.

It is now of interest to relate the electric moment as given by $m_e = dl\,I\epsilon^{j(\omega t-\beta r)}$ for an elementary linear radiator and the charge moment $m = dl\,q\epsilon^{j(\omega t-\beta r)}$ to the Hertzian vector Π as defined in (295) and the Poynting vector as expressed in (291) and (292). Choosing the elementary length dl, expressed in meters, along the Z axis of an XYZ system, from the origin of which we have the r coordinate of a spherical coordinate system, the θ coordinate with respect to the Z axis and the φ coordinate with respect of the projection of r into the XY plane and the X direction, since $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$. We have for the unit vectors i, j, and k along the X, Y, and Z directions,

the vector potential

$$\underset{\bullet}{\Lambda} = i\Lambda_x + j\Lambda_y + k\Lambda_z = k\Lambda_z \qquad \text{volt}\cdot\text{sec/m}$$

since there can be no components along the X and Y directions. The field point has, in this case, been chosen at xyz. Hence, (297) becomes

$$\Lambda_z = \int_0^{dz} \frac{\mu J_z \epsilon^{j(\omega t - \beta r)}}{4\pi r}\, dz = \frac{\mu J_z \epsilon^{j(\omega t-\beta r)}}{4\pi r}\, dz \qquad \text{volt}\cdot\text{sec/m} \qquad (299)$$

This assumes an elementary radiator of length dz expressed in meters. Even though we have only one component Λ_z as referred to the XYZ system, we have generally two components Λ_θ and Λ_r for spherical coordinates just as is the case for the Hertzian vector $\underset{\bullet}{\Pi}$. Therefore,

$$\underset{\bullet}{\Lambda} = \underset{\bullet}{\Lambda}_z$$

$$\underset{\bullet}{\Lambda} = \underset{\bullet}{\Lambda}_r + \underset{\bullet}{\Lambda}_\theta = \overbrace{\Lambda_z \cos\theta}^{\Lambda_r} - \overbrace{\Lambda_z \sin\theta}^{\Lambda_\theta}$$

$$\underset{\bullet}{\Pi} = \underset{\bullet}{\Pi}_z$$

$$\underset{\bullet}{\Pi} = \underset{\bullet}{\Pi}_r + \underset{\bullet}{\Pi}_\theta = \overbrace{\Pi_z \cos\theta}^{\Pi_r} - \overbrace{\Pi_z \sin\theta}^{\Pi_\theta}$$

Therefore, we have only the absolute amounts

$$\left.\begin{array}{llll} \Lambda_z \text{ or } +\Lambda_z\cos\theta & \text{and} & -\Lambda_z \sin\theta & \text{volt}\cdot\text{sec/m} \\ \Pi_z \text{ or } +\Pi_z\cos\theta & \text{and} & -\Pi_z\sin\theta & \text{m}\cdot\text{volts} \end{array}\right\} \qquad (300)$$

since $\Lambda_x = \Lambda_y = \Lambda_\varphi = \Pi_x = \Pi_y = \Pi_\varphi = 0$.

If m_0 and m_{e_0} denote the crest values of the respective instantaneous values m and m_e for the doublet and current moments, we have

$$\left.\begin{array}{ll} m = m_0\epsilon^{j(\omega t-\beta r)} & \text{retarded doublet moment in meter}\cdot\text{coulombs} \\ m_e = m_{e_0}\epsilon^{j(\omega t-\beta r)} & \text{retarded elementary current moment in} \\ & \text{meter}\cdot\text{amp} \\ m_0 = q\,dl \qquad \text{and} & m_{e_0} = I\,dl \end{array}\right\} \qquad (301)$$

According to (299), when $i\,dz = i\,dl$ has a fixed amplitude I along the elementary radiator, we have for $i = I\epsilon^{ja}$ for $a = \omega t - \beta r$

$$\left.\begin{array}{l} \Lambda = \Lambda_z = \dfrac{\mu}{4\pi r} I\,dl\,\epsilon^{ja} = \dfrac{\mu}{4\pi r} m_{e_0}\epsilon^{ja} = \dfrac{\mu}{4\pi r} m_e \\ \Lambda = \Lambda_z = \dfrac{\mu}{4\pi r} I\,dl\,\epsilon^{ja} = j\dfrac{\omega\mu}{4\pi r} m_0\epsilon^{ja} = j\dfrac{\omega\mu}{4\pi r} m \end{array}\right\} \text{volt}\cdot\text{sec/m} \qquad (302)$$

Since for $K = \mu/(4\pi r)$

$$\Lambda = Km_e = K\frac{dm}{dt} \tag{302a}$$

We can impose the condition

$$\text{div}\ \Lambda + \mu\kappa\frac{\partial V}{\partial t} = 0 \tag{303}$$

Inasmuch as for a spherical wave spread and a vertical radiator, we have only components $\Lambda_r = \Lambda_z \cos\theta = \Lambda\cos\theta$ and $\Lambda_\theta = -\Lambda\sin\theta$, and

$$\text{div}\ \Lambda \equiv \nabla\cdot\Lambda = \frac{1}{r^2}\frac{\partial(r^2\Lambda\cos\theta)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(-\Lambda\sin^2\theta)}{\partial\theta}$$

$$= \frac{d\Lambda}{dr}\cos\theta + \frac{2\Lambda}{r}\cos\theta - \frac{2\Lambda}{r}\cos\theta$$

$$\text{div}\ \Lambda = \frac{d\Lambda}{dr}\cos\theta \tag{304}$$

The value of the scalar potential V in (303) can be found from (298), where for propagation in the atmosphere the conductivity $\sigma = 0$. For a retarded Hertzian doublet moment $m = m_0\epsilon^{ja}$ and an elementary linear current source $dl\,I\epsilon^{j\omega t}$, we have the retarded effect $dl\,I\epsilon^{ja} = m_{e_0}\epsilon^{ja} = m_e$. The scalar potential then becomes

$$\left.\begin{aligned} V &= -m_0\frac{d(\epsilon^{ja}/4\pi\kappa r)}{dr}\cos\theta = -\frac{d(m/4\pi\kappa r)}{dr}\cos\theta \\ V &= -\frac{m_{e_0}}{j\omega}\frac{d(\epsilon^{ja}/4\pi\kappa r)}{dr}\cos\theta = -\frac{\cos\theta}{j\omega}\frac{d(m_e/4\pi\kappa r)}{dr} \end{aligned}\right\}\text{volts} \tag{305}$$

Since

$$\mu\kappa\frac{\partial V}{\partial t} = j\omega\mu\kappa V$$

we find from (302) to (305)

$$\text{div}\ \Lambda + \mu\kappa\frac{\partial V}{\partial t} = \frac{d\Lambda}{dr}\cos\theta + j\omega\mu\kappa V = 0$$

or

$$\left.\begin{aligned} \frac{d\Lambda}{dr}\cos\theta - j\omega\mu\cos\theta\frac{d(m/4\pi r)}{dr} &= 0 \\ \frac{d\Lambda}{dr}\cos\theta - \mu\cos\theta\frac{d(m_e/4\pi r)}{dr} &= 0 \end{aligned}\right\} \tag{306}$$

These results are satisfied by the respective relations

$$\left.\begin{aligned} \Lambda &= \frac{j\omega\mu}{4\pi r}m \\ \Lambda &= \frac{\mu}{4\pi r}m_e \end{aligned}\right\}\text{volt}\cdot\text{sec/meter} \tag{307}$$

square meter, since only one vector, whose time rate $\mathcal{J} = (J/j\omega)\epsilon^{j\omega t}$, defines the current and charge densities.

49. Notes on the Application of the Hertzian Vector in Connection with the General Kirchhoff Wave Equation. In Secs. 45 to 49, Eqs. (238), (298), (309), and (312), we find that the scalar potential V, vector potential Λ, and corresponding field associations $\mathcal{E}$ and H are

$$\left.\begin{aligned} V^{(\text{volts})} &= \iiint \frac{\operatorname{div} J\epsilon^{j(\omega t-\beta r)}}{4\pi(\sigma + j\omega\kappa)}\,d\tau \\ \Lambda^{(\text{volt-sec/m})} &= \iiint_\tau \frac{\mu J\epsilon^{j(\omega t-\beta r)}}{4\pi r}\,d\tau = \frac{\mu}{4\pi}\iiint_\tau \frac{dm_e}{r} = \mu\kappa\frac{\partial\Pi}{\partial t} \\ \mathcal{E}^{(\text{volts/m})} &= \operatorname{grad}\operatorname{div}\Pi - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} \\ H^{(\text{amp/m})} &= \kappa\nabla\times\frac{\partial\Pi}{\partial t} = \kappa\operatorname{curl}\frac{\partial\Pi}{\partial t} \end{aligned}\right\} \qquad (315)$$

where the Hertzian vector Π, vector potential Λ, and the instantaneous moment m_e of the electric current element $i\,dl = dl\,I\epsilon^{j\omega t}$ read

$$\left.\begin{aligned} \Pi^{(\text{m-volts})} &= \frac{m_e}{j4\pi\omega\kappa r} \\ \Lambda^{(\text{volt-sec/m})} &= \frac{\mu m_e}{4\pi r} \\ m_e^{(\text{m-amp})} &= I\,dl\,\epsilon^{j(\omega t-\beta r)} \end{aligned}\right\} \qquad (316)$$

This holds for an elementary radiator and zero conductivity of the medium through which waves pass. According to (244) and (316), we have

$$\mathcal{E}^{(\text{volts/m})} = \operatorname{grad}\operatorname{div}\frac{m_e}{j4\pi\omega\kappa r} - \frac{j\omega\mu m_e}{4\pi r}$$

$$H^{(\text{amp/m})} = \frac{\operatorname{curl}\Lambda}{\mu} = \frac{\operatorname{curl} m_e}{4\pi r}$$

When the elementary length dl is along the Z axis, we also have the electric moment m_e along Z directions as well as Λ and Π, since the current flow is only along $\pm Z$ directions.

In the spherical coordinate system m_e, Λ and Π can then be resolved into components $m_e \cos\theta$, $\Lambda\cos\theta$, and $\Pi\cos\theta$ along the r direction of the spherical r, φ, θ coordinates, and $-m_e\sin\theta$, $-\Lambda\sin\theta$, and $-\Pi\sin\theta$ along the θ direction, which is normal to the r direction. We have then at a particular space point

$$m_e = im_e \cos\theta - jm_e \sin\theta; \qquad \Lambda = i\Lambda \cos\theta - j\Lambda \sin\theta;$$
$$\Pi = i\Pi \cos\theta - j\Pi \sin\theta \quad (317)$$

for unity vectors i and j along r and tangent through the meridian, along θ, at a certain space point. If a common vector A represents the moment per cubic meter of the free charge, we have, in a medium of σ mhos per meter, μ henrys per meter, and κ farads per meter, the current density $J = \partial A/\partial t$ amperes per square meter and the charge density $\rho = -\nabla \cdot A$, the field relations

$$\left.\begin{aligned} &\nabla \times \mathcal{E} = -\mu\frac{\partial H}{\partial t}; \qquad \nabla \times H = \kappa\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial A}{\partial t}; \qquad \nabla \cdot \mathcal{E} \equiv \operatorname{div} \mathcal{E} \\ &\qquad = -\frac{1}{\kappa}\nabla \cdot A \equiv -\frac{\operatorname{div} A}{\kappa}; \qquad \nabla \cdot H \equiv \operatorname{div} H = 0; \\ &\nabla \times \mathcal{E} \equiv \operatorname{curl} \mathcal{E}; \quad \nabla \times H \equiv \operatorname{curl} H; \quad \text{Maxwell's field equations of a common vector } A = \textstyle\int J\,dt \end{aligned}\right\} \quad (318)$$

For Cartesian coordinates, we find then

$$\nabla^2\Pi - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} + \frac{A}{\kappa} = 0 \quad (319)$$

since Π is a solution of

$$\nabla \times \nabla \times \Pi - \nabla\nabla \cdot \Pi + \mu\kappa\frac{\partial^2\Pi}{\partial t^2} - \frac{A}{\kappa} = 0$$

and

$$\mathcal{E} = \nabla\nabla \cdot \Pi - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} = \nabla\nabla \cdot \Pi + \beta^2\Pi$$
$$H = \kappa\nabla \times \frac{\partial\Pi}{\partial t} = -j\omega\kappa\nabla \times \Pi; \qquad \beta = \omega\sqrt{\mu\kappa}$$

W. L. Barrow and F. M. Greene[1] have shown that the open rectangular wave guide acts as a useful wave-front radiator, based on the Huygens wavelet principle. For a unity vector j_y along the Y axis, the Hertzian vector Π within a rectangular guide excited as a $\mathrm{TE}_{m,0}$ wave has a value

$$\left.\begin{aligned} &\Pi = j_y K_1 \cos\left(\frac{m\pi}{a}x_0\right)\epsilon^{j(\omega t - \beta z)} \qquad \text{for } m = 1, 3, 5, 7, \cdots \\ &\Pi = j_y K_1 \sin\left(\frac{m\pi}{a}x_0\right)\epsilon^{j(\omega t - \beta z)} \qquad \text{for } m = 2, 4, 6, 8, \cdots \end{aligned}\right\} \text{m} \cdot \text{volts} \quad (320)$$

where a meters is the long side and b meters is the shorter side of the

[1] *Proc. IRE*, **26**, 1514, December, 1938.

rectangular guide cross section shown in Fig. 43, and the phase constant

$$\beta = \sqrt{\mu\kappa\omega^2 - \left(\frac{m\pi}{a}\right)^2} = \sqrt{\left(\frac{2\pi}{\lambda'}\right)^2 - \left(\frac{m\pi}{a}\right)^2}$$ circular radians per meter length of the guide

In this relation, β is the phase constant along the guide, $\omega/6.28$ is the operating frequency in cycles per second, and λ' is the wavelength in meters of the medium of μ henrys per meter; κ farads per meter, and σ = 0 mhos per meter within the guide if the metal boundary were removed.

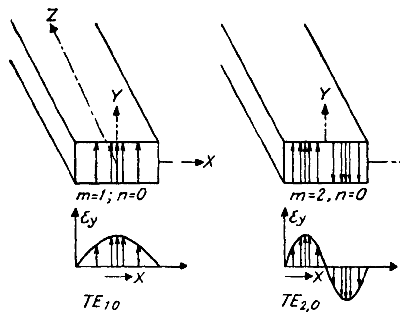


FIG. 43. Electric $\mathcal{E}_y$ field distribution for consecutive $TE_{m,n}$ wave excitations in rectangular wave guides.

This wavelength is found from $\lambda' = v/f = 1/(f\sqrt{\mu\kappa})$ meters for f in cycles per second and the phase velocity $v = 1/\sqrt{\mu\kappa}$ in meters per second. If, as customary, air is the medium within a guide, then $v \cong c = 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec and λ' is identical with the empty space wavelength $\lambda_0^{(m)} = 300/f^{(Mc)}$. It is this wavelength which we shall use in the above expression from now on.

The same value of $\prod$ is then also assumed to prevail at the very opening of the rectangular guide of finite length. The respective field associations $\mathcal{E}$ and H can be found from

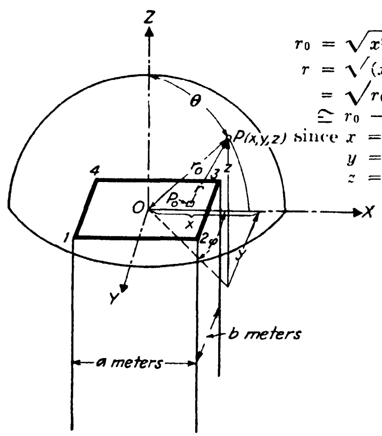
$$\mathcal{E} = \text{grad div} \prod - \mu\kappa \frac{\partial^2 \prod}{\partial t^2} \text{ volts/m;} \qquad H = \kappa \text{ curl} \frac{\partial \prod}{\partial t} \text{ amp/m} \qquad (321)$$

For this reason, mx_0/a trigonometric distributions prevail, because x_0, y_0 denote points P_0 in the guide opening of the XY plane. The very opening acts as a wave-front radiator, as is shown in Fig. 44. The Kirchhoff-Huygens solution refers to field points P a great distance of r meters away from any elementary wave-front surface dS of the guide opening. To obtain the resultant Hertzian vector $\prod_P$ at P, we must integrate over the area of the entire surface, *i.e.*, over the entire opening $S = ab$ square meters, and find

$$\Pi_P = \frac{1}{4\pi}\int\!\!\int \frac{1}{r}\left\{\left[\sqrt{\mu\kappa}\left(\frac{\partial\Pi}{\partial t}\right)_{t-r\sqrt{\mu\kappa}} + \frac{\Pi_{t-r\sqrt{\mu\kappa}}}{r}\right]\cos(n, r) + \left(\frac{\partial\Pi}{\partial n}\right)_{t-r\sqrt{\mu\kappa}}\right\} dS$$

$$\left.\begin{aligned}\Pi_P &= \frac{j_y K_1}{4\pi}\int\!\!\int\left[\epsilon^{j(\omega t-\beta_0 r)}\cos\frac{m\pi}{a}x_0\right][\beta_0\cos(n, r) + \beta]\,dS \\ &\qquad\qquad \text{for } m = 1, 3, 5, \cdots \\ \Pi_P &= \frac{j_y K_1}{4\pi}\int\!\!\int\left[\epsilon^{j(\omega t-\beta_0 r)}\sin\frac{m\pi}{a}x_0\right][\beta_0\cos(n, r) + \beta]\,dS \\ &\qquad\qquad \text{for } m = 2, 4, 6, \cdots\end{aligned}\right\} \quad (322)$$

where, strictly, the double integral is to be multiplied by a unity vector j_z acting along the Z axis. In these relations, β_0 denotes the phase con-



$$\begin{aligned} r_0 &= \sqrt{x^2 + y^2 + z^2} \quad \text{meters} \\ r &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2} \\ &= \sqrt{r_0^2 - 2r_0\sin\theta(x_0\cos\varphi + y_0\sin\varphi) + x_0^2 + y_0^2} \\ &\cong r_0 - \sin\theta(x_0\cos\varphi + y_0\sin\varphi) \\ \text{since } x &= r_0\sin\theta\cos\varphi \\ y &= r_0\sin\theta\sin\varphi \\ z &= r_0\cos\theta \end{aligned}$$

FIG. 44. Conditions for the radiation due to a rectangular wave-guide opening. Field point P a large distance r_0 m away from the symmetry point O of the opening of the rectangular guide 1-2-3-4-1. Elementary area $dS = dx_0\,dy_0$ sq m at $P_0(x_0, y_0)$. $\mathcal{E}_y$ volts/m and H_x amp/m are transverse aperture fields. H_z longitudinal field meets a discontinuity at aperture.

stant $\omega\sqrt{\mu\kappa}$ for the unbounded medium within the guide and angle n, r is the angle that direction r makes with the normal of any surface element dS square meters (the normal is in the $-Z$ direction). In this expression, K_1 is the constant also appearing in the Hertzian vector Π of Eq. (320), which holds for fixed field associations within the guide. We distinguish between a propagation constant Γ within the guide and a constant beyond the opening of the guide, since the guide wavelength λ_g is generally not the same as the unbounded (inside space) wavelength (essentially free-space wavelength λ_0 for air). This means that we have a phase constant $\beta = 2\pi/\lambda_g$ within and along the guide differing from

$\beta_0 = 2\pi/\lambda_0$. The wave-guide theory on pages 306–310 shows that for a $TE_{m,0}$ wave, the expression $\Gamma = \sqrt{(m\pi/a)^2 - \omega^2\mu\kappa}$ denotes the propagation constant Γ in generalized hyperbolic radians per meter within and along the guide. Since for atmosphere within the guide $\mu\kappa \cong \mu_0\kappa_0 = 1/c^2 = (10^{-8}/3)^2$ sec^2/m^2, we have $\omega^2\mu\kappa = (\omega/c)^2 = (2\pi f/c)^2 = (2\pi/\lambda_0)^2 = \beta_0^2$. Hence, for $TE_{m,0}$ waves,

$$\Gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\kappa} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \left(\frac{2\pi}{\lambda_0}\right)^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \beta_0^2}$$
$$= \alpha + j\beta \qquad \text{generalized hyperbolic radians/m} \qquad (323)$$

where the distance is in meters along the guide (along Z direction).

If we are interested in radiation from a wave front at the opening of the guide, this requires that electromagnetic field energy be propagated along the guide. This is possible only when the attenuation α in nepers per meter along the guide vanishes. This happens if $\Gamma = j\beta$ or

$$j\beta = j\sqrt{\omega^2\mu\kappa - \left(\frac{m\pi}{a}\right)^2}; \qquad \beta = \sqrt{\left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{m\pi}{a}\right)^2}$$
$$= \sqrt{\beta_0^2 - \left(\frac{m\pi}{a}\right)^2} \qquad \begin{array}{l}\text{circular radians/m}\\ \text{guide length}\end{array} \qquad (324)$$
$$\alpha = 0 \qquad \text{pure hyperbolic radians/m guide length;} \qquad \lambda_0^{(m)} = 300/f^{(\mathrm{Mc/sec})}$$

As in the general Kirchhoff wave equation on page 1060, for a wave-front radiator, we can again put the average distance $r = r_0$ meters outside the integral of (322), since the area $S = ab$ square meters has negligible dimensions compared with the long distance of r meters to field point for which the Kirchhoff relation (322) holds. We can also put $\cos(n, r) = \cos\theta$ and outside the integral, since it is only a slow cosinoidal variation compared with the very fast exponential variation $\epsilon^{j(\omega t - \beta_0 r)}$. In doing this, Eq. (322) yields

$$\left.\begin{aligned}
\Pi_P &= \frac{j_\nu\beta_0K_1\left(\cos\theta + \dfrac{\beta}{\beta_0}\right)\epsilon^{j(\omega t - r_0\sqrt{\mu\kappa})}}{4\pi r_0}A\\
A &= \int_{x_0=-a/2}^{x_0=+a/2}\int_{y_0=-b/2}^{y_0=+b/2}\cos\frac{m\pi}{a}x_0\epsilon^{j\psi}\,dx_0\,dy_0 \qquad \text{for } m = 1, 3, 5\\
A &= \int_{x_0=-a/2}^{x_0=+a/2}\int_{y_0=-b/2}^{y_0=+b/2}\sin\frac{m\pi}{a}x_0\epsilon^{j\psi}\,dx_0\,dy_0 \qquad \text{for } m = 2, 4, 6\\
\psi &= -\beta_0(r - r_0) = \beta_0(x_0\cos\varphi + y_0\sin\varphi)\sin\theta\\
\beta_0 &= \frac{2\pi}{\lambda_0} = \frac{2\pi f}{c} = \frac{\omega}{c}; \qquad c = 3\times10^8\ \text{m/sec}
\end{aligned}\right\} \qquad (325)$$

where $\omega/6.28$ is in cycles per second. For the abbreviation

$$K = \frac{j_y \beta_0 K_1 \left(\cos\theta + \frac{\beta}{\beta_0}\right) \epsilon^{j(\omega - r_0\sqrt{\mu\kappa})}}{4\pi r_0} \tag{326}$$

we have for odd values of m, the Hertzian vector

$$\prod_P = K \int_{x_0=-a/2}^{x_0=+a/2} \cos\frac{m\pi}{a} x_0 \epsilon^{j\beta_0 x_0 \cos\varphi \sin\theta}\, dx_0 \int_{y_0=-b/2}^{y_0=+b/2} \epsilon^{j\beta_0 y_0 \sin\varphi \sin\theta}\, dy_0$$

$$= K \underbrace{\int_{-a/2}^{+a/2} \cos\frac{m\pi}{a} x_0 \epsilon^{j\beta_0 g x_0}\, dx_0}_{C} \underbrace{\int_{-b/2}^{+b/2} \epsilon^{j\beta_0 h y_0}\, dy_0}_{D} = KCD \tag{327}$$

for $g = \cos\varphi \sin\theta$ and $h = \sin\varphi \sin\theta$. We find for D the solution

$$D = \int_{y_0=-b/2}^{y_0=+b/2} \epsilon^{j\beta_0 h y_0}\, dy_0 = \left|\frac{\epsilon^{j\beta_0 y_0}}{j\beta_0 h}\right|_{y_0=-b/2}^{y_0=b/2} = \frac{\epsilon^{jk} - \epsilon^{-jk}}{j\beta_0 h}$$

for $k = 0.5\beta_0 bh = 0.5\beta_0 b \sin\varphi \sin\theta$. Proceeding, we find

$$D = \frac{\epsilon^{jk} - \epsilon^{-jk}}{j\beta_0 h} = \frac{2j \sin k}{j\beta_0 h} = \frac{2 \sin k}{\beta_0 h} = \frac{2 \sin (0.5\beta_0 b \sin\varphi \sin\theta)}{\beta_0 \sin\varphi \sin\theta}$$

But $\beta_0 = 2\pi/\lambda_0$ and

$$D = \frac{\sin\left(\frac{\pi}{\lambda_0} b \sin\varphi \sin\theta\right)}{(\pi/\lambda_0) \sin\varphi \sin\theta} = \frac{b \sin [(\pi b/\lambda_0) \sin\varphi \sin\theta]}{(\pi b/\lambda_0) \sin\varphi \sin\theta} \tag{328}$$

Inserting the expression for C in Eq. (327)

$$\frac{m\pi}{a} = N \qquad \text{and} \qquad j\beta_0 g = j\beta_0 \cos\varphi \sin\theta = M$$

we find

$$\left.\begin{aligned} C &= \int_{x_0=-a/2}^{x_0=a/2} \epsilon^{x_0 M} \cos (x_0 N)\, dx_0 \qquad \text{for } m = 1, 3, 5, \cdots \\ C &= \int_{-a/2}^{a/2} \epsilon^{x_0 M} \sin (x_0 N)\, dx_0 \qquad \text{for } m = 2, 4, 6, \cdots \end{aligned}\right\} \tag{329}$$

But

$$\left.\begin{aligned} \int \epsilon^{x_0 M} \sin (x_0 N)\, dx_0 &= \frac{\epsilon^{x_0 M}}{M^2 + N^2} [M \sin (x_0 N) - N \cos (x_0 N)] \\ \int \epsilon^{x_0 M} \cos (x_0 N)\, dx_0 &= \frac{\epsilon^{x_0 M}}{M^2 + N^2} [M \cos (x_0 N) + N \sin (x_0 N)] \end{aligned}\right\} \tag{330}$$

rendering the factor C for odd values of m

$$C = C_{1,3,5,7,\ldots} = \int_{-a/2}^{+a/2} \epsilon^{x_0 M} \sin (x_0 N)\, dx_0$$

$$= \left\{ \frac{\epsilon^{0.5aM}}{M^2 + N^2} [M \sin (0.5aN) - N \cos (0.5aN)] \right\}$$

$$- \left\{ \frac{\epsilon^{-0.5aM}}{M^2 + N^2} [M \sin (-0.5aN) - N \cos (-0.5AN)] \right\}$$

Since $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$, we have

$$C_{1,3,5,7,\cdots} = \frac{\epsilon^{0.5aM}}{M^2+N^2}[M\sin(0.5aN) - N\cos(0.5aN)] + \frac{\epsilon^{-0.5aM}}{M^2+N^2}[M\sin(0.5aN) + N\cos(0.5aN)]$$

Because $M = j\beta_0 g$, we have $0.5aM = j0.5\beta_0 ga = j\delta$, and

$$\epsilon^{0.5aM} = \cos\delta + j\sin\delta; \qquad \epsilon^{-0.5aM} = \cos\delta - j\sin\delta$$

Hence, for $0.5aN = 0.5m\pi = e$, we find

$$C_{1,3,5,7,\cdots} = \frac{\left\{\begin{matrix} M\sin e\cos\delta - N\cos e\cos\delta + jM\sin e\sin\delta - jN\cos e\sin\delta \\ + M\sin e\cos\delta + N\cos e\cos\delta - jM\sin e\sin\delta - jN\cos e\sin\delta \end{matrix}\right\}}{M^2+N^2}$$

$$= \frac{2M\sin e\cos\delta - j2N\cos e\sin\delta}{M^2+N^2}$$

Since $M = j\beta_0 g$, we find

$$C_{1,3,5,7,\cdots} = 2j\frac{\beta_0 g\sin e\cos\delta - N\cos e\sin\delta}{N^2 - \beta_0^2 g^2}$$

But $N = m\pi/a$ for $m = 1, 3, 5, 7, \cdots$ in this expression, $g = \cos\varphi\sin\theta$, $e = 0.5aN = 0.5m\pi$, $\delta = 0.5\beta_0 ga = 0.5\beta_0 a\cos\varphi\sin\theta$, and $\beta_0 = 2\pi/\lambda_0$; hence

$$C_{1,3,5,7,\cdots} = 2j\frac{\frac{2\pi}{\lambda_0}g\sin e\cos\delta - \frac{m\pi}{a}\cos e\sin\delta}{\left(\frac{m\pi}{a}\right)^2 - \left(\frac{2\pi}{\lambda_0}g\right)^2}$$

$$= 2j\frac{\frac{1}{a}\left[\frac{2\pi a}{\lambda_0}g\sin e\cos\delta - m\pi\cos e\sin\delta\right]}{\frac{1}{a^2}\left[(m\pi)^2 - \left(\frac{2\pi ag}{\lambda_0}\right)^2\right]}$$

$$= 2ja\frac{\left\{\begin{matrix}\frac{2\pi a}{\lambda_0}\cos\varphi\sin\theta\sin\left(\frac{m\pi}{2}\right)\cos\left(\frac{\pi a}{\lambda_0}\cos\varphi\sin\theta\right) \\ - m\pi\cos\left(\frac{m\pi}{2}\right)\sin\left(\frac{\pi a}{\lambda_0}\right)\cos\varphi\sin\theta\end{matrix}\right\}}{(m\pi^2) - \left(\frac{2\pi a}{\lambda_0}\cos\varphi\sin\theta\right)^2} \tag{331}$$

which factor, for the $TE_{1,0}$ excitation, gives $C = C_1$, $m = 1$, $\sin(\frac{1}{2}m\pi) = \sin 90$ deg $= 1$, $\cos(\frac{1}{2}m\pi) = \cos 90$ deg $= 0$, so that the entire second term vanishes and

$$C = C_1 = 2ja\frac{\frac{2\pi a}{\lambda_0}\cos\varphi\sin\theta\cos\left(\frac{\pi a}{\lambda_0}\cos\varphi\sin\theta\right)}{(m\pi)^2 - \left(\frac{2\pi a}{\lambda_0}\cos\varphi\sin\theta\right)^2} \tag{332}$$

For even values of m such as 2, 4, 6, 8, · · · , we have to use the second expression of (330) and find, for $0.5aN = e$ and $0.5aM = j\delta$,

$$C = C_{2,4,6,8,\ldots} = \int_{-a/2}^{+a/2} \epsilon^{x_0 M} \cos(x_0 N)\, dx_0 = \frac{\epsilon^{j\delta}}{M^2 + N^2}[M \cos e + N \sin e] - \frac{\epsilon^{-j\delta}}{M^2 + N^2}[M \cos(-e) + N \sin(-e)]$$

$$= \frac{\epsilon^{j\delta}}{M^2 + N^2}(M \cos e + N \sin e) - \frac{\epsilon^{-j\delta}}{M^2 + N^2}(M \cos e - N \sin e)$$

$$= \frac{\left\{\begin{array}{l} M \cos e \cos \delta + N \sin e \cos \delta + jM \cos e \sin \delta + jN \sin e \sin \delta \\ \quad - M \cos e \cos \delta + N \sin e \cos \delta + jM \cos e \sin \delta - jN \sin e \sin \delta \end{array}\right\}}{M^2 + N^2}$$

$$= \frac{2N \sin e \cos \delta + 2jM \cos e \sin \delta}{M^2 + N^2} = 2\frac{N \sin e \cos \delta - \beta_0 g \cos e \sin \delta}{N^2 - \beta_0^2 g^2}$$

$$= 2\frac{\dfrac{m\pi}{a} \sin e \cos \delta - \dfrac{2\pi}{\lambda_0} g \cos e \sin \delta}{(m\pi/a)^2 - (2\pi g/\lambda_0)^2} = 2a\frac{m\pi \sin e \cos \delta - \dfrac{2\pi a}{\lambda_0} g \cos e \sin \delta}{(m\pi)^2 - (2\pi a g/\lambda_0)^2}$$

Generally, therefore, we have for $TE_{m,0}$ waves in the rectangular guide opening

$$\left.\begin{array}{l} C = C_{1,3,5,7,\ldots} = 2ja\dfrac{(2\pi a g/\lambda_0) \sin e \cos \delta - m\pi \cos e \sin \delta}{(m\pi)^2 - (2\pi a g/\lambda_0)^2} \\ \qquad\qquad\qquad\qquad \text{for } m\ 1, 3, 5, 7, \cdots \\ C = C_{2,4,6,8,\ldots} = 2a\dfrac{m\pi \sin e \cos \delta - (2\pi a g/\lambda_0) \cos e \sin \delta}{(m\pi)^2 - (2\pi a g/\lambda_0)^2} \\ \qquad\qquad\qquad\qquad \text{for } m\ 2, 4, 6, 8, \cdots \end{array}\right\} \quad (333)$$

$$e = \frac{m\pi}{2}; \qquad \delta = 0.5\beta_0 a \cos\varphi \sin\theta; \qquad \beta_0 = \frac{2\pi}{\lambda_0}; \qquad g = \cos\varphi \sin\theta$$

Therefore, from (328) and (333), we find for the double integral of (325)

$$CD = \int_{-a/2}^{+a/2}\int_{-b/2}^{+b/2} = j\frac{2ab}{\pi^2}\frac{\sin\left(\dfrac{\pi b}{\lambda_0}\sin\varphi\sin\theta\right)}{(b/\lambda_0)\sin\varphi\sin\theta}$$
$$\times\left[\frac{\dfrac{2a\cos\varphi\sin\theta}{\lambda_0}\sin\left(\dfrac{m\pi}{2}\right)\cos\left(\dfrac{\pi a}{\lambda_0}\cos\varphi\sin\theta\right) - m\cos\dfrac{m\pi}{2}\sin\left(\dfrac{\pi a}{\lambda_0}\cos\varphi\sin\theta\right)}{m^2 - (2a\cos\varphi\sin\theta/\lambda_0)^2}\right]$$

$$CD = \int_{-a/2}^{+a/2}\int_{-b/2}^{+b/2} = \frac{2ab}{\pi^2}\frac{\sin\left(\dfrac{\pi b}{\lambda_0}\sin\varphi\sin\theta\right)}{(b/\lambda_0)\sin\varphi\sin\theta}$$
$$\times\left[\frac{m\sin\left(\dfrac{m\pi}{2}\right)\cos\left(\dfrac{\pi a}{\lambda_0}\cos\varphi\sin\theta\right) - \dfrac{2a\cos\varphi\sin\theta}{\lambda_0}\cos\left(\dfrac{m\pi}{2}\right)\sin\left(\dfrac{\pi a}{\lambda_0}\cos\varphi\sin\theta\right)}{m^2 - (2a\cos\varphi\sin\theta/\lambda_0)^2}\right] \quad (334)$$

where the first formula holds for $m = 1, 3, 5, 7, \cdots$ and the second formula for $m = 2, 4, 6, 8, \cdots$. The $TE_{1,0}$ excitation ($m = 1$) is based on a half-period electric field distribution along the X direction (Fig. 44). According to (327), each formula for the product of CD must be multiplied by K as defined by (326), in order to give the $\prod_P$ value for the distant radiation region beyond the guide opening for any r, φ, θ orientation of field point P in Fig. 44 not occupying the space of the guide itself.

The Hertz vector then yields the results

$$\left.\begin{aligned} \prod_P^{(\text{meter-volts})} &= \frac{\tfrac{1}{2}abK_1(\beta_0 \cos\theta + \beta)\epsilon^{jd}}{\pi^3 r_0}\,\frac{\sin k}{(b/\lambda_0)h} \\ &\quad\times \frac{m\cos e \sin\delta - (2ag/\lambda_0)\sin e \cos\delta}{m^2 - (2ag/\lambda_0)^2} \quad \text{for } m = 1, 3, 5, \cdots \\ \prod_P^{(\text{meter-volts})} &= j\,\frac{\tfrac{1}{2}abK_1(\beta_0 \cos\theta + \beta)\epsilon^{jd}}{\pi^3 r_0}\,\frac{\sin k}{(b/\lambda_0)h} \\ &\quad\times \frac{m\sin e \cos\delta - (2ag/\lambda_0)\cos e \sin\delta}{m^2 - (2ag/\lambda_0)^2} \quad \text{for } m = 2, 4, 6, \cdots \end{aligned}\right\} \tag{335}$$

where ab square meters is the radiating surface of guide opening (Fig. 44), r_0 meters is the distance from the symmetry center of the guide opening to field point P, $\beta_0 = 2\pi/\lambda_0$, λ_0 meters $= 300/f$ megacycles per second, $\beta = \sqrt{\beta_0^2 - (m\pi/a)^2}$, a meters is the guide-opening dimension normal to $\mathcal{E}$ field excitations, $d = \omega t - r_0\sqrt{\mu\kappa}$, $\omega/6.28$ is in cycles per second; t in seconds, $\sqrt{\mu\kappa} \cong 10^{-8}/3$ sec/m for customary atmosphere, $g = \cos\varphi \sin\theta$, $h = \sin\varphi \sin\theta$, b in meters, dimension of guide opening along the $\mathcal{E}$ excitation of the guide, $k = 0.5\beta_0 b \sin\varphi \sin\theta$, m is the number of half-period distributions of $\mathcal{E}$ along the a dimension, $e = 0.5\pi m$, $\delta = 0.5\beta_0 a \cos\varphi \sin\theta$, θ field point P orientation with respect to the Z axis, and φ orientation with respect to the X axis, K_1 is a constant. We note from the results that only the portion

$$\left.\begin{aligned} &(\beta_0 \cos\theta + \beta)\,\frac{\sin k}{h}\,\frac{m\cos e \sin\delta - (2ag/\lambda_0)\sin e \cos\delta}{m^2 - (2ag/\lambda_0)^2} \\ &\qquad\qquad \text{for } m = 1, 3, 5, 7, \cdots \\ &(\beta_0 \cos\theta + \beta)\,\frac{\sin k}{h}\,\frac{m\sin e \cos\delta - (2ag/\lambda_0)\cos e \sin\delta}{m^2 - (2ag/\lambda_0)^2} \\ &\qquad\qquad \text{for } m = 2, 4, 6, 8, \cdots \end{aligned}\right\} \tag{336}$$

determines the direction of the Hertz vector $\prod_P$ and that for the $TE_{1,0}$ excitation, *i.e.*, $m = 1$, we have $\sin e = 1$ and $\cos e = 0$ so that the Hertz vector becomes

$$\prod_P = -\frac{0.5aK_1(\beta_0 \cos\theta + \beta)\epsilon^{jd}}{\pi^3 r_0}\,\frac{\sin k}{h/\lambda_0}\,\frac{(2ag/\lambda_0)\sin e \cos\delta}{1 - (2ag/\lambda_0)^2} \tag{337}$$

with a directivity given by the function

$$-(\beta_0 \cos\theta + \beta)\frac{\sin k}{h}\frac{g \sin e \cos\delta}{1 - (2ag/\lambda_0)^2} \tag{338}$$

In any homogeneous isotropic medium σ mhos per meter, μ henrys per meter, κ farads per meter with no sources, the vectors $\mathcal{E}$ volts per meter, H amperes per meter, $D = \kappa\mathcal{E}$ coulombs per square meter, $B = \mu H$ webers per square meter as well as the vector potential Λ volt · seconds per meter and the Hertz vector Π meter · volts satisfy a linear wave equation that can be expressed in two forms, namely,

$$\left.\begin{aligned} \nabla\nabla\cdot A - \nabla\times\nabla\times A + \Gamma^2 A &= 0 \\ \nabla^2 A - \mu\kappa\frac{\partial^2 A}{\partial t^2} - \mu\sigma\frac{\partial A}{\partial t} &= 0 \end{aligned}\right\}\text{general form} \tag{339}$$

The vector $A = A_0\epsilon^{j\omega t}$ then represents any of the above vectors and $\Gamma = j\omega\sqrt{\mu\kappa_e}$ for $\kappa_e = \kappa - j\sigma/\omega$ denotes the propagation constant in unbounded space for $\omega/6.28$ in cycles per second. We have then the respective relations

$$\left.\begin{aligned} &\nabla^2\Pi = \mu\kappa\frac{\partial^2\Pi}{\partial t^2} + \mu\sigma\frac{\partial\Pi}{\partial t} \\ &\nabla^2\Lambda = \mu\kappa\frac{\partial^2\Lambda}{\partial t^2} + \mu\sigma\frac{\partial\Lambda}{\partial t}; \qquad \nabla^2\mathcal{E} = \mu\kappa\frac{\partial^2\mathcal{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathcal{E}}{\partial t} \\ &\nabla^2 V = \mu\kappa\frac{\partial^2 V}{\partial t^2} + \mu\sigma\frac{\partial V}{\partial t}; \qquad \nabla^2 H = \mu\kappa\frac{\partial^2 H}{\partial t^2} + \mu\sigma\frac{\partial H}{\partial t} \end{aligned}\right\} \tag{339a}$$

where for perfect insulators, we have the general form

$$\nabla^2 A = \mu\kappa\frac{\partial^2 A}{\partial t^2}$$

The electric potential V volts and the electromagnetic field associations $\mathcal{E}$ volts per meter and H amperes per meter are then

$$\left.\begin{aligned} V &= -\operatorname{div}\Pi \equiv -\nabla\cdot\Pi \\ \mathcal{E} &= \operatorname{grad}\operatorname{div}\Pi - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} \\ \mathcal{E} &= -\nabla V - \mu\kappa\frac{\partial^2\Pi}{\partial t^2} = -\nabla V - \frac{\partial\Lambda}{\partial t} \\ H &= \kappa\operatorname{curl}\frac{\partial\Pi}{\partial t} \\ H &= \frac{\nabla\times\Lambda}{\mu} \equiv \frac{\operatorname{curl}\Lambda}{\mu} \end{aligned}\right\} \tag{340}$$

and

$$\Lambda = \mu\kappa \frac{\partial \Pi}{\partial t}; \qquad \text{curl } \Lambda = \mu\kappa \text{ curl} \frac{\partial \Pi}{\partial t} \tag{341}$$

According to the general form (339), it is then necessary only to resolve any of the above vectors designated A in (339) into rectangular components so that three independent differential equations are secured.

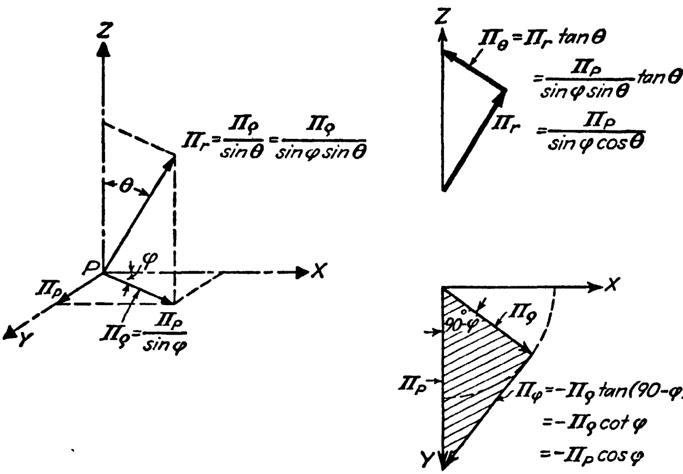


FIG. 45. Hertz vectors at the distant field point P.

We now have to find expressions for the field components of $\mathcal{E}$ and H at field point P in the spherical coordinate system. We have the unit vectors j_r, j_φ, and j_θ along the r, φ, and θ directions, where r now stands for the distance from field point P to the origin (or symmetry point of the guide opening). It is the distance r_0 (Fig. 44). Hence, the resultant vector

$$\Pi_P = j_r\Pi_r + j_\varphi\Pi_\varphi + j_\theta\Pi_\theta$$

For $TE_{m,0}$ excitations as indicated in Figs. 43 and 44, we have only the Y component for the transverse $\mathcal{E}$ wave, namely, $\mathcal{E}_y$. Hence, also $\Pi_P = j_y\Pi_y$; and according to Fig. 45, we have

$$\Pi_r = \frac{\Pi_P}{\sin\varphi \sin\theta}; \qquad \Pi_\theta = \frac{\Pi_P}{\sin\varphi\cos\theta}; \qquad \Pi_\varphi = -\Pi_P \cos\varphi \tag{342}$$

According to (321), we can then find the field associations $\mathcal{E}$ and H at a space point P. For the unit vectors j_r, j_φ, and j_θ, along the orthogonal

r, φ, and θ directions, we have

$$\text{curl} \prod_P = \text{curl}_r \prod_P + \text{curl}_\varphi \prod_P + \text{curl}_\theta \prod_P \tag{343}$$

with orthogonal components

$$\left.\begin{aligned} \text{curl}_r \prod_P &= \frac{j_r}{r \sin\theta}\left[\frac{\partial\left(\prod_\varphi \sin\theta\right)}{\partial\theta} - \frac{\partial\prod_\theta}{\partial\varphi}\right] \\ \text{curl}_\varphi \prod_P &= \frac{j_\varphi}{r}\left[\frac{\partial\left(r\prod_\theta\right)}{\partial r} - \frac{\partial\left(\prod_r\right)}{\partial\theta}\right] \\ \text{curl}_\theta \prod_P &= \frac{j_\theta}{r \sin\theta}\left[\frac{\partial\prod_r}{\partial\varphi} - \sin\theta\frac{\partial\left(r\prod_\varphi\right)}{\partial r}\right] \end{aligned}\right\} \tag{344}$$

According to (321), we have for a nondissipative medium ($\sigma = 0$), the resultant magnetic field component H_P at any space point P

$$\begin{aligned} H_P = \kappa \,\text{curl}\, \frac{\partial\prod_P}{\partial t} &= j\omega\kappa \,\text{curl} \prod_P \\ &= j\omega\kappa\left(\text{curl}_r \prod_P + \text{curl}_\varphi \prod_P + \text{curl}_\theta \prod_P\right) \end{aligned} \tag{345}$$

where (341) gives the values for $\prod_r$, $\prod_\varphi$, and $\prod_\theta$ that occur in respective curl terms along the r, φ, and θ directions and are, according to (344), to be differentiated. The components of H_P are then

$$\left.\begin{aligned} H_r &= j\omega\kappa \,\text{curl}_r \prod_P = j_r j\omega\kappa \frac{\dfrac{\partial\left(\prod_\varphi \sin\theta\right)}{\partial\theta} - \dfrac{\partial\prod_\theta}{\partial\varphi}}{r \sin\theta} \\ H_\varphi &= j_\varphi j\omega\kappa \frac{\dfrac{\partial\left(r\prod_\theta\right)}{\partial r} - \dfrac{\partial\prod_r}{\partial\theta}}{r} \\ H_\theta &= j_\theta j\omega\kappa \frac{\dfrac{\partial\prod_r}{\partial\varphi} - \sin\theta\dfrac{\partial\left(r\prod_\varphi\right)}{\partial r}}{r \sin\theta} \end{aligned}\right\} \text{amp/m} \tag{346}$$

In a similar way, the corresponding electric field components in volts per meter can be found by means of (321) or (340). Quite often only horizontal and vertical patterns are of concern. The formula for horizontal XY patterns are obtained by putting $\theta = 90$ deg. For the vertical patterns, we have two main planes, namely, the $\varphi = 0$ deg and the $\varphi = 90$ deg.

For experimental data and wave-guide patterns, the reader is referred to the article by W. L. Barrow and F. M. Greene.[1]

50. Potential, Current or Flow, and Hertz Functions Applied to Space Radiation. Suppose we have a linear radiator along the vertical Z axis of a Cartesian XYZ system and unobstructed radiation in atmosphere ($\sigma = 0$, and only μ and κ effective). We have then the magnetic field component H_z along the Z direction equal to zero. The Maxwellian main equations (1) have then, according to (137), the six components for the curl of $\mathcal{E}$ and curl of H

$$\left.\begin{aligned}
&-\frac{\partial(\mu H_x)}{\partial t} = \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z}; \qquad -\frac{\partial(\mu H_y)}{\partial t} = \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x};\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\underbrace{\frac{\partial(\mu H_z)}{\partial t}}_{=0} = \underbrace{\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}}_{\text{hence, } =0}\\
&\underbrace{\sigma\mathcal{E}_x}_{=0} + \frac{\partial(\kappa\mathcal{E}_x)}{\partial t} = \overbrace{\frac{\partial H_z}{\partial y}}^{=0} - \frac{\partial H_y}{\partial z}; \qquad \underbrace{\sigma\mathcal{E}_y}_{=0} + \frac{\partial(\kappa\mathcal{E}_y)}{\partial t} = \frac{\partial H_x}{\partial z} - \overbrace{\frac{\partial H_z}{\partial x}}^{=0};\\
&\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{\sigma\mathcal{E}_z}_{=0} + \frac{\partial(\kappa\mathcal{E}_z)}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}
\end{aligned}\right\} \quad (347)$$

These relations, therefore, simplify greatly. Since $H_z = 0$, we have also the time derivative of μH_z equal to zero, and there can be *no* magnetic displacement current in volts per square meter across surfaces normal to the Z axis. *For such a condition,* we may define the electric field $\mathcal{E}$ in terms of a potential $\Phi = V$ in volts and the magnetic field H by means of a *current* or *flow* function ψ in amperes. These functions are expressed by

$$\mathcal{E}_x = -\frac{\partial \Phi}{\partial x} \text{ volts/m}; \qquad \mathcal{E}_y = -\frac{\partial \Phi}{\partial y} \text{ volts/m}; \qquad H_x = \frac{\partial \psi}{\partial y} \text{ amp/m};$$
$$H_y = -\frac{\partial \psi}{\partial x} \text{ amp/m} \quad (348)$$

The meaning of the current or flow function ψ becomes evident if we assume a curve $\psi(x, y) = k$, where k is a constant. We find then

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0; \qquad \frac{dy}{dx} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} = \frac{H_y}{H_x}$$

This means that each $\psi(x, y) = k$ curve is *identical* with a magnetic line of force, giving ψ the properties of a flow or current function. If the

[1] *Ibid.*, pp. 1510, 1511.

radiator has a finite length along the Z axis, we have space radiation and the different cross sections for which Φ and ψ hold are no longer equally entitled. For an infinite length, we have guided electromagnetic radiation. For a finite length, we have then Φ and ψ, *both dependent* on z. If then Π meter · volts denotes the Hertz function for the space, we have

$$\Phi = -\frac{\partial \Pi}{\partial z} \qquad \text{volts} \tag{349}$$

and find the following details:

Using (348) and (349), we have

$$\mathcal{E}_x = -\frac{\partial \Phi}{\partial x} = -\frac{\partial\left(-\partial\Pi/\partial z\right)}{\partial x} = +\frac{\partial^2 \Pi}{\partial x\,\partial z}; \qquad \mathcal{E}_y = -\frac{\partial\left(-\partial\Pi/\partial z\right)}{\partial y} = +\frac{\partial^2 \Pi}{\partial y\,\partial z}$$

Since propagation in air is propagation in an isotropic medium, for which μ and κ are constant, so that $\partial(\mu H) \equiv \mu\,\partial H$ and $\partial(\kappa\mathcal{E}) \equiv \kappa\,\partial\mathcal{E}$, and

$$\frac{\partial(\kappa\mathcal{E}_x)}{\partial t} = \kappa\frac{\partial \mathcal{E}_x}{\partial t} = \kappa\frac{\partial\left[\partial^2\Pi/(\partial x\,\partial z)\right]}{\partial t} = \kappa\frac{\partial^3 \Pi}{\partial x\,\partial z\,\partial t};$$

$$\frac{\partial(\kappa\mathcal{E}_y)}{\partial t} = \kappa\frac{\partial\left[\partial^2\Pi/(\partial y\,\partial z)\right]}{\partial t} = \kappa\frac{\partial^3 \Pi}{\partial y\,\partial z\,\partial t}$$

$$\frac{\partial H_y}{\partial z} = \frac{\partial(-\partial\psi/\partial x)}{\partial z} = -\frac{\partial^2 \psi}{\partial x\,\partial z}; \qquad \frac{\partial H_x}{\partial z} = \frac{\partial(\partial\psi/\partial y)}{\partial z} = \frac{\partial^2 \psi}{\partial y\,\partial z}$$

The first two curl H components of (347), which read

$$\frac{\partial(\kappa\mathcal{E}_x)}{\partial t} = -\frac{\partial H_y}{\partial z} \qquad \text{and} \qquad \frac{\partial(\kappa\mathcal{E}_y)}{\partial t} = +\frac{\partial H_x}{\partial z}$$

are then in terms of the Hertz and current functions

$$\kappa\frac{\partial^3 \Pi}{\partial x\,\partial z\,\partial t} = \frac{\partial^2 \psi}{\partial x\,\partial z}; \qquad \kappa\frac{\partial^3 \Pi}{\partial y\,\partial z\,\partial t} = \frac{\partial^2 \psi}{\partial y\,\partial z} \tag{350}$$

Both expressions are satisfied by

$$\psi = \kappa\frac{\partial \Pi}{\partial t}. \tag{351}$$

since

$$\frac{\partial^2 \psi}{\partial x\,\partial z} = \frac{\partial^2\left(\kappa\dfrac{\partial \Pi}{\partial t}\right)}{\partial x\,\partial z} = \kappa\frac{\partial^3 \Pi}{\partial x\,\partial z\,\partial t} \qquad \text{and} \qquad \frac{\partial^2 \psi}{\partial y\,\partial z} = \frac{\partial^2\left(\kappa\dfrac{\partial \Pi}{\partial t}\right)}{\partial y\,\partial z} = \kappa\frac{\partial^3 \Pi}{\partial y\,\partial z\,\partial t}$$

By means of (348) and (351), we find for the third curl H component of (347)

$$\underbrace{\frac{\partial(\kappa\mathcal{E}_z)}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}}_{\text{remaining curl } H \text{ component}} = \frac{\partial\left(-\frac{\partial\psi}{\partial x}\right)}{\partial x} - \frac{\partial\left(\frac{\partial\psi}{\partial y}\right)}{\partial y} = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$$

$$= -\left[\frac{\partial^2\left(\kappa\frac{\partial\Pi}{\partial t}\right)}{\partial x^2} + \frac{\partial^2\left(\kappa\frac{\partial\Pi}{\partial t}\right)}{dy^2}\right]$$

$$= -\kappa\left(\frac{\partial^3\Pi}{\partial x^2\,\partial t} + \frac{\partial^3\Pi}{\partial y^2\,\partial t}\right)$$

or

$$\mathcal{E}_z = -\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right) \tag{352}$$

There are *only two* curl $\mathcal{E}$ components in (347), namely,

$$-\frac{\partial(\mu H_x)}{\partial t} = \frac{\partial\mathcal{E}_z}{\partial y} - \frac{\partial\mathcal{E}_y}{\partial z} \quad \text{and} \quad -\frac{\partial(\mu H_y)}{\partial t} = \frac{\partial\mathcal{E}_x}{\partial z} - \frac{\partial\mathcal{E}_z}{\partial x}$$

since the magnetic displacement current $[\partial(\mu H_z)]/\partial t$ vanishes. Introducing the Hertz function (349) and the current function ψ of (348), we find

$$\frac{\partial(\mu H_x)}{\partial t} = \mu\frac{\partial(\partial\psi/\partial y)}{\partial t} = \mu\frac{\partial^2\psi}{\partial y\,\partial t}; \qquad \frac{\partial\mathcal{E}_y}{\partial z} = \frac{\partial(-\partial\Phi/\partial y)}{\partial z} = -\frac{\partial^2\Phi}{\partial y\,\partial z}$$

$$= -\frac{\partial^2\left(-\partial\Pi/\partial z\right)}{\partial y\,\partial z} = +\frac{\partial^3\Pi}{\partial y\,\partial z^2}$$

$$\frac{\partial(\mu H_y)}{\partial t} = \mu\frac{\partial(-\partial\psi/\partial x)}{\partial t} = -\mu\frac{\partial^2\psi}{\partial x\,\partial t}; \qquad \frac{\partial\mathcal{E}_x}{\partial z} = \frac{\partial(-\partial\Phi/\partial x)}{\partial z} = -\frac{\partial^2\Phi}{\partial x\,\partial z}$$

$$= -\frac{\partial^2\left(-\partial\Pi/\partial z\right)}{\partial x\,\partial z} = +\frac{\partial^3\Pi}{\partial x\,\partial z^2}$$

Therefore,

$$-\mu\frac{\partial^2\psi}{\partial y\,\partial t} = \frac{\partial\mathcal{E}_z}{\partial y} - \frac{\partial^3\Pi}{\partial y\,\partial z^2}; \qquad +\mu\frac{\partial^2\psi}{\partial x\,\partial t} = \frac{\partial^3\Pi}{\partial x\,\partial z^2} - \frac{\partial\mathcal{E}_z}{\partial x} \tag{353}$$

But according to (351),

$$\frac{\partial\psi}{\partial t} = \kappa\frac{\partial^2\Pi}{\partial t^2} \quad \text{and} \quad \mu\frac{\partial^2\psi}{\partial y\,\partial t} = \mu\frac{\partial\left(\frac{\partial\psi}{\partial t}\right)}{\partial y} = \mu\frac{\partial\left(\kappa\frac{\partial^2\Pi}{\partial t^2}\right)}{\partial y} = \mu\kappa\frac{\partial^3\Pi}{\partial y\,\partial t^2} \tag{354}$$

In the same way, we find

$$\mu\frac{\partial^2\psi}{\partial x\,\partial t} = \mu\frac{\partial\left(\frac{\partial\psi}{\partial t}\right)}{\partial x} = \mu\frac{\partial\left(\kappa\frac{\partial^2\Pi}{\partial t^2}\right)}{\partial x} = \mu\kappa\frac{\partial^3\Pi}{\partial x\,\partial t^2} \tag{355}$$

According to (352),

$$\frac{\partial \mathcal{E}_z}{\partial y} = -\frac{\partial\left(\dfrac{\partial^2\Pi}{\partial x^2} + \dfrac{\partial^2\Pi}{\partial y^2}\right)}{\partial y}; \qquad \frac{\partial \mathcal{E}_z}{\partial x} = -\frac{\partial\left(\dfrac{\partial^2\Pi}{\partial x^2} + \dfrac{\partial^2\Pi}{\partial y^2}\right)}{\partial x}$$

$$\mu\kappa \frac{\partial^3\Pi}{\partial y\,\partial t^2} = \mu\kappa \frac{\partial\left(\partial^2\Pi/\partial t^2\right)}{\partial y}; \qquad \mu\kappa \frac{\partial^3\Pi}{\partial x\,\partial t^2} = \mu\kappa \frac{\partial\left(\partial^2\Pi/\partial t^2\right)}{\partial x} \tag{356}$$

Using the results of (356) in (353), in addition to the transformations

$$\frac{\partial^3\Pi}{\partial y\,\partial z^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial z^2}\right) \quad \text{and} \quad \frac{\partial^3\Pi}{\partial x\,\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2\Pi}{\partial z^2}\right)$$

we find

$$-\mu\kappa \frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial t^2}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2} + \frac{\partial^2\Pi}{\partial z^2}\right);$$

$$+\mu\kappa \frac{\partial}{\partial x}\left(\frac{\partial^2\Pi}{\partial t^2}\right) = +\frac{\partial}{\partial x}\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2} + \frac{\partial^2\Pi}{\partial z^2}\right) \tag{357}$$

Either one of the Π relations of (357) gives the same result, since respective partial differentiations with respect to x and y cancel out, and we have

$$\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2} + \frac{\partial^2\Pi}{\partial z^2} - \mu\kappa \frac{\partial^2\Pi}{\partial t^2} = 0; \qquad \frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2} + \frac{\partial^2\Pi}{\partial z^2} = \nabla^2\Pi;$$

$$\mu\kappa = \frac{1}{v^2} \tag{358}$$

which shows that we can use the Hertz vector Π in the wave equation because $\mu\kappa$ is the reciprocal of the square of the velocity of propagation. For a purely dielectric medium ($\sigma = 0$), the same relation holds also for the electric field vector $\mathcal{E}$ or the magnetic field vector H in place of Π.

51. Notes on the Hertz Solution for an Elementary Space Radiator. The result of (358) can be expressed in other coordinate systems. If we use the Laplace transformation for spherical and Cartesian coordinates, which are $x = r \sin\theta \cos\varphi$; $y = r \sin\theta \sin\varphi$ and $z = r\cos\theta$, we find for the wave equation

$$\frac{1}{r^2 \sin\theta}\left[\frac{\partial\left(r^2 \sin\theta \dfrac{\partial\Pi}{\partial r}\right)}{\partial r} + \frac{\partial\left(\sin\theta \dfrac{\partial\Pi}{\partial\theta}\right)}{\partial\theta} + \frac{\partial\left(\dfrac{1}{\sin\theta}\dfrac{\partial\Pi}{\partial\varphi}\right)}{\partial\varphi}\right]$$

$$= \mu\kappa \frac{\partial^2\Pi}{\partial t^2} \tag{359}$$

Assuming that the Hertz function $\prod$ in meter · volts depends only on $r = \sqrt{x^2 + y^2 + z^2}$, *not* on the bearing φ with the X axis and *not* on the elevation $90 - \theta$ with respect to the XY plane, Eq. (359) simplifies to

$$\underbrace{\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial\prod}{\partial r}\right)}{\partial r}}_{\frac{1}{r^2}r\frac{\partial^2(r\prod)}{\partial r^2}} = \mu\kappa\frac{\partial^2\prod}{\partial t^2} \tag{360}$$

yielding the well-known differential equation

$$\frac{\partial^2(r\prod)}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2(r\prod)}{\partial t^2} \quad \text{our result;}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{well-known differential equation with solution} \quad y = F\underset{\longrightarrow}{\left(t - \frac{x}{v}\right)} + G\underset{\longleftarrow}{\left(t + \frac{x}{v}\right)} \tag{361}$$

where for unobstructed radiation space, we have only the outgoing wave function with argument $t - (x/v)$. Therefore, $r\prod = F[t - (r/v)]$ and the diverging outgoing spherical electromagnetic wave has a solution for the Hertz function of the form

$$\prod = \frac{F\left(t - \frac{r}{v}\right)}{r} = \frac{F\left(t - \frac{\sqrt{x^2 + y^2 + z^2}}{v}\right)}{\sqrt{x^2 + y^2 + z^2}} \quad \text{m · volts} \tag{362}$$

By means of Eqs. (349) and (351), we find the potential $\Phi = V$ volts and the current function ψ in amperes with the solutions

$$V = -\frac{\partial\prod}{\partial z} = -\frac{\partial\left\{\frac{F\left[t - \frac{(x^2+y^2+z^2)^{1/2}}{v}\right]}{(x^2+y^2+z^2)^{1/2}}\right\}}{\partial z} \quad \text{volts;}$$

$$\psi = \kappa\frac{\partial\prod}{\partial t} = \kappa\frac{\partial\left[\frac{F\left(t - \frac{r}{v}\right)}{r}\right]}{\partial t} \quad \text{amp} \tag{363}$$

with the following details:

The value of $r = \sqrt{x^2 + y^2 + z^2}$ contains z, for which reason the r term in the spherical function $F[t - (r/v)]$ has to be expressed with its Cartesian or other suitable components for bringing about differentiation of the V relation. This is not needed for the differentiation of the current function ψ, since r does not contain the time function t. We then have V of the form

$$V = -\frac{\partial(u/w)}{\partial z} = -\frac{u'w - uw'}{w^2} = \frac{uw' - u'w}{w^2}$$

where

$$u = F\left[t - \frac{(x^2 + y^2 + z^2)^{1/2}}{v}\right]; \qquad w = r = (x^2 + y^2 + z^2)^{1/2}$$

$$w' = \frac{\partial w}{\partial z} = \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial z} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}2z = \frac{z}{r}$$

$$u' = \frac{\partial u}{\partial z} = \frac{\partial\left\{F\left[t - \frac{(x^2 + y^2 + z^2)^{1/2}}{v}\right]\right\}}{\partial z} = \underbrace{\frac{\partial[F(t)]}{\partial z}}_{F(0)} - \frac{\partial F\left[\frac{(x^2 + y^2 + z^2)^{1/2}}{v}\right]}{\partial z}$$

$$= F(0) - F\left[\frac{1}{v}\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}2z\right]$$

$$= -\frac{z}{vr}F'\left(t - \frac{r}{v}\right).$$

The solutions of (363) for the potential V and the current function ψ are then

$$V^{(\text{volts})} = \frac{uw' - u'w}{w^2} = \frac{\frac{z}{r}F\left(t - \frac{r}{v}\right) + r\frac{z}{vr}F'\left(t - \frac{r}{v}\right)}{r^2}$$

$$= \frac{z}{r}\left[\frac{F\left(t - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t - \frac{r}{v}\right)}{vr}\right]$$

$$\psi^{(\text{amp})} = \kappa\frac{\partial\left[\frac{F\left(t - \frac{r}{v}\right)}{r}\right]}{\partial t} = \frac{\kappa}{r}\left[\underbrace{\frac{\partial F(t)}{\partial t}}_{=F(1)} - \underbrace{\frac{\partial F\left(\frac{r}{v}\right)}{\partial t}}_{=F(0)}\right]$$

$$= \kappa\frac{\partial F'\left(t - \frac{r}{v}\right)}{r} \tag{364}$$

The V solution and the ψ solution give us, according to (348), a way to find the electric and magnetic field components $\mathcal{E}_x = -\partial V/\partial x$, $\mathcal{E}_y = -\partial V/\partial y$, $H_x = \partial\psi/\partial y$, $H_y = -\partial\psi/\partial x$, with the following details:

For the current function ψ, we have

$$H_x^{(\text{amp/m})} = \frac{\partial\psi}{\partial y} = \kappa\frac{\partial\left[\frac{F'\left(t - \frac{r}{v}\right)}{r}\right]}{\partial y} = \kappa\frac{\partial\left(\frac{u}{w}\right)}{\partial y} = \kappa\frac{u'w - uw'}{w^2}$$

where

$$u = F'\left(t - \frac{r}{v}\right); \qquad w = r; \qquad w' = \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial y}$$

$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}2y = \frac{y}{r}$$

$$u' = \frac{\partial\left\{F'\left[t - \dfrac{(x^2+y^2+z^2)^{1/2}}{v}\right]\right\}}{\partial y} = \underbrace{\frac{\partial[F'(t)]}{\partial y}}_{=F'(0)} - \underbrace{\frac{\partial\left\{F'\left[\dfrac{(x^2+y^2+z^2)^{1/2}}{v}\right]\right\}}{\partial y}}_{-F''\left[\frac{1}{v}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\right]2y}$$

$$= -\frac{y}{vr}F''\left(t - \frac{r}{v}\right)$$

$$H_x = \kappa\frac{u'w - uw'}{w^2} = \kappa\left[\frac{-\dfrac{y}{vr}rF''\left(t - \dfrac{r}{v}\right) - \dfrac{y}{r}F'\left(t - \dfrac{r}{v}\right)}{r^2}\right]$$

The two magnetic field components then are

$$H_x = -\frac{y\kappa}{r^2}\left[\frac{F'\left(t - \dfrac{r}{v}\right)}{r} + \frac{F''\left(t - \dfrac{r}{v}\right)}{v}\right];$$

$$H_y = +\frac{x\kappa}{r^2}\left[\frac{F'\left(t - \dfrac{r}{v}\right)}{r} + \frac{F''\left(t - \dfrac{r}{v}\right)}{v}\right] \quad \text{amp/m} \quad (365)$$

where the H_y detailed solution is the same as for H_x if we use x instead of y and a minus sign before our result, since $H_y = -\partial\psi/\partial x$. For the electric field components, we have, according to (348), (364), and (352), the following details:

$$\mathcal{E}_x = -\frac{\partial V}{\partial x} = -\frac{\partial\left\{\dfrac{z}{r}\left[\dfrac{F\left(t - \dfrac{x}{v}\right)}{r^2} + \dfrac{F'\left(t - \dfrac{r}{v}\right)}{vr}\right]\right\}}{\partial x} = -\underbrace{\frac{\partial\left[\dfrac{zF\left(t - \dfrac{x}{v}\right)}{r^3}\right]}{\partial x}}_{a}$$

$$-\underbrace{\frac{\partial\left[\dfrac{zF'\left(x - \dfrac{r}{v}\right)}{vr^2}\right]}{\partial x}}_{b}$$

$$a = \frac{u'w - uw'}{w^2} \text{ for } u = zF\left[t - \frac{(x^2 + y^2 + z^2)^{1/2}}{v}\right], \qquad w = r^3 = (x^2 + y^2 + z^2)^{3/2}$$

$$u' = z\left\{\underbrace{\frac{\partial[F(t)]}{\partial x}}_{=F(0)} - \underbrace{\frac{\partial\left[\dfrac{F(x^2+y^2+z^2)^{1/2}}{v}\right]}{\partial x}}_{F\left[\frac{1}{v}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\right]2x}\right\} = z\left[F(0) - F\left(\frac{x}{vr}\right)\right]$$

$$= -\frac{zx}{vr}F'\left(t - \frac{r}{v}\right)$$

$$w' = \frac{\partial(x^2+y^2+z^2)^{3/2}}{\partial x} = \frac{3}{2}(x^2+y^2+z^2)^{1/2}2x = 3rx$$

$$a = \frac{u'w - uw'}{w^2} = \frac{r^3\dfrac{zx}{vr}F'\left(t-\dfrac{r}{v}\right) - 3rxzF\left(t-\dfrac{r}{v}\right)}{r^6}$$

$$= \frac{-\dfrac{rzx}{v}F'\left(t-\dfrac{r}{v}\right) - 3zxF\left(t-\dfrac{r}{v}\right)}{r^5}$$

In the same way, we find for b the solution

$$b = \frac{u'w - uw'}{w^2} \text{ for } u = zF'\left[t - \frac{(x^2+y^2+z^2)^{1/2}}{v}\right]; \qquad w = vr^2 = v(x^2+y^2+z^2)$$

$$u' = z\left\{\underbrace{\frac{\partial[F'(t)]}{\partial x}}_{=F'(0)} - \underbrace{\frac{\partial\left[\dfrac{F'(x^2+y^2+z^2)^{1/2}}{v}\right]}{\partial x}}_{=F'\left[\frac{1}{v}\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\right]2x}\right\} = z\left[F'(0) - F'\left(\frac{x}{vr}\right)\right]$$

$$= -\frac{zx}{vr}F''\left(t - \frac{r}{v}\right)$$

$$w' = \frac{\partial[v(x^2+y^2+z^2)]}{\partial x} = 2vx$$

$$b = \frac{u'w - uw'}{w^2} = \frac{vr^2\dfrac{-zx}{vr}F''\left(t-\dfrac{r}{v}\right) - 2vxzF'\left(t-\dfrac{r}{v}\right)}{v^2r^4}$$

$$= \frac{-rzxF''\left(t-\dfrac{r}{v}\right) - 2vxzF'\left(t-\dfrac{r}{v}\right)}{v^2r^4}$$

$$\mathcal{E}_x = -a - b = \frac{3xz}{r^5}F\left(t-\frac{r}{v}\right) + \underbrace{\frac{xz}{vr^4}F'\left(t-\frac{r}{v}\right) + \frac{2xz}{vr^4}F'\left(t-\frac{r}{v}\right)}_{\frac{3xz}{vr^4}F'\left(t-\frac{r}{v}\right)} + \frac{xz}{v^2r^3}F''\left(t-\frac{r}{v}\right)$$

Since the relation for the component along the Y axis is $\mathcal{E}_y = -\partial V/\partial y$, the solution is symmetrical with the above derivations and we have only to use y in place of x in our expressions.

The electric components along the X and Y axes then read

$$\mathcal{E}_x = \frac{xz}{r^3}\underbrace{\left[\frac{F\left(t - \frac{r}{v}\right)}{r^2} + 3\frac{F'\left(t - \frac{r}{v}\right)}{rv} + \frac{F''\left(t - \frac{r}{v}\right)}{v^2}\right]}_{P};$$

$$\mathcal{E}_y = \frac{yz}{r^3}P \qquad \text{volts/m} \qquad (366)$$

The component of the electric field along the Z axis cannot be found directly from the potential V, but from the expression (352), since $\prod$ is given by (362). We have then the following details:

From (352) and (362), we find

$$\mathcal{E}_z = -\left(\frac{\partial^2\prod}{\partial x^2} + \frac{\partial^2\prod}{\partial y^2}\right) \text{ volts/m}; \qquad \prod = -\frac{F\left(t - \frac{r}{v}\right)}{r} \text{ m} \cdot \text{volts}$$

$$\frac{\partial\prod}{\partial x} = -\frac{x}{r}\left[\frac{F\left(t - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t - \frac{r}{v}\right)}{vr}\right]; \qquad \frac{\partial\prod}{\partial y} = -\frac{y}{r}\left[\frac{F\left(t - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t - \frac{r}{v}\right)}{vr}\right]$$

These results are similar to the derivations leading to the V expression of (364), since $V = -\partial\prod/\partial z$. We have, therefore, only to use x for z or y for z We find then

$$\frac{\partial^2\prod}{\partial x^2} = -\frac{\partial\left\{\frac{x}{r}\left[\frac{F\left(t - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t - \frac{r}{v}\right)}{vr}\right]\right\}}{\partial x} = -\left\{\underbrace{\frac{\partial\left[\frac{xF\left(t - \frac{r}{v}\right)}{r^3}\right]}{\partial x}}_{=A_x} + \underbrace{\frac{\partial\left[\frac{xF'\left(t - \frac{r}{v}\right)}{vr^2}\right]}{\partial x}}_{=B_x}\right\}$$

$$A_x = \frac{U'W - UW'}{W^2}; \qquad U = xF\left(t - \frac{r}{v}\right), \qquad W = (x^2 + y^2 + z^2)^{3/2}$$

$$U = uw; \qquad U' = u'w + uw', \qquad u = x; \qquad w = F\left(t - \frac{r}{v}\right)$$

$$u' = \frac{\partial x}{\partial x} = 1; \qquad w' = \frac{\partial\left\{F\left[t - \frac{(x^2 + y^2 + z^2)^{1/2}}{v}\right]\right\}}{\partial x} = -\frac{x}{vr}F'\left(t - \frac{r}{v}\right)$$

$$W' = \frac{\partial(x^2 + y^2 + z^2)^{3/2}}{\partial x} = \frac{3}{2}(x^2 + y^2 + z^2)^{1/2}2x = 3rx$$

$$U' = u'w + uw' = F\left(t - \frac{r}{v}\right) - \frac{x^2}{vr}F'\left(t - \frac{r}{v}\right)$$

$$A_x = \frac{\left[F\left(t - \frac{r}{v}\right) - \frac{x^2}{vr}F'\left(t - \frac{r}{v}\right)\right]\overbrace{(x^2 + y^2 + z^2)^{3/2}}^{r^3} - 3rx^2F\left(t - \frac{r}{v}\right)}{\underbrace{(x^2 + y^2 + z^2)^3}_{r^6}}$$

$$= \frac{F\left(t - \frac{r}{v}\right) - \frac{x^2}{vr}F'\left(t - \frac{r}{v}\right)}{r^3} - \frac{3x^2F\left(t - \frac{r}{v}\right)}{r^5}$$

For the term B_x of $\partial^2\prod/\partial x^2$, we find

$$B_x = \frac{U'W - UW'}{W^2}; \qquad U = xF'\left(t - \frac{r}{v}\right); \qquad W = v(x^2 + y^2 + z^2)$$

$$U = uw; \qquad U' = u'w + uw'; \qquad u = x; \qquad w = F'\left(t - \frac{r}{v}\right), \qquad u' = \frac{\partial x}{\partial x} = 1$$

$$w' = \frac{\partial F'\left(t - \frac{r}{v}\right)}{\partial x} = -\frac{x}{vr}F''\left(t - \frac{r}{v}\right); \qquad W' = \frac{\partial[v(x^2 + y^2 + z^2)]}{\partial x} = 2vx$$

$$U' = u'w + uw' = F'\left(t - \frac{r}{v}\right) - \frac{x^2}{vr}F''\left(t - \frac{r}{v}\right)$$

$$B_x = \frac{\left[F'\left(t - \frac{r}{v}\right) - \frac{x^2}{vr}F''\left(t - \frac{r}{v}\right)\right]vr^2 - 2vx^2F'\left(t - \frac{r}{v}\right)}{v^2r^4}$$

$$= \frac{F'\left(t - \frac{r}{v}\right) - \frac{x^2}{vr}F''\left(t - \frac{r}{v}\right)}{vr^2} - \frac{2x^2F'\left(t - \frac{r}{v}\right)}{vr^4}$$

In exactly the same manner, we find the solution for the second partial differentiation of $\prod$ with respect to the Y direction. We can use the above results by replacing the x factor by the y factor and find

$$\frac{\partial^2\prod}{\partial y^2} = -\frac{\partial\left\{\frac{y}{r}\left[-\frac{F\left(t - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t - \frac{r}{v}\right)}{vr}\right]\right\}}{\partial y} = -\underbrace{\frac{\partial\left[\frac{yF\left(t - \frac{r}{v}\right)}{r^3}\right]}{\partial y}}_{A_y} + \underbrace{\frac{\partial\left[\frac{yF'\left(t - \frac{r}{v}\right)}{vr^2}\right]}{\partial y}}_{B_y}$$

$$A_y = \frac{F\left(t - \frac{r}{v}\right) - \frac{y^2}{vr}F'\left(t - \frac{r}{v}\right)}{r^3} - \frac{3y^2F\left(t - \frac{r}{v}\right)}{r^5};$$

$$B_y = \frac{F'\left(t - \frac{r}{v}\right) - \frac{y^2}{vr}F''\left(t - \frac{r}{v}\right)}{vr^2} - \frac{2y^2F'\left(t - \frac{r}{v}\right)}{vr^4}$$

We have now all the terms to find the value for the field intensity along the Z axis in volts per meter. For the sake of simplicity, let us operate with F instead of the complete function $F[t - (r/v)]$ and with F' and F'' for the first and second derivatives ∂F and $\partial^2 F$. These derivatives are the outcome with respect to r, since in our differential equation (360), it is assumed that the Hertz function $\prod$ depends only upon r but not on φ or θ. Strictly, we deal with r, t functions and in many applications F' and F'' are t derivatives. We have then

$$\frac{\partial^2\prod}{\partial x^2} = -A_x - B_x; \qquad \frac{\partial^2\prod}{\partial y^2} = -A_y - B_y; \qquad \mathcal{E}_z = -\frac{\partial^2\prod}{\partial x^2} - \frac{\partial^2\prod}{\partial y^2} = A_x + B_x + A_y + B_y$$

$$\mathcal{E}_z = \underbrace{\frac{F}{r^3} - \frac{x^2F'}{vr^4} - \frac{3x^2F}{r^5}}_{A_x} + \underbrace{\frac{F'}{vr^2} - \frac{x^2F''}{v^2r^3} - \frac{2x^2F'}{vr^4}}_{B_x} + \underbrace{\frac{F}{r^3} - \frac{y^2F'}{vr^4} - \frac{3y^2F}{r^5}}_{A_y} + \underbrace{\frac{F'}{vr^2} - \frac{y^2F''}{v^2r^3} - \frac{2y^2F'}{vr^4}}_{B_y}$$

Since $r^2 = x^2 + y^2 + z^2$, we have also $-(x^2 + y^2) = z^2 - r^2$ and

$$\mathcal{E}_z = \underbrace{-3(x^2 + y^2)}_{+3(z^2-r^2)}\frac{F}{r^5} - \underbrace{3(x^2 + y^2)}_{+3(z^2-r^2)}\frac{F'}{vr^4} - \underbrace{(x^2 + y^2)}_{+z^2-r^2}\frac{F''}{v^2r^3} + \underbrace{\frac{2F}{r^3}}_{\frac{2r^2F}{r^5}} + \underbrace{\frac{2F'}{vr^2}}_{\frac{2r^2F'}{vr^4}}$$

$$= \frac{3z^2 - r^2}{r^5}F + \frac{3z^2 - r^2}{vr^4}F' + \frac{z^2 - r^2}{v^2r^3}F''$$

The electric field component along the Z axis, then becomes

$$\left.\begin{aligned} \mathcal{E}_z &= \frac{3z^2 - r^2}{r^5}F\left(t - \frac{r}{v}\right) + \frac{3z^2 - r^2}{vr^4}F'\left(t - \frac{r}{v}\right) \\ &\qquad + \frac{z^2 - r^2}{v^2r^3}F''\left(t - \frac{r}{v}\right) \quad \text{volts/m} \\ r &= \sqrt{x^2 + y^2 + z^2} \quad \text{m}; \qquad z = r\cos\theta \quad \text{m}; \\ &\qquad v = \frac{1}{\sqrt{\mu\kappa}} \quad \text{m/sec} \end{aligned}\right\} \qquad (367)$$

where t is in seconds, μ in henrys per meter, κ in farads per meter, and r is the direction of propagation under consideration, as is indicated in Fig. 46 for any meridian plane of the spherical wave expansion. Rela-

tions (366) and (367) are then the three Cartesian components of the resultant electric field vector at the field point, and relations (365) are the associated magnetic field vectors. For the *distant* radiation region,

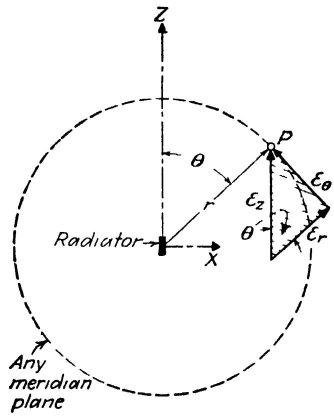


FIG. 46. Electric field components of which only the $\mathcal{E}_\theta$ value plays a part in the distant radiation region. $|\mathcal{E}_t|$ volts/m $= \sqrt{|\mathcal{E}_\theta|^2 + |\mathcal{E}_r|^2}$. $\mathcal{E}_z/\mathcal{E}_r = -\tan\theta$. P is field point in distant radiation region.

only fields of the lowest inverse distance power can be significant, and we have

$$\left.\begin{aligned}
\mathcal{E}_x &= \frac{xz}{r^3v^2}F''\left(t - \frac{r}{v}\right); \qquad \mathcal{E}_y = \frac{yz}{r^3v^2}F''\left(t - \frac{r}{v}\right); \\
&\qquad \mathcal{E}_z = \frac{z^2 - r^2}{r^3v^2}F''\left(t - \frac{r}{v}\right) \quad \text{volts/m} \\
H_x &= -\frac{y\kappa}{r^2v}F''\left(t - \frac{r}{v}\right); \qquad H_y = +\frac{x\kappa}{r^2v}F''\left(t - \frac{r}{v}\right) \quad \text{amp/m}
\end{aligned}\right\} \tag{368}$$

These expressions show, for instance, that the resultant associated fields are

$$\mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2 + \mathcal{E}_z^2}; \qquad H = \sqrt{H_x^2 + H_y^2}$$

where $\mathcal{E}$ and H are taken as absolute values, yielding the ratio

$$\begin{aligned}
\frac{\mathcal{E}}{H} &= \frac{vr^2}{\kappa v^2 r^3}\sqrt{\frac{z^2(x^2 + y^2) + (z^2 - r^2)^2}{x^2 + y^2}} = \frac{1}{\kappa v r}\sqrt{\frac{z^2(r^2 - z^2) + (z^2 - r^2)^2}{r^2 - z^2}} \\
&= \frac{\sqrt{\mu\kappa}}{\kappa r}\sqrt{\frac{z^2(z^2 - r^2) - (z^2 - r^2)^2}{z^2 - r^2}} = \frac{\sqrt{\mu/\kappa}}{r}\sqrt{z^2 - z^2 + r^2} \\
&= \sqrt{\frac{\mu}{\kappa}} \quad \text{ohms}
\end{aligned} \tag{369}$$

which is the three-dimensional wave impedance in an unobstructed

medium for a zero conductivity. It would be about 377 ohms for air. Since $z = r\cos\theta$, $x = r\sin\theta\cos\varphi$, and $y = r\sin\theta\sin\varphi$, we find

$$\left.\begin{aligned}
\mathcal{E}_z &= \frac{r^2\cos^2\theta - r^2}{r^3v^2}F'' = -\frac{\sin^2\theta}{rv^2}F'' \\
\mathcal{E}_x &= \frac{r\sin\theta\cos\varphi\cdot r\cos\theta}{r^3v^2}F'' = \frac{\sin\theta\cos\theta\cos\varphi}{rv^2}F'' \\
\mathcal{E}_y &= \frac{r\sin\theta\sin\varphi\cdot r\cos\theta}{r^3v^2}F'' = \frac{\sin\theta\cos\theta\sin\varphi}{rv^2}F'' \\
\mathcal{E}_r &= \frac{\sin\theta\cos\theta}{rv^2}F'' \\
\frac{\mathcal{E}_z}{\mathcal{E}_r} &= -\tan\theta
\end{aligned}\right\} \qquad (370)$$

where F'' is abbreviation for $F''\left(t - \frac{r}{v}\right)$. Since all lines of force in meridian planes hold for φ = constant, the radial field component $\mathcal{E}_r$ can be found from the $\mathcal{E}_x$ or the $\mathcal{E}_y$ relations, by putting $\cos\varphi = 1 = \sin\varphi$. In a similar way, we find for the distant radiation region, the magnetic field H_φ normal to any meridian plane, for $Z_0 = \sqrt{\mu/\kappa}$ ohms

$$H_\varphi = \kappa\frac{\sin\theta}{vr}F''\left(t - \frac{r}{v}\right) = \frac{Z_0\sin\theta}{r}F''\left(t - \frac{r}{v}\right) \quad \text{amp/m} \qquad (371)$$

and in the equatorial plane (XY plane), also $\sin\theta = 1$ in this expression. Inasmuch as the electric potential V volts is

$$V^{(\text{volts})} = -\frac{\partial\Pi}{\partial z} = -\frac{z}{r}\left[\frac{F\left(t - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t - \frac{r}{v}\right)}{vr}\right] \quad \text{for any } z \text{ and } t \text{ value}$$

$$V^{(\text{volts})} = \frac{\partial\Pi}{\partial z} = -\frac{z_1}{r}\left[\frac{F\left(t_1 - \frac{r}{v}\right)}{r^2} + \frac{F'\left(t_1 - \frac{r}{v}\right)}{vr}\right] = \text{constant} \begin{cases}\text{for a fixed plane } z = z_1 \text{ and} \\ \text{fixed instant } t = t_1\end{cases}$$

where the fixed plane is either the XY plane or a plane parallel thereto, V = constant depends only upon r and represents equipotential circular lines. The lines of electric force must be radial then in such $z = z_1$ planes. For such conditions, we find from the current function

$$\psi^{(\text{amp})} = \kappa\frac{\partial\Pi}{\partial t} = \kappa\frac{F'\left(t_1 - \frac{r}{v}\right)}{r} = \text{constant } (t = t_1)$$

showing that for the XY plane or any plane parallel thereto ($z = z_1$), the magnetic lines of force are also circles.

From the general solutions (365) to (367), we note that the functions F, F', and F'' occur, where the portion $F(t) = \epsilon^{j\omega t}$ for harmonic excitations as is the case in most practical applications. Since $r/v = \sqrt{\mu\kappa}\, r$ is in seconds in the argument $[t - (r/v)]$ in the function F, the first and second derivatives F' and F'', we find

$$\left.\begin{aligned} F\left(t - \frac{r}{v}\right) &= F\left[\epsilon^{j\omega\left(t-\frac{r}{v}\right)}\right] = F[\epsilon^{j(\omega t - \beta r)}] = \underbrace{F[\cos(\omega t - \beta r)]}_{\substack{\text{real part is usually}\\ \text{taken as the instan-}\\ \text{taneous value of func-}\\ \text{tion } F}} \\ &\qquad + jF[\sin(\omega t - \beta r)] \\ \beta &= \frac{\omega}{v} = \frac{2\pi f}{v} = \frac{2\pi}{\lambda}; \qquad v = \frac{1}{\sqrt{\mu\kappa}} \text{ m/sec} \end{aligned}\right\} \quad (372)$$

It is normally a current of form[1] $I \sin \omega t$ that excites a radiator unless we deal with wave-front radiators that have an electromagnetic field as the source. We operate then with $F(t) = I \sin \omega t$, $F'(t) = d[F(t)]/dt = \omega I \cos \omega t$, $F'' = d^2[F(t)]/dt^2 = -\omega^2 I \cos \omega t$. This is taken up in connection with the electric and magnetic dipole theory, as is also the case with respect to near-field and between-field regions of the electromagnetic field.

In connection with the $\prod$ function as defined by (349) and given by (362), we must realize that a radiator has to be oriented. For instance, an elementary or other radiator must have a certain position in space, for example, an elementary radiator $i\,dl$, where $i = I \sin \omega t$ and the length dl may be along the Z axis. Therefore, this is also the axis of the field. This direction or polarization can be taken care of by a unit vector $\underset{\cdot}{v}$ lying in the Z axis, and the product $\underset{\cdot}{v}\prod$ is then the *Hertz vector*. The unit vector is, for simplicity's sake, often omitted in this text, but understood just the same. This means that $\prod$ meter · volts acts, in this case, along the dl direction as would also be the case for the magnetic vector potential Λ volt · seconds per meter. We also see that the Hertz function $\prod$ is a *meter · volt* concept just as the electric moment of the $i\,dl$ radiator is a *meter · ampere* concept, both with the maximum effect along the dl directions.

[1] Rigorously, $I \cos \omega t$ would be the excitation, according to the real part of (372). However, it does not matter, since $I \sin \omega t$ is the same variation, with a reference $t = 0$ value a quarter period shifted.

52. Potential and Current Functions and Hertzian Space Function for Guided and for Space Electromagnetic Radiation (a Comparison).

Maxwell's equations for space without any sources in Cartesian x, y, z coordinates, for the current densities J in amperes per square meter, electric field intensity $\mathcal{E}$ in volts per meter, and magnetic field intensity H in amperes per meter, in a medium of σ mhos per meter, μ henrys per meter, κ farads per meter, for the electric displacement $D = \kappa\mathcal{E}$ coulombs per square meter and magnetic flux density $B = \mu H$ in webers per square meter, are

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{\partial D_x}{\partial t} = J_x \quad \text{(I)}$$
$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \frac{\partial D_y}{\partial t} = J_y \quad \text{(II)}$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{\partial D_z}{\partial t} = J_z \quad \text{(III)}$$
$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} + \frac{\partial B_x}{\partial t} = 0 \quad \text{(IV)}$$
$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} + \frac{\partial B_y}{\partial t} = 0 \quad \text{(V)}$$
$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} + \frac{\partial B_z}{\partial t} = 0 \quad \text{(VI)}$$

The six Maxwellian component equations for a space point are often expressed by $\nabla \times H - \frac{\partial D}{\partial t} = J$ amp/m² and $\nabla \times \mathcal{E} + \frac{\partial B}{\partial t} = 0$; where $\nabla \times F \equiv \text{curl } F$; $B = \mu H$; $D = \kappa\mathcal{E}$; $J + \frac{\partial D}{\partial t} = (\sigma + j\omega\kappa)\mathcal{E}$; $\frac{\partial B}{\partial t} = j\omega\mu H$ for $\epsilon^{j\omega t}$ variations

For μ and κ independent of the time, we have for harmonic fields

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + j\omega\kappa)\mathcal{E}_x \ \text{(Ia)}; \qquad \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -j\omega\mu H_x \ \text{(IVa)}$$
$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = (\sigma + j\omega\kappa)\mathcal{E}_y \ \text{(IIa)}; \qquad \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -j\omega\mu H_y \ \text{(Va)}$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (\sigma + j\omega\kappa)\mathcal{E}_z \ \text{(IIIa)}; \qquad \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -j\omega\mu H_z \ \text{(VIa)}$$

Maxwellian curl components for harmonic fields

Suppose we compare guided TEM waves (TEM waves of customary double lines) with a TEM wave in the distant radiation region. Assume that a transmission line along the Z axis exhibits perfect conductivity; then no electromagnetic field can exist within the conductors. Even for customary metallic conductivities, the small $\mathcal{E}_z$ field tangential to the surface of the respective conductors is essentially zero. The total energy flux is then essentially only due to two components p_z and $p_{x,y}$, where the xy coordinates are in planes perpendicular across the guided wave transmission. We have then

$$p_z = (\mathcal{E}_{x\,y} \cdot H_{x,y}); \qquad p_{x,y} = (\mathcal{E}_z \cdot H_{x,y})$$

since $H_z = 0$ for a power transfer along the double line. The xy components are in the cross section, and $\mathcal{E}_z$ is negligibly small (along the surface of either conductor). Table 8 gives the comparisons.

TABLE 8

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
1	$\mathcal{E}_z$ along the surface vanishes hence, Eqs. (I*a*) to (VI*a*) reduce, for $H_z = \mathcal{E}_z = 0$, to $-\dfrac{\partial H_y}{\partial z} = (\sigma + j\omega\kappa)\mathcal{E}_x = j\omega\kappa\mathcal{E}_x$ (I*a*) since $\sigma = 0$ for space next to conductor $\dfrac{\partial H_x}{\partial z} = j\omega\kappa\mathcal{E}_y$ (II*a*) $\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = 0$ (III*a*) $-\dfrac{\partial \mathcal{E}_y}{\partial z} = -j\omega\mu H_x$ (IV*a*) $\dfrac{\partial \mathcal{E}_x}{\partial z} = -j\omega\mu H_y$ (V*a*) $\dfrac{\partial \mathcal{E}_y}{\partial x} - \dfrac{\partial \mathcal{E}_x}{\partial y} = 0$ (VI*a*)	We deal with unhindered space radiation and no conductive guide. Hence, $\mathcal{E}_z \neq 0$; $H_z = 0$; $\sigma = 0$, and Eqs. (I*a*) to (VI*a*) reduce to $-\dfrac{\partial H_y}{\partial z} = j\omega\kappa\mathcal{E}_x$ (I*a*) $\dfrac{\partial H_x}{\partial z} = j\omega\kappa\mathcal{E}_y$ (II*a*) $\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = j\omega\kappa\mathcal{E}_z$ (III*a*) $\dfrac{\partial \mathcal{E}_z}{\partial y} - \dfrac{\partial \mathcal{E}_y}{\partial z} = -j\omega\mu H_x$ (IV*a*) $\dfrac{\partial \mathcal{E}_x}{\partial z} - \dfrac{\partial \mathcal{E}_z}{\partial x} = -j\omega\mu H_y$ (V*a*) $\dfrac{\partial \mathcal{E}_y}{\partial x} - \dfrac{\partial \mathcal{E}_x}{\partial y} = 0$ (VI*a*)	$H_z = 0$ defines transversability in either case for a propagation along the Z axis
2	For the potential V volts in cross-sectional planes, we have then $\mathcal{E}_x = -\dfrac{\partial V}{\partial x}$ and $\mathcal{E}_y = -\dfrac{\partial V}{\partial y}$ We may assume a current function ψ such that $H_x = +\dfrac{\partial \psi}{\partial y}$ and $H_y = -\dfrac{\partial \psi}{\partial x}$ referring to a curve $\psi(x, y)$ equal to a constant, yielding $\dfrac{\partial \psi}{\partial x} dx + \dfrac{\partial \psi}{\partial y} dy = 0$ or	Also here, in each plane $z =$ const., the magnetic displacement current vanishes and $\mathcal{E}$ depends again on potential V in the cross section, so that $\mathcal{E}_x = -\dfrac{\partial V}{\partial x}$ and $\mathcal{E}_y = -\dfrac{\partial V}{\partial y}$ with a current function ψ and $H_x = +\dfrac{\partial \psi}{\partial y}$ and $H_y = -\dfrac{\partial \psi}{\partial x}$ Unlike guided (conductor-guided or directed) radiation, the planes $z =$ const. remain no longer alike and	

TABLE 8.—(*Continued*)

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
	$\frac{dy}{dx} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} = +\frac{H_y}{H_x}$ proving that curve $\psi(x, y) = \text{constant}$ means actually magnetic lines of force, and ψ is a current function as postulated above. For any time function (I*a*) reads, since for air $\sigma = 0$ mhos/m $-\frac{\partial H_y}{\partial z} = \kappa\frac{\partial \mathcal{E}_x}{\partial t}$ $= \kappa\frac{\partial(-\partial V/\partial x)}{\partial t} = -\kappa\frac{\partial^2 V}{\partial x\,\partial t}$ or $\kappa\frac{\partial^2 V}{\partial x\,\partial t} = \frac{\partial H_y}{\partial z} = \frac{\partial(-\partial\psi/\partial x)}{\partial z}$ $= -\frac{\partial^2\psi}{\partial x\,\partial z}$ (I*b*) Similarly we find from (II*a*) to (VI*a*) $\kappa\frac{\partial^2 V}{\partial y\,\partial t} = -\frac{\partial^2\psi}{\partial y\,\partial z}$ (II*b*) $-\frac{\partial^2 V}{\partial y\,\partial z} = \mu\frac{\partial^2\psi}{\partial y\,\partial t}$ (IV*b*) $-\frac{\partial^2 V}{\partial x\,\partial z} = \mu\frac{\partial^2\psi}{\partial x\,\partial t}$ (V*b*)	both V and ψ become functions of z, taken care of by the Hertz function Π in the entire space, and $V = -\frac{\partial\Pi}{\partial z}$ This is evident, since for no charges in a medium $\frac{\partial\mathcal{E}_x}{\partial x} + \frac{\partial\mathcal{E}_y}{\partial y} + \frac{\partial\mathcal{E}_z}{\partial z} = 0$ yields, for $\mathcal{E}_x = -\partial V/\partial x$ and $\mathcal{E}_y = -\partial V/\partial y$, the relation $\frac{\partial\mathcal{E}_z}{\partial z} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$ which suggests making V the z derivative of some function Π, so that $\frac{\partial\mathcal{E}_z}{\partial z} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$ can be integrated. Hence, $\mathcal{E}_x = -\frac{\partial V}{\partial x} = -\frac{\partial(-\partial\Pi/\partial z)}{\partial x}$ $= +\frac{\partial^2\Pi}{\partial x\,\partial z}$ and $\frac{\partial\mathcal{E}_x}{\partial t} = +\frac{\partial^3\Pi}{\partial x\,\partial z\,\partial t}$. We also have $\frac{\partial H_y}{\partial z} = \frac{\partial\left(-\frac{\partial\psi}{\partial z}\right)}{\partial z} = -\frac{\partial^2\psi}{\partial x\,\partial z}$ and (I*a*) becomes for any time function of $\mathcal{E}_x$ on the right-hand side $\kappa\partial\mathcal{E}_x/\partial t$, and	

TABLE 8.—(*Continued*)

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
		we have $\kappa \frac{\partial^3\Pi}{\partial x\,\partial z\,\partial t} = \frac{\partial^2\psi}{\partial x\,\partial z}$ (I*b*) In the same way we find from (II*a*) to (V*a*), the expressions $\kappa \frac{\partial^3\Pi}{\partial y\,\partial z\,\partial t} = \frac{\partial^2\psi}{\partial y\,\partial z}$ (II*b*) $\kappa \frac{\partial \mathcal{E}_z}{\partial t} = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$ (III*b*) $-\mu \frac{\partial^2\psi}{\partial y\,\partial t} = \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial^3\Pi}{\partial y\,\partial z^2}$ (IV*b*) $\mu \frac{\partial^2\psi}{\partial x\,\partial t} = \frac{\partial^3\Pi}{\partial x\,\partial z^2} - \frac{\partial \mathcal{E}_z}{\partial x}$ (V*b*)	
3	These relations are satisfied by $\kappa \frac{\partial V}{\partial t} = \frac{\partial\psi}{\partial z}$ and $\mu \frac{\partial\psi}{\partial t} = \frac{\partial V}{\partial z}$	The first two expressions are satisfied by $\psi = \kappa \frac{\partial\Pi}{\partial t}$ and $\frac{\partial\psi}{\partial z} = -\kappa \frac{\partial V}{\partial t}$ and $\kappa \frac{\partial \mathcal{E}_z}{\partial t} = -\kappa \left(\frac{\partial^3\Pi}{\partial t\,\partial x^2} + \frac{\partial^3\Pi}{\partial t\,\partial y^2}\right)$ Hence, $\mathcal{E}_z = -\left(\frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2}\right)$ leading to $-\frac{\mu\kappa\partial^3\Pi}{\partial y\,\partial t^2} = \frac{\partial P_d}{\partial y}$; $\mu\kappa \frac{\partial^3\Pi}{\partial x\,\partial t^2} = -\frac{\partial P_d}{\partial x}$ for the differential parameter $P_d = \frac{\partial^2\Pi}{\partial x^2} + \frac{\partial^2\Pi}{\partial y^2} + \frac{\partial^2\Pi}{\partial z^2}$	

TABLE 8.—(*Continued*)

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
4	yielding $$\frac{\partial^2 V}{\partial t^2} = \frac{1}{\mu\kappa}\frac{\partial^2 V}{\partial z^2}$$ being a potential function moving with a velocity $v = c' = 1/\sqrt{\mu\kappa}$ along the Z axis, since $$V = V_0(x, y)[F_1(z - c't) + F_2(z + c't)]$$ is the general solution	Therefore $$\frac{\partial^2 \Pi}{\partial t^2} = \frac{1}{\mu\kappa}\left(\frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2} + \frac{\partial^2 \Pi}{\partial z^2}\right)$$ where Π depends on the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the origin and the time t. The quantity Π is the *Hertzian function for the entire space* of electromagnetic constants μ, κ, and σ. This function moves likewise out (a wave spread) with a velocity $v = 1/\sqrt{\mu\kappa}$ m/sec. The parameter P_d does not depend on the choice of the coordinate system. If V is the potential function of the field, we have $$\mathcal{E}_x = -\frac{\partial V}{\partial x}; \quad \mathcal{E}_y = -\frac{\partial V}{\partial y}, \quad \mathcal{E}_z = -\frac{\partial V}{\partial z}$$	
5	In each case, the medium of the transverse electromagnetic wave motion depends on the electromagnetic constants μ, κ, and σ, since the actual energy is entirely in the field medium and for guided transmission-line motion, the conductors are only instrumental in guiding the energy. The conductors absorb energy only if the conductor is imperfect (σ not ∞) in terms of joulean heat loss. For ordinary air $c' = 1/\sqrt{\mu\kappa} \cong 1/\sqrt{\mu_0\kappa_0} = 1/\sqrt{1.257 \times 10^{-6} \times 8.854 \times 10^{-12}} = c \cong 3 \times 10^8$ m/sec. For distilled water, we still have $\sigma = 0$ mhos/m and $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m but $\kappa \cong 80\kappa_0 = 80 \times 8.854 \times 10^{-12}$ depending on the temperature of the water. For room temperature, we find a velocity of propagation of		

TABLE 8.—(*Continued*)

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
	about one-ninth as large as for air. A dipole in water is a case of guided transmission as far as the respective components of the standing wave for resonance are concerned. The dipole in water has then only roughly one-ninth of the length as when located in atmosphere, depending on the operating frequency		
6	For plane TEM wave propagation, the equiphase surfaces form parallel planes of finite size and $\mathcal{E}$ as well as H are independent of the location in the planes, since $\partial/\partial x = \partial/\partial y = 0$ for propagation along the Z axis and $\mathcal{E}_z = H_z = 0$, yielding $\frac{\partial \mathcal{E}_x}{\partial z} = -j\omega\mu H_y;$ $\frac{\partial \mathcal{E}_y}{\partial z} = +j\omega\mu H_x$ $\frac{\partial H_x}{\partial z} = +a\mathcal{E}_y;$ $\frac{\partial H_y}{\partial z} = -a\mathcal{E}_x$ for $a = \sigma + j\omega\kappa$ as already found in step 1 of this table. Equations for $\partial\mathcal{E}_x/\partial z$ and $\partial H_y/\partial z$ have now a form as in expressions used for transmission lines (which are not derived by Maxwell's equations), and we have the comparisons $\frac{\partial \mathcal{E}_x}{\partial z} = -j\omega\mu H_y;$ $\frac{\partial H_y}{\partial z} = -(\sigma + j\omega\kappa)\mathcal{E}_x$ $\frac{\partial E}{\partial z} = -j\omega LI;$ $\frac{\partial I}{\partial z} = -(G + j\omega C)E$	Generally wave propagation is then expressed by $\mu\kappa \frac{\partial^2 \mathcal{E}}{\partial t^2} = \nabla^2 \mathcal{E};$ $\mu\kappa \frac{\partial^2 H}{\partial t^2} = \nabla^2 H$ where we have $\mathcal{E} = \mathcal{E}_x, = \mathcal{E}_y, = \mathcal{E}_z$ and $H = H_x, = H_y, = H_z$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ with a solution for $c' = 1/\sqrt{\mu\kappa} = v$ m/sec $w = F_1(r - c't) + F_2(r + c't)$ where r is the length of the normal between the origin to the plane wave front in the distant radiation region and $r = \sqrt{x^2 + y^2 + z^2}$ for components x, y, and z and direction cosines $l = \cos\alpha = x/r$, $m = \cos\beta = y/r$, and $n = \cos\gamma = z/r$, satisfying $l^2 + m^2 + n^2 = 1$, so that $r = lx + my + nz$. The *portion* $F_1(r - c't)$ accounts for radiated energy in *unobstructed* space, and w stands either for any $\mathcal{E}$ or any H components. According to step 4 of this table the Hertz function $\prod$ is a *single* vector function	

TABLE 8.—(*Continued*)

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
	where the resistance R of the conductor is assumed zero in order to make it consistent with the assumption made for guided TEM transmission. The upper equation is then that of a transmission line with a series inductance μ, a shunt capacitance κ, and shunt conductance σ if taken per unit length. In equivalence, $\mathcal{E}_x$ behaves like a transverse electromotive force and H_y like a transverse magnetomotive force causing a longitudinal current I. As far as L and μ are concerned, they both have the dimensions of henrys per unit length (per meter), just as C and κ are in farads per meter, since in transmission-line theory L and C are taken per unit length. $\mathcal{E}_x$ is, however, in volts/m while E is in volts, H_y in amp/m, while I is in amp. The comparison can be made identical by imagining the transmission line made up of two parallel perfectly conducting sheets of width w m and a separation of s m, then $\mathcal{E}_x = E/s$ and $H_y = I/w$. Putting this in the upper comparison relations, we find $$\frac{\partial \mathcal{E}_x}{\partial z} = \frac{\partial (E/s)}{\partial z} = -j\omega\mu \frac{I}{w};$$ $$\frac{\partial (I/w)}{\partial z} = -(\sigma + j\omega\kappa)\frac{E}{s};$$	expression of the electromagnetic field. The vector potential Λ is then $$\Lambda = \mu\kappa \frac{\partial \prod}{\partial t}$$ for a medium with no conduction currents and no charges. This is treated in mks units in this volume; while in classical texts using cgs absolute systems and $c = 3 \times 10^{10}$ cm/sec, we find the relations for the magnetic vector potential Λ, the Hertz vector $\prod$ by means of the relations $$\Lambda = \iiint_\tau \mu \frac{[J]}{4\pi r}\,d\tau$$ $$V = \iiint_\tau \frac{[\rho]}{4\pi\kappa r}\,d\tau$$ $$\mathcal{E} = -\operatorname{grad} V - \frac{\partial \Lambda}{\partial t}$$ $$\mu H = \operatorname{curl} \Lambda$$ $$\Lambda = \mu\kappa \frac{\partial \prod}{\partial t} = \iiint_\tau \mu \frac{[J]}{4\pi r}\,d\tau$$ where $[J]$ and $[\rho]$ are the *retarded* values $$\prod = \frac{F\left(t - \frac{r}{v}\right)}{r} = \frac{F(t - r\sqrt{\mu\kappa})}{r} \text{ meter}\cdot\text{volts}$$ for the *outgoing* electromag-	

TABLE 8.—(*Continued*)

Consecutive steps	Transmission-line field conditions (guided TEM radiation)	Space-field display (unguided space radiation) free TEM	Remarks
	$\frac{\partial E}{\partial z} = -j\omega \frac{\mu s I}{w}$ from Maxwell $\frac{\partial E}{\partial z} = -j\omega L I$ from line theory and $\partial I =$ $-\left(\frac{\sigma w}{s} + j\omega \frac{\kappa w}{s}\right) E$ Maxwell $\partial I = -(G + j\omega C)I$ line if the capacitance per unit length is C and $L = \frac{\mu s}{w}; \quad C = \frac{\kappa w}{s}; \quad G = \frac{\sigma w}{s}$ Since for the line theory, the propagation constant is $\Gamma = \sqrt{YZ}$ $= \sqrt{(G + j\omega C) j\omega L}$ and the surge impedance $Z_0 = \sqrt{Z/Y}$ we find $\Gamma = \sqrt{j\omega\mu(\sigma + j\omega\kappa)}$ hyp. radians/m $Z_0 = \frac{s}{w}\sqrt{\frac{j\omega\mu}{G + j\omega\kappa}}$ ohms	netic wave in unobstructed radiation space	since we assumed $R = 0$

53. Remarks on TE, TM, and TEM Waves with the Electromagnetic Field Components in Spherical Coordinates. For source-free field regions, we have, for the possible fields of Fig. 47, the following relations:

$$\left.\begin{aligned}
&\frac{\partial(\mathcal{E}_\varphi \sin\theta)}{\partial\theta} - \frac{\partial\mathcal{E}_\theta}{\partial\varphi} + j\omega\mu r\, H_r \sin\theta = 0; \\
&\qquad \frac{\partial(H_\varphi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\varphi} - (\sigma + j\omega\kappa) r\, \mathcal{E}_r \sin\theta = 0 \\
&\frac{\partial\mathcal{E}_r}{\partial\varphi} - \sin\theta \frac{\partial(r\mathcal{E}_\varphi)}{\partial r} + j\omega\mu r\, H_\theta \sin\theta = 0; \\
&\qquad \frac{\partial H_r}{\partial\varphi} - \sin\theta \frac{\partial(rH_\varphi)}{\partial r} - (\sigma + j\omega\kappa) r\, \mathcal{E}_\theta \sin\theta = 0 \\
&\frac{\partial(r\mathcal{E}_\theta)}{\partial r} - \frac{\partial\mathcal{E}_r}{\partial\theta} + j\omega\mu r H_\varphi = 0; \\
&\qquad \frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta} - (\sigma + j\omega\kappa) r\mathcal{E}_\varphi = 0
\end{aligned}\right\} \quad (373)$$

We have also the symmetrical forms

$$\left.\begin{aligned}
\frac{1}{r^2}\frac{\partial(r^2\mathcal{E}_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\mathcal{E}_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\mathcal{E}_\varphi}{\partial\varphi} = 0 \\
\frac{1}{r^2}\frac{\partial(r^2 H_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(H_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial H_\varphi}{\partial\varphi} = 0
\end{aligned}\right\} \quad (374)$$

for the respective $\mathcal{E}$ and H fields. These forms satisfy the general wave equation (page 321) for spherical coordinates, which, for the wave function w and the propagation constant $\Gamma = j\omega\sqrt{\mu(\kappa - j\sigma/\omega)} = j\omega\sqrt{\mu\kappa_e}$, yields

$$\underbrace{\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial w}{\partial\theta}\right)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2}}_{\nabla^2 w} - w\Gamma^2 = 0;$$

$$\nabla^2 w - w\Gamma^2 = 0 \quad (375)$$

We can then take any radius of the sphere as a possible direction of wave propagation. *Three* different types of waves are then possible, namely, (1) TE, or transverse electric, waves, (2) TM, or transverse magnetic, waves, (3) TEM, or transverse electromagnetic, waves. For the TE wave, the longitudinal electric component $\mathcal{E}_r$ vanishes; for $H_r = 0$, we have TM waves; and for $\mathcal{E}_r = H_r = 0$, we have TEM waves in the true radiation region or anywhere for certain shapes of radiators.

We now deal with spherical coordinates r, φ, and θ for any space point P and can no longer expect an *exponential* variation of $\mathcal{E}_r$, $\mathcal{E}_\varphi$, $\mathcal{E}_\theta$,

H_r, H_φ, and H_θ with the range. This means that an exponential function $\epsilon^{j(\omega t \pm r\Gamma_r \pm \varphi\Gamma_\varphi \pm \theta\Gamma_\theta)}$ would not be correct, since, as we shall learn, Bessel functions will play a part that only for long ranges (great values of r) approximate an exponential variation.

Inasmuch as a TE wave with $\mathcal{E}_r = 0$, but a longitudinal H_r component, can also be called a *longitudinal magnetic wave*, we can treat H_r as a scalar function of respective coordinates r, φ, and θ and put the general wave function w of (375) equal to rH_r. In a similar way, for the TM wave with $H_r = 0$, we have actually a longitudinal electric $\mathcal{E}_r$ wave and can assume $w = r\mathcal{E}_r$.

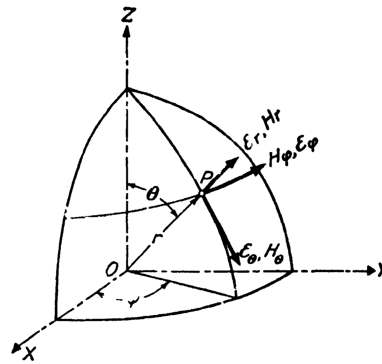


FIG. 47. Possible field components in the spherical coordinate system. O is the radiation center, P is the field point in the unobstructed radiation region.

For TEM waves with no $\mathcal{E}_r$ and H_r components, we may have two combinations $\mathcal{E}_\theta$, H_φ, and $\mathcal{E}_\varphi$, H_θ respectively. The field associations of either combination (Fig. 47) are then only functions of the range r and the elevation $90 - \theta$ above the XY plane. The decrease of either field is then proportional to $1/r$.

The field of the *electric dipole starts out with the principal longitudinal* $\mathcal{E}_r$ *wave* (*principal TM wave*), while the field of a *small-loop radiator* representing a magnetic dipole *contains* the principal longitudinal H_r wave (*principal TE wave*). The field of the biconical radiator contains the principal TEM wave *to begin with.* The customary linear radiator is made up of electric dipoles and develops, therefore, only a TEM wave in the distant radiation field but starts out as a TM wave.

Since we can refer all spherical components to the general wave equation (375), we have for the TE and TM waves, for $w = rH_r$ and $w = r\mathcal{E}_r$, respectively,

$$\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial\left(\sin\theta\frac{\partial w}{\partial \theta}\right)}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2} + \mu w(\kappa\omega^2 - j\sigma\omega) = 0 \qquad (376)$$

which, for the *nondissipative* medium ($\sigma = 0$), gives

$$\nabla^2 w - w\Gamma^2 = \nabla^2 w + \mu\kappa\omega^2 w = \nabla^2 w + \frac{\omega^2}{v^2} w \qquad (377)$$

where v denotes the phase velocity $1/\sqrt{\mu\kappa}$. The ratio ω/v is then equal

TABLE 9

For $\Gamma = j\omega\sqrt{\mu\kappa_e}$ generalized radians/m (holds for general wave medium which is dissipative)

<table>
<tr><th>Longitudinal H_r (or TE) wave with a wave function $w = rH_r$</th><th>Longitudinal $\mathcal{E}_r$ (or TM) wave with a wave function $w = r\mathcal{E}_r$</th><th>TEM wave with wave functions $w = rH_\varphi$ and $w = r\mathcal{E}_\theta$</th></tr>
<tr><td colspan="2">According to Eq. (428) of p. 321 the general wave equation is
$$\frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial\left(\sin\theta\frac{\partial w}{\partial\theta}\right)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} + \overbrace{\mu w(\kappa\omega^2 - j\sigma\omega)}^{\mu\kappa_e w\omega^2} = 0$$
which can be solved by the product solution $w = \Phi_1(r)\cdot\Phi_2(\theta)\cdot\Phi_3(\varphi)$
where $\Gamma^2 = \mu w(\kappa\omega^2 - j\sigma\omega)$
The separated solutions are then for the separation constants p and q
$$\frac{1}{r^2}\frac{d\left(r^2\frac{d\Phi_1}{dr}\right)}{dr} + \left[\Gamma^2 - \frac{q(q+1)}{r^2}\right]\Phi_1 = 0$$
or
$$\frac{d^2(r\Phi_1)}{dr^2} + \left[\Gamma^2 - \frac{q(q+1)}{r^2}\right]r\Phi_1 = 0 \quad \text{(I)}$$
(Bessel equations)
$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{d\Phi_2}{d\theta}\right)}{d\theta} + \left[q(q+1) - \frac{p^2}{\sin^2\theta}\right]\Phi_2 = 0 \quad \text{(II)}$$
$$\frac{d^2\Phi_3}{d\varphi^2} + p^2\Phi_3 = 0 \quad \text{(III)}$$</td><td rowspan="2">Since $\mathcal{E}_r = H_r = 0$ and for $\mathcal{E}$ tangent to the meridian or θ circle, H must be tangent to the perpendicular φ circle. We have then only $\mathcal{E}_\theta$ and H_φ fields. Equations (373) and (374) then reduce to
$$\frac{\partial(H_\varphi\sin\theta)}{\partial\theta} = 0 \text{ (IV)}; \quad \frac{1}{r}\frac{(rH_\varphi)}{\partial r} + (\sigma + j\omega\kappa)\mathcal{E}_\theta = 0 \quad \text{(V)}$$
$$\frac{1}{r}\frac{\partial(r\mathcal{E}_\theta)}{\partial r} + j\omega\mu H_\varphi = 0 \quad \text{(VI)}$$
Equations (374) reduce to
$$\frac{\partial(\mathcal{E}_\theta\sin\theta)}{\partial\theta} = 0 \quad \text{(VII)}$$
Instead of V and VI, we have also the forms
$$\frac{\partial(rH_\varphi)}{\partial r} = -(\sigma + j\omega\kappa)r\mathcal{E}_\theta \quad \text{(V}a\text{)}$$
$$\frac{\partial(r\mathcal{E}_\theta)}{\partial r} = -j\omega\mu rH_\varphi \quad \text{(VI}a\text{)}$$
From (Va) upon differentiation and (VIa), we find
$$\frac{\partial^2(rH_\varphi)}{\partial r^2} = -(\sigma + j\omega\kappa)\frac{\partial(r\mathcal{E}_\theta)}{\partial r} = +j\omega\mu(rH_\varphi)(\sigma + j\omega\kappa)$$</td></tr>
<tr><td>From (373) and (374), we have
$$H_r = \frac{w}{r}$$</td><td>For (373) and (374), we have the relations
$$\mathcal{E}_r = \frac{w}{r}$$</td></tr>
</table>

TABLE 9.—(*Continued*)

Longitudinal H_r (or TE) wave with a wave function $w = rH_r$	Longitudinal $\mathcal{E}_r$ (or TM) wave with a wave function $w = r\mathcal{E}_r$	TEM wave with wave functions $w = rH_\varphi$ and $w = r\mathcal{E}_\theta$
$H_\theta = \frac{1}{q(q+1)} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial}{\partial r}(rw) = \frac{1}{j\omega\mu r} \frac{\partial(r\mathcal{E}_\varphi)}{\partial r}$ $H_\varphi = \frac{1}{q(q+1)} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial r}(rw) = -\frac{1}{j\omega\mu r} \frac{\partial(r\mathcal{E}_\theta)}{\partial r}$ $\mathcal{E}_r = 0;\ \mathcal{E}_\theta = -\frac{j\mu\omega}{q(q+1)} \frac{1}{r \sin\theta} \frac{\partial(rw)}{\partial \varphi}$ $\mathcal{E}_\varphi = \frac{j\mu\omega}{q(q+1)} \frac{1}{r} \frac{\partial(rw)}{\partial \theta}$	$\mathcal{E}_\theta = \frac{1}{q(q+1)} \frac{1}{r} \frac{\partial}{\partial \theta} \frac{\partial}{\partial r}(rw) = -\frac{1}{j\omega\kappa_e r} \frac{\partial(rH_\varphi)}{\partial r}$ $\mathcal{E}_\varphi = \frac{1}{q(q+1)} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial r}(rw) = -\frac{1}{j\omega\kappa_e r} \frac{\partial(rH_\theta)}{\partial r}$ $H_r = 0;\ H_\theta = \frac{j\omega\kappa_e}{q(q+1)} \frac{1}{r \sin\theta} \frac{\partial(rw)}{\partial \varphi}$ $H_\varphi = -\frac{j\omega\kappa_e}{q(q+1)} \frac{1}{r} \frac{\partial(rw)}{\partial \theta}$	yielding $\frac{\partial^2(rH_\varphi)}{\partial r^2} - (j\omega\mu\sigma - \omega^2\mu\kappa)(rH_\varphi) = 0 \quad \text{(VIII)}$ Differentiating (VI*a*) with respect to r and substituting the value of (V*a*) yields the symmetrical form $\frac{\partial^2(r\mathcal{E}_\theta)}{\partial r^2} - (j\omega\mu\sigma - \omega^2\mu\kappa)(r\mathcal{E}_\theta) = 0 \quad \text{(IX)}$ In these expressions, for the wave functions $w = rH_\varphi$ and $w = r\mathcal{E}_\theta$, respectively, we have $\frac{\partial^2 w}{\partial r^2} - w\Gamma^2 = 0 \quad \text{for} \quad \Gamma = \pm\sqrt{j\omega\mu\sigma - \omega^2\mu\kappa}$ These respective wave functions depend, therefore, on $\epsilon^{\pm jr\Gamma}$, and the amplitude according to (VII) decreases with $r \sin\theta$. From (V*a*) and (VI*a*), we note that the impedance $Z_0 = \frac{\mathcal{E}_\theta}{H_\varphi} = \sqrt{\frac{j\mu\omega}{\sigma + j\omega\kappa}}$ which for a nondissipative medium ($\sigma = 0$) degenerates to $\sqrt{\mu/\kappa}$ ohms and is pure resistive
Eq. (I) can be brought into the form $\frac{d^2(\sqrt{r}\,\Phi_1)}{dr^2} + \frac{1}{r} \frac{d(\sqrt{r}\,\Phi_1)}{dr} = \left[\frac{\Gamma^2}{r^2} - \left(q + \frac{1}{2}\right)^2\right] \sqrt{r}\,\Phi_1$ and is satisfied by $J_{q+\frac{1}{2}}(r\Gamma)$ and $J_{-(q+\frac{1}{2})}(r\Gamma)$ For $q + \frac{1}{2}$ an integer, we have to use *in addition*, the Neumann functions This leads to Hankel functions as is discussed on p. 322, and we arrive for $x = r\Gamma$ at expressions $\sqrt{\frac{0.5\pi}{x}}\, J_{q+\frac{1}{2}}(x) \quad \text{and} \quad \sqrt{\frac{0.5\pi}{x}}\, N_{q+\frac{1}{2}}(x)$ where the latter form stands for the Neumann and the former for the Bessel function solution. $J_0(x) = (\sin x)/x$; $N_0(x) = -(\cos x)/x$; $J_1(x) = [(\sin x)/x^2] - [(\cos x)/x]$; $N_1(x) = -[(\sin x)/x] - [(\cos x)/x^2]$; etc.		

to the phase constant $\beta = 2\pi/\lambda$, if λ meters is the wavelength in medium μ, κ, $\sigma = 0$. Table 9 gives the derivation for TE, TM, and TEM waves for a propagation constant Γ for which $\Gamma^2 = -\mu\kappa_e\omega^2$, where $\kappa_e = \kappa - j\sigma/\omega$ denotes the equivalent dielectric constant in farads per meter for a *dissipative* medium (σ also effective) and $\omega/6.28$ is in cycles per second.

54. Remarks on Field Equations for Spaces with Exciters. Generally, the stimulus of an electromagnetic field may be taken as a distribution of electric charge and current that for isotropic mediums is uniform and expressed by the volume charge density ρ coulombs per cubic meter and current density J amperes per square meter. We then have the well-known curl equations

$$\text{curl } \mathcal{E} \equiv \nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t} \quad \text{volts/m}^2;$$

$$\text{curl } H \equiv \nabla \times H = \kappa \frac{\partial \mathcal{E}}{\partial t} + \underbrace{\sigma \mathcal{E}}_{J} \quad \text{amp/m}^2 \qquad (378)$$

for source-*free* regions. For harmonic excitations, we have curl $\mathcal{E} = -j\omega\mu H$ and curl $H = (\sigma + j\omega\kappa)\mathcal{E}$. For a space, such as in a cavity resonator, with impressed currents J_ι amperes per square meter and impressed charges of ρ_ι coulombs per cubic meter, we have

$$\text{curl } \mathcal{E} = -\mu \frac{\partial H}{\partial t}; \qquad \text{curl } H = \kappa \frac{\partial \mathcal{E}}{\partial t} + \sigma\mathcal{E} + J_\iota \qquad (379)$$

for a coupling of the exciter to the resonating space loose enough so that no appreciable back action exists on the exciter (excludes an arbitrary excitation). Differentiating the second relation with respect to t, we find, by means of the first relation,

$$\nabla^2\mathcal{E} - \mu\kappa \frac{\partial^2 \mathcal{E}}{\partial t^2} - \mu\sigma \frac{\partial \mathcal{E}}{\partial t} = \mu \frac{\partial J_\iota}{\partial t} + \frac{1}{\kappa} \text{grad } \rho_\iota$$

which, for an $\epsilon^{j\omega t}$ excitation, yields

$$\nabla^2\mathcal{E} + (\mu\kappa\omega^2 - j\mu\sigma\omega)\mathcal{E} = \overbrace{j\omega\mu J_\iota + \frac{1}{\kappa} \text{grad } \rho_\iota}^{\text{excitation of the space resonator}} \qquad (380)$$

But $\mu\kappa\omega^2 - j\mu\sigma\omega$ is equal to the square of the propagation constant Γ, and

$$\nabla^2\mathcal{E} + \mathcal{E}\Gamma^2 = \overbrace{j\omega\mu J_\iota + \frac{1}{\kappa} \text{grad } \rho_\iota}^{\text{equal to a vector field}} \qquad (381)$$

For natural oscillations (cavity resonance), we have the propagation constant $\Gamma = \Gamma_r$ and $\nabla^2\mathcal{E} + \mathcal{E}\Gamma_r^2 = 0$, where Γ_r has several values, since we can excite a space resonator inside a metallic cavity at different natural modes.

55. Similitude Relations of Mediums of Different Size. This discussion refers to problems in connection with smaller antenna models and models of mediums of different size in which electromagnetic phenomena are investigated, instead of the actual antenna of different dimensions.

Inasmuch as such relations must satisfy Maxwell's equations

$$\text{curl } \mathcal{E} \equiv \nabla \times \mathcal{E} = -\frac{\partial B}{\partial t}; \qquad \text{curl } H \equiv \nabla \times H = J + \frac{\partial D}{\partial t} \quad (382)$$

where $B = \mu H$, $D = \kappa\mathcal{E}$, and $J = \sigma\mathcal{E}$, we must also consider the electromagnetic medium parameters σ mhos per meter, μ henrys per meter, and κ farads per meter for the mediums of *actual* and *reduced* size if the model is smaller. This is a requirement, since the current density J in amperes per square meter is equal the product of the specific conductivity σ and the electric field intensity $\mathcal{E}$ in volts per meter, the displacement density D in coulombs per square meter is equal to the product $\kappa\mathcal{E}$, and the magnetic flux density B in webers per square meter is equal to μH, where H is in amperes per meter.

Hence, if a large antenna array with parasitic radiators due to structures in close proximity is to be replaced by a model N times smaller than the actual radiators, we must also generally bring about changes in the parameters σ, μ, and κ of all mediums concerned.

For an actual system A and a model system M, we have, for a linear decrease N, the relation $M = A/N$, since the model M is N times smaller than the actual system A. Hence, for any point $P(x, y, z)$ in the actual system A, there must be a corresponding point $P(x_m, y_m, z_m)$ in the model system M and such that the Cartesian coordinates $x = x_mN$, $y = y_mN$, and $z = z_mN$. Moreover, in the actual system A, we generally have three space components as well as a time variation, which would be $\epsilon^{j\omega t}$ for a harmonic wave excitation. Hence, both $\mathcal{E}$ and H depend on four variables of which x, y, z are the space coordinates and t is the time function. Therefore, we have

$$\left.\begin{array}{lll} \mathcal{E}(x, y, z, t) & \text{and} \quad H(x, y, z, t) & \left\{ \begin{array}{l}\text{for the actual}\\ \text{system } A\end{array}\right. \\ \mathcal{E}_m(x_m, y_m, z_m, t_m) & \text{and} \quad H_m(x_m, y_m, z_m, t_m) & \left\{ \begin{array}{l}\text{for the model}\\ \text{system } M\end{array}\right. \end{array}\right\} \quad (382a)$$

since t_m in the model system corresponds to the time t of the actual system. Maxwell's equations (381) for the actual system A then read

$$\left.\begin{aligned}\operatorname{curl} \mathcal{E} &= -\mu(x, y, z)\,\frac{\partial H(x, y, z, t)}{\partial t} \\ \operatorname{curl} H &= \kappa(x, y, z)\,\frac{\partial \mathcal{E}(x, y, z, t)}{\partial t} + \sigma(x, y, z)\mathcal{E}(x, y, z, t)\end{aligned}\right\} \qquad (383)$$

Moreover, as far as respective field associations $\mathcal{E}$ and H of the actual and model electromagnetic fields are concerned, we can use the proportionality relations

$$\mathcal{E}_m(x_m, y_m, z_m, t_m) = k_1\mathcal{E}(x, y, z, t); \qquad H_m(x_m, y_m, z_m, t_m) = k_2H(x, y, z, t) \qquad (384)$$

where k_1 and k_2 are proportionality factors. This leads to

$$k_1 \operatorname{curl} \mathcal{E} = \frac{\operatorname{curl} \mathcal{E}_m}{N}; \qquad k_2 \operatorname{curl} H = \frac{\operatorname{curl} H_m}{N} \qquad (385)$$

The curl relations (383), for the actual system A, read for the model system M

$$\operatorname{curl} \mathcal{E}_m = -\frac{k_1}{k_2} N \frac{\partial t_m}{\partial t}\,\mu(x, y, z)\,\frac{\partial H_m}{\partial t_m}; \qquad \operatorname{curl} H_m = \frac{k_2}{k_1} N \frac{\partial t_m}{\partial t}\,\kappa(x, y, z)\,\frac{\partial \mathcal{E}_m}{\partial t_m} + \frac{k_2}{k_1} N\sigma(x, y, z)\mathcal{E}_m \qquad (386)$$

The conductivity of the model system M must be equal to the multiplier of $\mathcal{E}_m$, or

$$\sigma_m(x_m, y_m, z_m) = \frac{k_2}{k_1} N\sigma(x, y, z) \qquad (387)$$

which is the formula for computing the equivalent conductivity. From (386), the equivalences for the dielectric constant and the permeability of the model system M, in terms of the actual system A, become

$$\kappa_m(x_m, y_m, z_m) = \frac{k_2}{k_1} N \frac{\partial t_m}{\partial t}\,\kappa(x, y, z); \qquad \mu_m(x_m, y_m, z_m) = \frac{k_1}{k_2} N \frac{\partial t_m}{\partial t}\,\mu(x, y, z) \qquad (388)$$

The application of formulas (387) and (388) is not difficult, since in many cases simplifications can be made. One simplification is that $k_1 = k_2$ for an isotropic radiation field; hence, $k_1/k_2 = k_2/k_1 = 1$. This simplification eliminates the knowledge requirement of k_1 and k_2. If the medium for the actual case is air, then $\sigma(x, y, z) = 0$, $\mu(x, y, z) = 1.257 \times 10^{-6}$ henry/m, and $\kappa(x, y, z) = 8.854 \times 10^{-12}$ farad/m. For customary wave propagation, $\mathcal{E} = \mathcal{E}_0\epsilon^{j\omega t}$ and $H = H_0\epsilon^{j\omega t}$ are harmonic time variations; and in (383), we have

$$\frac{\partial[H(x, y, z, t)]}{\partial t} = j\omega H(x, y, z, t); \qquad \frac{\partial[\mathcal{E}(x, y, z, t)]}{\partial t} = j\omega\mathcal{E}(x, y, z, t)$$

$$\frac{\partial H_m}{\partial t_m} = j\omega_m H_m; \qquad \frac{\partial \mathcal{E}_m}{\partial t_m} = j\omega_m \mathcal{E}_m$$

We have also

$$\frac{\partial t_m}{\partial t} = \frac{t_m}{t} = \frac{\omega}{\omega_m} = \frac{f}{f_m} = \frac{T_m}{T} \tag{389}$$

where f and f_m are the frequencies of the *actual* and the *model* system, respectively. For isotropic mediums, σ, μ, and κ are constant everywhere in the medium, and (387) and (388) yield

$$\left.\begin{aligned} \sigma_m &= \frac{k_2}{k_1} N\sigma = N\sigma && \text{mhos/m} \\ \kappa_m &= \frac{k_2}{k_1} N \frac{\partial t_m}{\partial t} \kappa = N \frac{f}{f_m} \kappa && \text{farads/m} \\ \mu_m &= \frac{k_1}{k_2} N \frac{\partial t_m}{\partial t} \mu = N \frac{f}{f_m} \mu && \text{henrys/m} \end{aligned}\right\} \tag{390}$$

Since $f_m = Nf$, we see that $\mu_m = \mu$ and $\kappa_m = \kappa$, *but* $\sigma_m = N\sigma$. It is, therefore, necessary to use in the model experiments only a frequency f_m, which is as many times higher than the frequency f of the actual system as the model is smaller than the actual system. The permeability μ_m and the dielectric constant κ_m of the model system are the same as for the actual system, but the conductivity σ_m of the model system is to be chosen N times larger than for the actual system. Suppose we desire to model a ship with all of its structures and radiators and the actual ship is meant for ocean service. Choosing a model that is one-fiftieth as large as the actual ship, $N = 50$, requires then model frequencies that are fifty times the actual ship frequencies. Since for ocean $\sigma = 5$ mhos/m, $\mu = 1.257 \times 10^{-6}$ henry/m, and $\kappa = 7.0832 \times 10^{-10}$ farad/m, the model would have to rest on a medium of $\sigma_m = 5 \times 50 = 250$ mhos/m, $\mu_m = \mu$, and $\kappa_m = \kappa$. If the actual ship's operating frequency is $f = 100$ Mc/sec, the model experiments would have to be carried out with a frequency of $f_m = 50 \times 100 = 5000$ Mc/sec. To find a medium with values of $\kappa = 7.0832 \times 10^{-10}$ that has, in addition, a conductivity as *high as* 250 *mhos* across square-meter surfaces 1 m apart is not an easy practical task. To meet the permeability value of vacuum is not difficult, since we have only to use a medium that is not ferromagnetic.

56. Application of the Poisson and Laplace Equations and the Flow Function of an Electromagnetic Field. For a homogeneous medium, we have, for a potential V volts, a volume charge density ρ coulombs per cubic meter, a magnetic vector potential Λ volt · seconds per meter, and a

current density J amperes per meter, at a point in the field region the Poisson equation

$$\nabla^2 V = -\frac{\rho}{\kappa} \quad \text{and} \quad \nabla^2 \Lambda = -\mu J$$

In Cartesian x, y, z coordinates, we have then

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\kappa}; \qquad \frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial^2 \Lambda}{\partial y^2} + \frac{\partial^2 \Lambda}{\partial z^2} = -\mu J$$

Poisson equations (391)

If the electromagnetic field is free of sources and sinks (absorbers), we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0; \qquad \frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial^2 \Lambda}{dy^2} + \frac{\partial^2 \Lambda}{\partial z^2} = 0 \qquad \text{for Cartesian } x, y, z \text{ coordinates}$$

$$\frac{1}{r}\frac{\partial\left(r\frac{\partial V}{\partial r}\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{and so for } \Lambda \text{ in place of } V, \text{ for cylindric } r, \varphi, z \text{ coordinates}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial\left(r^2\frac{\partial V}{\partial r}\right)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial\left(\sin\theta\frac{\partial V}{\partial \theta}\right)}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0$$

and so for Λ in place of V; in spherical r, φ, θ coordinates (391a)

Therefore

$$\nabla^2 V = 0; \qquad \nabla^2 \Lambda = 0 \qquad \text{Laplace equations} \tag{392}$$

For every volume element $d\tau$ of a total volume of τ cubic meters, we deal with point sources with a certain current value of I amperes at the symmetry center of $d\tau$ From the symmetry center of each volume element, the current flows outward uniformly in all directions, causing, over a sphere about such symmetry centers, a current density $J_r = I/(4\pi r^2)$ amperes per square meter, where r in meters denotes the radius of such spheres and J_r is normal to surface elements of such spheres. This can also be written as $J_r = \frac{\partial(-I/4\pi r)}{\partial r}$. In exactly the same way, we can express the volume density ρ, which is in coulombs per cubic meter, and refer it to volume elements $d\tau$. If r meters then denotes the distance of volume elements to a certain space point $P(x, y, z)$, where an electromagnetic field exists, we have, for the retarded values, $[J]$ and $[\rho]$ acting at P in place of the actual values J and ρ, which initiate at a radiator. The corresponding integral relations are then

$$V_{(x,y,z)} = V^{(\text{volts})} = \frac{1}{4\pi\kappa}\int\int\int_\tau \frac{[\rho]}{r}\,d\tau;$$

$$\Lambda_{(x,y,z)} = \Lambda^{(\text{volt sec/m})} = \frac{\mu}{4\pi}\int\int\int_\tau \frac{[J]}{r}\,d\tau \tag{392a}$$

which confirm relations (239) on page 222.

From this discussion, we note that for an electromagnetic field free of sources and sinks (absorbers) and independent of the z coordinate, the

Laplace equation applies, and we find

$$\left.\begin{aligned} &\text{Laplace equation } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 && \text{in } x, y, z \text{ coordinates} \\ &\text{Laplace equation } r\frac{\partial\left(r\frac{\partial V}{\partial r}\right)}{\partial r} + \frac{\partial^2 V}{\partial \varphi^2} = 0 && \text{in cylindrical } r, \varphi, z \text{ coordinates} \end{aligned}\right\} \quad (393)$$

But the electric field intensity is equal to the potential gradient, *i.e.*, $\mathcal{E} = -\operatorname{grad} V$, and we obtain

$$\left.\begin{aligned} &\text{Gradients} = -\frac{\partial V}{\partial x} = \mathcal{E}_x; \quad -\frac{\partial V}{\partial y} = \mathcal{E}_y && \text{for Cartesian coordinates} \\ &\text{Gradients} = -\frac{\partial V}{\partial r} = \mathcal{E}_r; \quad -\frac{1}{r}\frac{\partial V}{\partial \varphi} = \mathcal{E}_\varphi && \text{for cylindrical coordinates} \end{aligned}\right\} \text{volts/m} \quad (394)$$

It is convenient to make use of the magnetic vector potential Λ since the magnetic field components are caused by a current flow. The two-dimensional Laplace equation is then of great value when we deal with guided wave propagation such as lines and wave guides. Assuming the flow along the Z axis, the components $\Lambda_x = 0$ and $\Lambda_y = 0$, and only Λ_z remains. This is then the condition when a wave guide is along the Z axis. For the two-dimensional field of (393), we find for the flow function ψ in amperes

$$\text{Magnetic field components for the energy flow along the } Z \text{ axis} \left\{\begin{aligned} &H_x = \frac{\partial \psi}{\partial y}; \quad H_y = -\frac{\partial \psi}{\partial x} && \text{in Cartesian coordinates} \\ &H_r = \frac{\partial \psi}{r\, d\varphi}; \quad H_\varphi = -\frac{\partial \psi}{\partial r} && \text{in cylindrical coordinates} \end{aligned}\right\} \text{amp/m} \quad (395)$$

According to the Poisson equation, we have for a conduction current of current density J amperes per square meter

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -J \quad (396)$$

For TM waves, whether in a wave guide or otherwise, along the Z axis, $(\partial H_x/\partial x) + (\partial H_y/\partial y)$ must vanish, since $H_z = 0$, and the transverse

electric field intensity must be equal to $-$ grad V. The values of H_x and H_y are defined by the flow function relations (395), since μH = curl Λ. But $\Lambda_x = \Lambda_y = 0$ and $\Lambda_z = \mu\psi$ volt · seconds per meter, where μ is in henrys per meter and henrys in volt · seconds per ampere, and the flow function ψ is in amperes. For the space propagation constant $\Gamma = j\omega\sqrt{\mu\kappa_e}$ in generalized radians per meter and $\kappa_e = \kappa - j\sigma/\omega$ farads per meter, the flow function ψ satisfies the wave equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \psi\Gamma^2 \tag{397}$$

We also have

$$\mathcal{E}^{(\text{volts/m})} = \frac{\text{curl } H}{\sigma + j\omega\kappa}; \qquad V^{(\text{volts})} = -\frac{\partial\psi/\partial z}{\sigma + j\omega\kappa} \tag{398}$$

We also have for the longitudinal electric vector $\mathcal{E}_z$ and the transverse vector $\mathcal{E}_t$

$$\mathcal{E}_z = -\frac{(\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2)}{\sigma + j\omega\kappa}; \qquad \mathcal{E}_t = \frac{\text{grad } \partial\psi/\partial z}{\sigma + j\omega\kappa} \quad \text{volts/m} \tag{399}$$

since the longitudinal component along the Z direction is

$$\mathcal{E}_z = \frac{\dfrac{\partial^2\psi}{\partial z^2} - \psi\Gamma^2}{\sigma + j\omega\kappa} \quad \text{volts/m} \tag{400}$$

The flow function ψ can then be used to express any field *as long as* the component $H_z = 0$, *i.e.*, as long as one of the three Cartesian magnetic field components vanishes. If the field in parallel planes with the XY plane is constant, the flow function ψ has the form

$$\psi = F_1(x, y) \cdot F_2(z)$$

for plane waves, which, together with (397), yields

$$\underbrace{\frac{1}{F_1F_2}\left[\frac{\partial^2(F_1F_2)}{\partial x^2} + \frac{\partial^2(F_1F_2)}{\partial y^2} + \frac{\partial^2(F_1F_2)}{\partial z^2}\right]}_{=\frac{1}{F_1}\frac{\partial^2 F_1}{\partial x^2} + \frac{1}{F_1}\frac{\partial^2 F_1}{\partial y^2} + \frac{1}{F_2}\frac{\partial^2 F_2}{\partial z^2}} = \underbrace{F_1F_2\Gamma^2\frac{1}{F_1F_2}}_{=\Gamma^2}$$

We have then

$$\underbrace{\frac{1}{F_1(x, y)}\left\{\frac{\partial^2[F_1(x, y)]}{\partial x^2} + \frac{\partial^2[F_1(x, y)]}{\partial y^2}\right\}}_{\substack{\text{not dependent upon } z\\ \text{(must be a constant)}}} + \underbrace{\frac{1}{F_2(z)}\frac{\partial^2[F_2(z)]}{\partial z^2}}_{\substack{\text{not dependent upon}\\ x \text{ and } y \text{ (must be a}\\ \text{constant)}}} = \underbrace{\Gamma^2}_{\text{a constant}} \tag{401}$$

Such equations are then solved by the method of the *separation of variables.* We have for the case of

$$\frac{\partial^2\psi_z}{\partial x^2} + \frac{\partial^2\psi_z}{\partial y^2} = -p\psi_z$$

a procedure with X and Y functions, with a solution $\psi_z = XY$. Hence,

$$\frac{\partial(XY)}{\partial x} = Y\frac{\partial X}{\partial x}; \qquad \frac{\partial^2(XY)}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}; \qquad \frac{\partial(XY)}{\partial y} = X\frac{\partial Y}{\partial y}; \qquad \frac{\partial^2(XY)}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

and

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = -pXY$$

Dividing by XY yields

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -p = \text{constant}$$

Hence, *either part* of the sum on the left side *must* also *be constant,* or

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_1; \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_2 \qquad \text{or} \qquad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_1 = 0; \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_2 = 0$$

But differential equations of such forms read

$$\left.\begin{aligned}\frac{d^2X}{dx^2} + k_1X &= 0\\ \frac{d^2Y}{dy^2} + k_2Y &= 0\end{aligned}\right\} \text{with the well-known solutions} \left\{\begin{aligned}X &= A_1\sin(x\sqrt{k_1} + \varphi_1)\\ Y &= A_2\sin(y\sqrt{k_2} + \varphi_2)\end{aligned}\right.$$

For k_1 a negative value, such as $(d^2X/dx^2) - k_1X = 0$, we have $X = A_1\sinh(x\sqrt{k_1} + \varphi_1)$ since

$$X = A\epsilon^{-x\sqrt{k_1}} + B\epsilon^{+x\sqrt{k_1}}$$

The discussion just given shows that Eq. (401) can be solved in a similar way, since

$$\frac{1}{F_1}\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2}\right) = \text{constant}; \qquad \frac{1}{F_2}\frac{\partial^2 F_2}{\partial z^2} = \text{constant}$$

and we can write for one constant K^2 and find

$$\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} = -K^2F_1 \tag{402}$$

which, introduced in (401), yields

$$-K^2 + \frac{1}{F_2}\frac{\partial^2 F_2}{\partial z^2} = \Gamma^2 \qquad \text{or} \qquad \frac{\partial^2 F_2}{\partial z^2} = (K^2 + \Gamma^2)F_2 \tag{403}$$

For the abbreviation $\Gamma_z^2 = K^2 + \Gamma^2$, we find a differential equation of the form

$$\frac{\partial^2 F_2}{\partial z^2} - F_2\Gamma_z^2 = 0 \tag{403a}$$

which is well known and has the solution

$$F_2(z) = \underbrace{A\epsilon^{-z\Gamma_z}}_{\substack{\text{wave along} \\ \text{the } +Z \text{ axis}}} + \underbrace{B\epsilon^{+z\Gamma_z}}_{\substack{\text{wave along} \\ \text{the } -Z \text{ axis}}} \tag{404}$$

This leaves for a progressive wave only the first term $F_2(z) = A\epsilon^{-z\Gamma_z}$, with a propagation constant $\Gamma_z = \sqrt{K^2 + \Gamma^2}$ along the wave guide. We

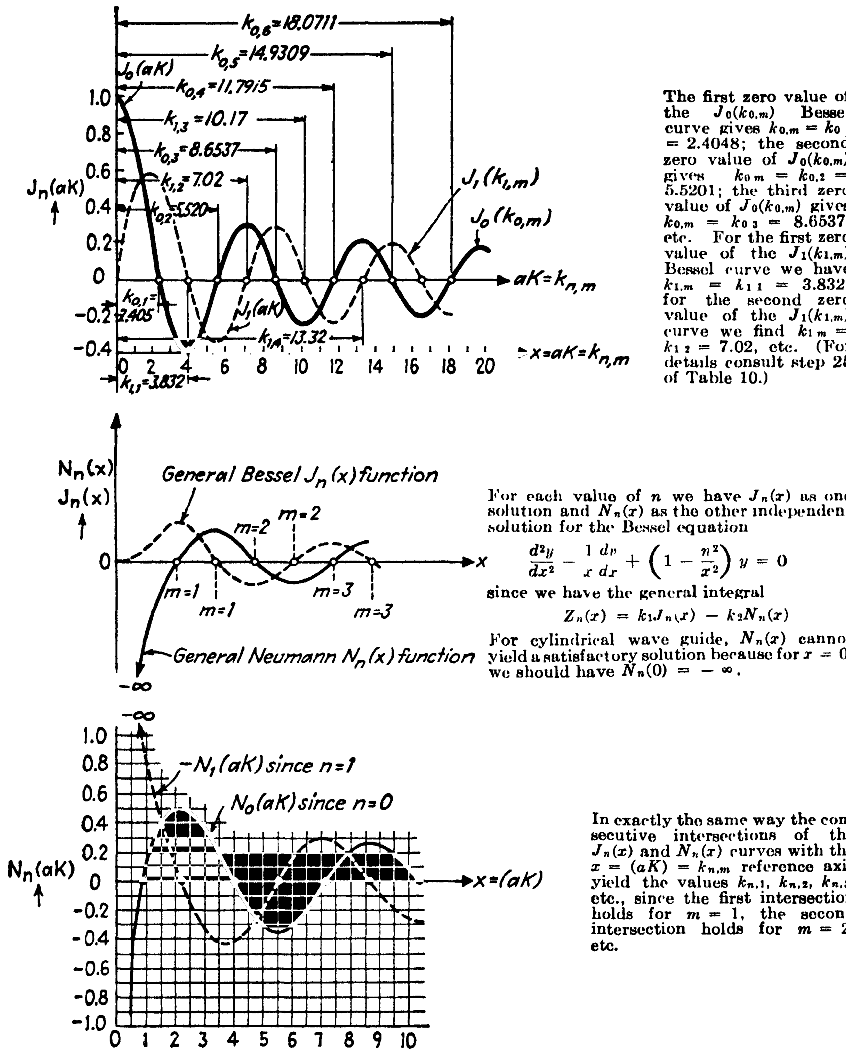


FIG. 48. Bessel and Neumann curves by means of which we can find graphically the various $k_{n,m}$ values.

have then also a characteristic impedance $\Gamma_z/(\sigma + j\omega\kappa)$ ohms. The flow function ψ in amperes is, therefore, a means of expressing the electromagnetic field by a *single scalar* wave function, since ψ represents such a function. Table 10 gives the case of a TE wave in comparison with a TM wave, both being propagated along the $+Z$ axis. It also contains applications to rectangular and circular wave guides with the necessary tables for computing the various types of possible wave characteristics. These tables show, for instance, that the $TE_{1,1}$ wave gives the smallest R' value, since for $n = 1$ and $m = 1$, we find $R' = 1.84 = k'_{n,m}$. The R table does *not* give such a *small* value. We find then a cutoff wavelength of $2\pi a/1.84$ meters, which is $3.4a$ meters. This shows that the largest cutoff wavelength is 1.7 times the diameter of the guide. But $2\pi a$ in the formula $\lambda_c = 2\pi a/1.84$ represents the inside circumference of the guide. Therefore, the longest cutoff wavelength is about half the inside circumference. In a rectangular guide, the longest wavelength for TE waves is exactly twice the longest side of the rectangular cross section. For a along the X direction as the largest dimension and a corresponding wave excitation, we have then $\lambda_c = 2a$ meters. It is the $TM_{1,1}$ wave, according to Step 19, that yields the longest natural wave. These wavelengths (λ_c) correspond to the unbounded-space wavelength, based on the relation $\lambda f = 1/\sqrt{\mu\kappa}$ meters per second, with λ in meters and f in cycles per second. It should not be confused with the wave-guide wavelength λ_g, which is always greater than the unbounded-space wavelength, using the same dielectric medium μ,κ. For rectangular guides, it is

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (m\lambda/2a)^2 - (n\lambda/2b)^2}} \quad \text{meters} \tag{405}$$

and for guides with a circular cross section

$$\left.\begin{aligned} \lambda_g &= \frac{6.28}{\sqrt{(6.28/\lambda)^2 - K^2}} \quad \text{meters} \\ K &= \frac{k_{n,m}}{a} \quad \text{for TM waves} \\ K &= \frac{k'_{n,m}}{a} \quad \text{for TE waves} \end{aligned}\right\} \tag{406}$$

Moreover, according to Step 24 of Table 10, we find for a cylindrical guide and the $TE_{1,1}$ wave, a frequency of $47.8 \times 1.84/a = 88/a$ megacycles per second. It is the $TE_{1,0}$ wave in a rectangular wave guide that is the dominant wave with the electric lines of force along the shorter side b.

TABLE 10

Steps	TE waves	TM waves	Explanations
1	$\mathcal{E}_z = 0$ but also longitudinal H_z component	$H_z = 0$ but also longitudinal $\mathcal{E}_z$ component	Longitudinal $\mathcal{E}_z$ and H_z components along the Z axis
2	No longitudinal electric current in amperes	No longitudinal magnetic current in volts	$\mathcal{E}_t$ and H_t are transverse field components, V is the electric scalar potential in volts, and U the magnetic scalar potential in amperes
3	Hence, $H_t = -\operatorname{grad} U$ amp/m	$\mathcal{E}_t = -\operatorname{grad} V$ volt/m	
4	Divergence yields $\dfrac{\partial \mathcal{E}_x}{\partial x} + \dfrac{\partial \mathcal{E}_y}{\partial y} = 0$	$\dfrac{\partial H_x}{\partial x} + \dfrac{\partial H_y}{\partial y} = 0$	Field medium is μ henrys/m; κ farads/m; σ mhos/m. Longitudinal $\mathcal{E}_z$ field is proportional to the amplitude of the distribution function $F_1(x, y)$ and three-dimensional propagation constant $\Gamma = \sqrt{j\omega\mu(\sigma + j\omega\kappa)} = j\omega\sqrt{\mu\kappa_e};\ \kappa_e = \kappa - j\dfrac{\sigma}{\omega}$ A_m is magnetic vector potential in volt · sec/m A_e is electric vector potential in amp · sec/m
5	$\mathcal{E}$ found from flow function Φ in volts	H found from flow function ψ in amps	
6	$\mathcal{E}_x = -\dfrac{\partial \Phi}{\partial y};\ \mathcal{E}_y = \dfrac{\partial \Phi}{\partial x}$	$H_x = \dfrac{\partial \psi}{\partial y};\ H_y = -\dfrac{\partial \psi}{\partial x}$	
7	$\kappa\mathcal{E} = -\operatorname{curl} A_e$ $A_{ex} = 0 = A_{ey};\ A_e = \kappa\Phi$	$\mu H = \operatorname{curl} A_m = \operatorname{curl} A$ $A_x = 0 = A_y;\ A_z = \mu\psi$	
8	$H_z = -\dfrac{\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2}}{j\omega\mu}$ amp/m	$\mathcal{E}_z = -\dfrac{\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}}{\sigma + j\omega\kappa}$ volt/m	
9	$U = -\dfrac{\dfrac{\partial \Phi}{\partial z}}{j\omega\mu}$ amp	$V = -\dfrac{\dfrac{\partial \psi}{\partial z}}{\sigma + j\omega\kappa}$ volt	
10	Flow function $\Phi^{(\text{volts})} = F_1(x, y) \cdot F_2(z)$ for any field with $\mathcal{E}_z = 0$	Flow function $\psi^{(\text{amp})} = F_1(x, y) \cdot F_2(z)$ for any field with $H_z = 0$	For waves with fixed field patterns in planes parallel with the XY plane
11	The F_1 and F_2 functions satisfy the same conditions for TE and TM waves.		$F_1(x, y)$ is the *amplitude* distribution which vanishes, for instance, at the boundary of the wave guide (at its conducting walls)
12	$\dfrac{\partial^2 \Phi}{\partial z^2} - \Phi\Gamma^2 = j\omega\mu H_z$	$\dfrac{\partial^2 \psi}{\partial z^2} - \psi \cdot \Gamma^2 = (\sigma + j\omega\kappa)\mathcal{E}_z$	Equations of the electromagnetic field expressed by respective flow functions (scalar wave functions) for a region without sources and sinks show that we can have transverse electromagnetic, magnetic, and electric waves, since TEM waves for $\mathcal{E}_z = 0 = H_z$ TM waves for $H_z = 0$ TE waves for $\mathcal{E}_z = 0$ depending upon the boundary and type of excitation
13	If we compute from the respective flow functions Φ volts and ψ amp, the sum of the fields and subtract from it the original field, we find the transverse electromagnetic field. If we compute the field given by ψ amp and subtract from it the total field, we find the TM field, just as the field computed from the flow function of Φ volts subtracted from the total field, yields the TE field. For useful guides $\sigma = 0$, $\kappa_e = \kappa$, and $\Gamma = j\omega\sqrt{\mu\kappa}$		

TABLE 10.—(*Continued*)

Steps	TE waves	TM waves	Explanations
14	The tangential $\mathcal{E}$ components along the wave-guide walls *must vanish*. Hence, also the normal H component must vanish at the wall. But for TE waves $H_n = -\partial U/\partial n$ according to step 3. This is the case because U amp is proportional to $F_1(x, y)$ or to $F_1(r, \varphi)$. Hence, $\frac{\partial F_1}{\partial n} = 0$ boundary condition where n expresses along the normal of the and at the boundary F_1 is abbreviation for $F_1(x, y)$ or $F_1(r, \varphi)$	$\mathcal{E}_z$ is proportional to the amplitude distribution function $F_1(x, y)$. At the walls (boundary) of the guide, $\mathcal{E}_z = 0$, *i.e.*, $\mathcal{E}_z$ must vanish, hence $F_1(x, y) = 0$ boundary condition Hence, also $V = 0$ at the boundary. Since $\mathcal{E}_z$ vanishes at the boundary, the normal component of H must also vanish at the boundary	Application to wave guides with perfectly conducting walls for transverse plane waves propagated along the $+Z$ axis (guide axis). The planes parallel to the XY plane are the planes of the cross section of the guide For cylindric guides, the r, φ coordinates define such cross sections. Note that the wavelength in a wave guide is always longer than the wavelength in an unguided (unbounded) medium for one and the same frequency
15	Axial (along Z) *electric* field is zero	Axial *magnetic* field is zero	Dominant wave has the lowest cutoff frequency (is the only wave that will carry energy when f is between the lowest cutoff frequency f_c and the next higher cutoff frequency). If the lines of force that bound the cross section of a guide are such lines for which one coordinate remains constant in a curvilinear coordinate system, for which we can separate the variables of the wave equation, we have TE and TM waves followed with two subscripts giving the solutions in each of the two coordinates
16	A circular electric wave has circular electric lines of force. $TE_{n,m}$ waves in a circular guide are such that m denotes the number of cylinders (including the boundary of the guide) at which the $\mathcal{E}$ field vanishes. A $TE_{0,1}$ type is a circular wave with the lowest cutoff frequency. $TE_{n,m}$ waves for $n \neq 0$ are *noncircular* waves; n denotes the number of axial planes, and m the number of cylinders (including the boundary) of the guide to which the $\mathcal{E}$ vector is normal The dominant $TE_{1,1}$ wave has the electric $\mathcal{E}$ lines approximately parallel with a diameter	A circular magnetic wave has circular magnetic lines of force. $TM_{m,n}$ waves in a circular guide are waves for which m is the number of cylinders (including the boundary to which the $\mathcal{E}$ vector is normal and the $TM_{0,1}$ wave is the circular transverse magnetic wave with the lowest cutoff frequency). $TM_{n,m}$ waves for $n \neq 0$ are *noncircular* waves with n giving the number of axial planes to which the $\mathcal{E}$ vector is normal and m as before.	
17	In $TE_{n,m}$ waves of a circular guide, the *axial magnetic* field is proportional to $J_n(k'_m r/a)$ cos $(n\varphi)$, where n is the order of the Bessel function, k'_m is the mth root of $\frac{d[J(x)]}{dx} = 0$, a is the radius of the cylinder, r the distance from the axis, and φ is the angular cylinder coordinate	In $TM_{n,m}$ waves of circular guides the *axial electric* field is proportional to $J(k_m r/a)$ cos $(n\varphi)$, where n is the order of the Bessel function and k_n is the mth root of $J_n(x) = 0$ and a, r and φ as to the left	These conventions of classifying wave-guide waves, are according to the standards of the IRE, 1945

TABLE 10.—(*Continued*)

Steps	TE waves	TM waves	Explanations
18	For *rectangular* guides, we have $TE_{m,n}$ waves, where m is the number of half-period variations of the field along the x coordinate and n is the number of such variations along the y coordinate (which is assumed to coincide with the *smaller* dimension of the wave-guide cross section)	For *rectangular guides* we have $TM_{m,n}$ waves, where m and n are as to the left	a is width of guide along the x coordinate and b the height of the guide along the y coordinate. The y dimension gives the smallest dimension of the wave-guide cross section
19	At the walls of the guide, we have the boundary requirement $\frac{\partial F_1}{\partial x} = 0$, at $x = 0$ and $x = a$ $\frac{\partial F_1}{\partial y} = 0$, at $y = 0$ and $y = b$ satisfied by amplitude distribution $F_1(x, y) = \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$ For any $TE_{m,n}$ wave, we have the possibilities of $m = 0, 1, 2, 3, \cdots$, and $n = 0, 1, 2, 3, \cdots$, with $TE_{0,0}$ *not* possible, but $TE_{0,1}$ and $TE_{1,0}$ waves $\lambda_c = \frac{2\pi}{K} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$ m cutoff wavelength. For $m = 0$ and $n = n$, we would have a cutoff wavelength of $\lambda_c = 2b/n$ m TE or longitudinal H waves ($\mathcal{E}_z = 0$) are of the form $H_z = H_{max} \cos \frac{2\pi x}{a} \times \cos \frac{n\pi y}{b} \epsilon^{-z\Gamma_z}$ amp/m	Boundary requirement: $F_1(x, y) = 0 = F_1(0, y) = F_1(a, y) = F_1(x, 0) = F_1(x, b)$ satisfied by the amplitude distribution $F_1(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b}$ For $m = 1, 2, 3, \cdots$, and $n = 1, 2, 3, \cdots$ cutoff wavelength $\lambda_c = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}$, since $\lambda_c = \frac{2\pi}{K}$ and longest wavelength in meters occurs for $m = 1 = n$ TM or longitudinal $\mathcal{E}$ waves ($H_z = 0$) are then of form $\mathcal{E}_z = \mathcal{E}_{max} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \epsilon^{-z\Gamma_z}$ volt/m	Application to possible modes for *rectangular guides*, bounded by conducting planes $x = 0$, $x = a$, $y = 0$, $y = b$. **Propagation constant** along the guide is $\Gamma_z = \sqrt{K^2 + \Gamma^2}$ $K = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ $TM_{1,1}$ wave has *longest* wavelength $\lambda_{1,1} = \frac{2ab}{\sqrt{a^2 + b^2}}$ meters (for details, see p. 1190) if a and b are in meters. Since for $TE_{m,n}$ waves, we have $m = 0$, for $n = 1, 2, 3, \cdots$, as well as $n = 0$ for $m = 1, 2, 3, \cdots$, we have more possible TE than TM waves The solutions are possible only for which the frequency in cycles/sec $f_{m,n} > \frac{\sqrt{(m/a)^2 + (n/b)^2}}{2\sqrt{\mu\kappa}}$ which for air as the inside medium of the guide, requires that the frequency must be larger than $1.5 \times 10^8 \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ cycles/sec. This is evident, since for $\sigma = 0$ mhos/m, we have $\Gamma = j\omega\sqrt{\mu\kappa}$, and for the pass band $\Gamma_z = \underbrace{\alpha_z}_{=0} + j\beta_z$

TABLE 10.—(*Continued*)

Steps	TE waves	TM waves	Explanations
			$= j\sqrt{\omega^2\mu\kappa - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$ and $\omega^2\mu\kappa$ must be larger than $\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ for finite values of Γ_z. We have also $\beta_z = \frac{2\pi}{\lambda_z} = \frac{\omega}{v_z}$ for λ_z as the wavelength and v_z as the velocity along the guide
20	Boundary requirement at $r = a$ m: $\frac{\partial[J_n(aK)]}{\partial(aK)} = 0 = J_n'(aK)$	Boundary requirement: $J_n(aK) = 0$	Application to possible modes of *cylindrical guides* (circular cross section) with perfectly conducting walls: Guide along Z direction of radius of a m and coordinates r, φ, and z. Here r denotes any value between $r = 0$ and $r = a$ m Equations (401) and (403) change then from $F_1(x, y)$ to $F_1(r, \varphi)$. The solutions then are $F_1(r, \varphi) = J_n(rK) \cos(n\varphi)$ $F_1(r, \varphi) = J_n(rK) \sin(n\varphi)$ since the respective Neumann amplitude factors $N_n(rK)$ would yield an infinite value at the axis $(r = 0)$ of the guide. According to step 21, we have for $n = 0$, a Bessel function derivative of zeroth order, *i.e.*, $J'_0(k'_{0,m})$. We have, therefore, only to find the intersections of the J''_0 Bessel curve with the aK reference axis, *i.e.*, the aK values that make the J'_0 value zero Figure 48 shows how such values are found from Bessel curves
20*a*	Step 20 is based on the fact that, besides $\mathcal{E}_z = 0$, as holding for TE waves, also at the boundary $(r = a)$ all other tangential components must vanish. Hence, $\mathcal{E}_\varphi = 0$. But $\mathcal{E}_\varphi$ is proportional to $\frac{\{\partial[F_1(r, \varphi)]\}}{\partial r}$ Consequently $\partial F_1/\partial r = 0$	Step 20 is as explained to the left as far as all tangential $\mathcal{E}$ components are concerned. Consequently $F_1(r, \varphi) = 0$ is the boundary condition	
21	Since $J_n'(aK)$ denotes an infinite number of derivatives on Bessel curves, we have theoretically an infinite number of $k'_{n,m}$ values for $k'_{n,m} = aK$. In exactly the same way, as is explained for the TM waves to the right, we find the consecutive $k'_{n,m}$ values for the $n = 0$ case as $k'_{0,1} = 3.83$, $k'_{0,2} = 7.02$, $k'_{0,3} = 10.17$, etc. The longest cutoff wavelength (above which no energy transfer is possible) is then $\lambda_c = \frac{2\pi}{K} = \frac{2\pi a}{k'_{0,1}} = \frac{2\pi a}{3.83}$ for the $TE_{0,1}$ wave. Hence, the longest TM wave, *i.e.*, the $TM_{0,1}$ wave, is for a circular wave guide the longest wave that can ever transfer field energy along the guide as far as these two compari-	Here, the boundary condition for $r = a$ m requires that $J_n(aK) = 0$ for $k_{n,m} = aK$. Hence, K can have values only for which the Bessel curve $J_n(aK)$ cuts the aK axis, since such a point on the curve means the Bessel amplitude $J_n(aK) = 0$. For $n = 0$, the J_0 curve will cut the aK reference axis several times, since it resembles a damped wave train about the aK zero axis with aK as the argument and $J_0(aK)$ as the corresponding amplitude. It starts out with the amplitude $J_0(aK) = J_0(0) = 1$ and has successive intersections with the aK axis for $aK = 2.405$, 5.52, 8.65, 11.79, etc. We have, therefore, $k_{0,1} = 2.4$, $k_{0,2} = 5.52$, $k_{0,3} = 8.65$, $k_{0,4} = 11.79$, etc. Therefore, the longest cutoff wavelength is, according to step 19 and this step	

TABLE 10.—(*Continued*)

Steps	TE waves	TM waves	Explanations
	sons are concerned. We shall find that the $TE_{1,1}$ wave is still more favorable	$\lambda_c = \frac{2\pi}{K} = \frac{2\pi a}{k_{0,1}} = \frac{2\pi a}{2.405}$ for the $TM_{0,1}$ wave	
22	According to the $k'_{n,m}$ values of step 21, we find for the consecutive TE waves the cutoff wavelengths $\lambda_c = \frac{2\pi a}{7.02}$ m for the $TE_{0,2}$ wave $\lambda_c = \frac{2\pi a}{10.17}$ m for the $TE_{0,3}$ wave	According to the $k_{n,m}$ values, given in step 21, we find the cutoff wavelengths above which a particular type of wave can no longer play a part as $\lambda_c = \frac{2\pi a}{5.52}$ m for the $TM_{0,2}$ wave $\lambda_c = \frac{2\pi a}{8.65}$ m for the $TM_{0,3}$ wave $\lambda_c = \frac{2\pi a}{11.79}$ m for the $TM_{0,4}$ wave	Since for TM waves, we have $K = k_{n,m}/a = K_{n,m}$ and for TE waves $K = k'_{n,m}/a = K'_{n,m}$, we have for the general guide wavelength of a circular guide $\lambda_g = \frac{6.28}{\sqrt{(6.28/\lambda)^2 - K^2}}$ where λ is the wavelength in the unbounded medium. We have then only to introduce the $K_{n,m}$ or the $K'_{n,m}$ value for the TM or the TE wave, respectively. The wavelength $\lambda = 1/(f\sqrt{\mu\kappa})$ m, if f is in cycles/sec; μ in henrys/m, and κ in farads/m. The cutoff frequency below which no energy can be transferred is found from $\beta_z = \sqrt{\omega^2\mu\kappa - K^2}$ as $\omega_c^2\mu\kappa = K^2$ $\omega_c^2 = (2\pi f_c)^2 = \frac{K^2}{\mu\kappa}$ For air $\frac{1}{\mu\kappa} \cong 9 \times 10^{16}$ m²/sec² R is the root of the particular Bessel function and R' the root of the derivative
23	TE waves are longitudinal H waves ($\mathcal{E}_z = 0$) of the form $H_z = H_{max}J'_n(rK) \times \cos(n\varphi)\epsilon^{-z\Gamma_z}$ amp/m	TM waves are longitudinal $\mathcal{E}$ waves ($H_z = 0$) of the form $\mathcal{E}_z = \mathcal{E}_{max}J_n(rK)\cos(n\varphi)\epsilon^{-z\Gamma_z}$ volts/m	
24	The cutoff frequency is $f_c^{(\text{cycles/sec})} = \frac{K'_{n,m}}{2\pi\sqrt{\mu\kappa}}$ $f_c^{(\text{Mc/sec})} = 47.8\,K'_{n,m}$ for air	The cutoff frequency f_c is $f_c^{(\text{cycles/sec})} = \frac{K_{n,m}}{2\pi\sqrt{\mu\kappa}}$ $f_c^{(\text{Mc/sec})} = 47.8K_{n,m}$ for air as dielectric	
25	$R' = aK'_{n,m} = k'_{n,m}$ Formula	$R = aK_{n,m} = k_{n,m}$ Formula	

Values of R' for TE waves

n/m	0	1	2	3
1	3.83	1.84	3.05	4.20
2	7.02	5.33	6.705	8.02
3	10.17	8.54	9.965	11.35

$$K'_{n,m} = \frac{R'}{a}$$

Example

$$K'_{3,2} = \frac{8.02}{a^{(\text{meters})}}$$

Values of R for TM waves

n/m	0	1	2	3
1	2.405	3.832	5.135	6.38
2	5.52	7.02	8.42	9.76
3	8.654	10.17	11.62	13.02
4	11.79	13.32	14.8	16.22

$$K_{n,m} = \frac{R}{a}$$

Example:

$$K_{2,4} = \frac{14.8}{a^{(\text{meters})}}$$

57. General Wave Equation[1] and Relations in Spherical Harmonics for Electromagnetic Wave Propagation. In Sec. 53, we note that for a scalar wave function w, we have $w(x, y, z, t)$ as a solution for

$$\nabla^2 w + w\Gamma^2 = 0 \tag{407}$$

if Γ denotes the propagation constant, which is generally complex. If $\nabla^2 w$ expresses the D'Alembertian of w, we also have the fundamental relation

$$\underbrace{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}}_{\nabla^2 w} - \frac{1}{v^2}\frac{\partial^2 w}{\partial t^2} = -F(x, y, x, t) \tag{408}$$

if v stands for the velocity of propagation. Equation (407) is continuous. For a closed space, Eq. (407) also has a continuous first derivative anywhere within a closed space. Calling w_S and $(\partial w/\partial n)_S$ the value on a bounding surface S and its normal gradient on such a surface, for all interior points x', y', z', we have the integral

$$w'(x', y', z') = \frac{1}{4\pi}\int\int_S \left[\left(\frac{\partial w}{\partial n}\right)_S \frac{\epsilon^{-r\Gamma}}{r} - w_S \frac{\partial(\epsilon^{-r\Gamma}/r)}{\partial n}\right] dS \tag{409}$$

This represents the Kirchhoff relation, a *discontinuous* variation, since at all interior points x', y', z', it is the function equal to w, and at all points outside the boundary the function vanishes. This means that the functions of w_S and $(\partial w/\partial n)_S$, therefore, *cannot* be used independently of each other, since w and w' must be the same at all points within the bounding surface S. This is what *conflicts* in the Kirchhoff diffraction formula, since the values of w' and $\partial w'/\partial n$ on the surface of S will usually differ from the values given to w_S and $(\partial w/\partial n)_S$. The only conditions that are generally satisfied by Eq. (407) and are defined by Eq. (409) are the values of w_S and $(\partial w/\partial n)_S$ for points within the boundary S.

It is Maxwell's equations

$$\text{curl } \mathcal{E} + \frac{\partial B}{\partial t} = 0; \qquad \text{div } D = \rho; \qquad \text{curl } H - J - \frac{\partial D}{\partial t} = 0;$$
$$\text{div } B = 0 \tag{410}$$

which, for atmospheric wave propagation, read

$$\text{curl } \mathcal{E} = -\frac{\partial(\mu H)}{\partial t}; \qquad \text{curl } H = \frac{\partial(\kappa \mathcal{E})}{\partial t} \tag{411}$$

[1] HERTZ, H., Electric Waves, *Ann. Physik.*, **36,** 1 (1888). HEAVISIDE, O., "Electric Papers," Vol. I, pp. 437–450. SILBERSTEIN, L., *Ann. Physik.*, **22,** 24 (1907). WHITTAKER, *Proc. Math. Soc.* (*London*), Ser. 2, 1 (1903). MIE, G., *Ann. Physik.*, **25,** 377 (1908). LAMOR, J., *Proc. Math. Soc.* (*London*), Ser. 2, **8,** 1. SOMMERFELD, A., *Jahrb. deut. math. Ver.*, **21** (1913). MATHIEU, E., "Cours de physique mathematique," 1873. STOKES, *Phil. Trans.*, **158,** 447 (1868). DEBYE, P., *Ann. Physik.*, **30,** 57 (1909). NIELSEN, "Handbuch der Cylinderfunktionen," B. G. Teubner, Leipzig, 1904.

Inasmuch as we deal in practical radio applications with harmonic fields $\mathcal{E}$ and H, which are subject to the time variation $\epsilon^{\pm j\omega t}$, we can use a common complex vector C. According to Sec. 44, it is given by

$$C = \mu H \pm j\sqrt{\mu\kappa}\,\mathcal{E} = B \pm jD\sqrt{\frac{\mu}{\kappa}} \tag{412}$$

where $B = \mu H$ webers per square meter, $D = \kappa\mathcal{E}$ coulombs per square meter, H is in amperes per meter, and $\mathcal{E}$ in volts per meter. Instead of (410), we have only one curl and one divergence relation

$$\text{curl } C = \underbrace{\pm j\sqrt{\mu\kappa}}_{\substack{1/v\\ \text{sec/m}}}\frac{\partial C}{\partial t} - \underbrace{\mu\sigma\mathcal{E}}_{\substack{\text{vanishes}\\ \text{for air}\\ \text{propagation}}} = 0; \qquad \text{div } C = \underbrace{j\rho\sqrt{\frac{\mu}{\kappa}}}_{\substack{\text{vanishes}\\ \text{for air}}} \tag{413}$$

Hence, curl $C = \mp j(1/v)\ \partial C/\partial t$ is the *single* and only equivalent electromagnetic field relation for propagation in air with div $C = 0$ (since there is a solenoidal or sourceless condition). Hence, C represents our single-wave function w in Eq. (407).

It is also often possible to express the twofold electromagnetic field (twofold on account of $\mathcal{E}$ and H associations) by means of a single Hertz vector as is also shown in Sec. 44. This is then a substitution of the scalar potential and vector potential method and consists of employing polarization and magnetization potentials that are vector functions. They are called "Hertz" vector and "Fitzgerald" vector, respectively although some writers call either function a Hertz vector. We have then

$$\left.\begin{aligned}
&\nabla^2\dot{\Pi}_c = \frac{1}{v^2}\frac{\partial^2\dot{\Pi}_c}{\partial t^2}; \quad v = \frac{1}{\sqrt{\mu\kappa}}\ \text{m/sec};\\
&\dot{\Pi}_c = \dot{\Pi} - jZ_o\dot{\Pi}_m; \quad Z_o = \sqrt{\frac{\mu}{\kappa}}\ \text{ohms}\cdot\\
&J = 0; \quad \rho = 0;
\end{aligned}\right\} \tag{414}$$

$\dot{\Pi}$ customary Hertz vector, $\dot{\Pi}_m$ Fitzgerald vector — consult also Appendix, Sec. 245 and p. 222

In these relations, Π_c denotes the complex Hertz vector with a real part Π, which is the *electric* polarization potential. The part Π_m is the *magnetic* polarization potential. The source of the Hertz vector Π as well as of the electromagnetic field associated with Π represents a distribution of *electric* polarization. The source of Π_m as well as the electromagnetic field found by means of it represents a distribution of *magnetic* polariza-

tion. Hence, the field contribution derived from Π_m is the effect of elementary *magnetic* dipoles.

It is, therefore, the complex Hertz vector Π_c that represents the single-wave function w in Eq. (407), just as the complex C vector defined above. Nevertheless, we have to meet the boundary conditions for $\mathcal{E}$ and H, which are behind the respective complex quantities Π_c and C, even though we have now two alternate procedures with either a single vector Π_c or a single vector C for a space-time condition (x, y, z, t) with the stipulation that the time function follows an exponential $j\omega t$ law.

With respect to $\Pi = \Pi_e$, dependent on *electric* polarization, and Π_m, dependent on *magnetic* polarization (magnetization), defined in Sec. 245 of the Appendix, and the volume density P of *electric* polarization in coulombs per square meter, and the volume density M of *magnetization* in amperes per square meter, in *charge-free* wave regions, we find for $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, the vectors $D = \kappa\mathcal{E}$ coulombs per square meter and $B = \mu H$ webers per square meter solenoidal, and have

$$P = D - \kappa_0\mathcal{E}; \qquad M = \frac{1}{\mu_0}B - H$$

since P and M must vanish for vacuum, for which $v = 1/\sqrt{\mu_0\kappa_0} = c$ meters per second. This holds also practically so for air. We find for a perfect dielectric wave medium (μ and κ) the velocity

$$v = \frac{1}{\sqrt{\mu\kappa}} \quad \text{m/sec}$$

and

$$B = \mu\kappa\nabla \times \frac{\partial\Pi}{\partial t} \equiv \mu\kappa \text{ curl } \frac{\partial\Pi}{\partial t}; \quad \mathcal{E} = \nabla\nabla \cdot \Pi - \left(\frac{1}{v^2}\right)\frac{\partial^2\Pi}{\partial t^2} = \text{grad div } \Pi - \mu\kappa\frac{\partial^2\Pi}{\partial t^2};$$

since $\nabla\nabla = \nabla^2 \equiv$ grad div. Moreover,

$$\nabla^2\Pi = \left(\frac{1}{v^2}\right)\frac{\partial^2\Pi}{\partial t^2}; \quad D = -\mu\kappa\nabla \times \frac{\partial\Pi_m}{\partial t} \equiv -\mu\kappa \text{ curl } \frac{\partial\Pi_m}{\partial t};$$

$$H = \nabla\nabla \cdot \Pi_m - \left(\frac{1}{v^2}\right)\frac{\partial^2\Pi_m}{\partial t^2} = \text{grad div } \Pi_m - \mu\kappa\frac{\partial^2\Pi_m}{\partial t^2}$$

Hence,

$$\nabla \times H \equiv \text{curl } H - \kappa_0\frac{\partial\mathcal{E}}{\partial t} = \frac{\partial P}{\partial t}; \quad \text{div } \mathcal{E} \equiv \nabla \cdot \mathcal{E} = -\frac{1}{\kappa_0}\nabla \cdot P;$$

$$\nabla^2\Pi = \mu\kappa_0\frac{\partial^2\Pi}{\partial t^2} = -\frac{1}{\kappa_0}P \quad \text{(electromagnetic field derived from distribution of } P\text{)}$$

$$\nabla^2\Pi_m = \mu_0\kappa\frac{\partial^2\Pi_m}{\partial t^2} = -M \quad \text{(electromagnetic field derived from magnetization } M\text{)}$$

Moreover,

$$\underset{\cdot}{\mathcal{E}} = \text{grad div} \underset{\cdot}{\Pi} - \mu_0\kappa_0 \frac{\partial^2 \underset{\cdot}{\Pi}}{\partial t^2} - \text{curl} \frac{\partial \underset{\cdot}{\Pi}_m}{\partial t} = \text{grad div} \underset{\cdot}{\Pi} + \frac{\omega^2}{c^2} \underset{\cdot}{\Pi} - j\omega \text{ curl} \underset{\cdot}{\Pi}_m;$$

$$\underset{\cdot}{B} = \mu_0\kappa_0 \text{ curl} \underset{\cdot}{\dot{\Pi}} + \text{curl curl} \underset{\cdot}{\Pi}_m = j\frac{\omega}{c^2} \text{ curl} \underset{\cdot}{\Pi} + \text{curl curl} \underset{\cdot}{\Pi}_m;$$

Since $\underset{\cdot}{\Pi}$ and $\underset{\cdot}{\Pi}_m$ do not depend on each other, we can separate them, as is also evident from the real and imaginary value for the complex Hertz vector $\underset{\cdot}{\Pi}_c$ in Eqs. (414), and

$$\left.\begin{aligned} \underset{\cdot}{\mathcal{E}} &= \text{grad div} \underset{\cdot}{\Pi} - \mu_0\kappa_0 \frac{\partial^2 \underset{\cdot}{\Pi}}{\partial t^2} \quad \text{volts/m} \\ \underset{\cdot}{B} &= \mu_0\kappa_0 \text{ curl} \frac{\partial \underset{\cdot}{\Pi}}{\partial t} \quad \text{weber/m}^2 \end{aligned}\right\} \quad \textit{electric type}$$

$$\begin{aligned} \underset{\cdot}{\mathcal{E}} &= -\text{curl} \frac{\partial \underset{\cdot}{\Pi}_m}{\partial t} \quad \text{volt/m} \\ \underset{\cdot}{B} &= \text{curl curl} \underset{\cdot}{\Pi}_m \quad \text{weber/m}^2 \end{aligned} \quad \textit{magnetic type}$$

With respect to a dissipative wave medium (σ mhos per meter also effective), reference is made to Eqs. (339) and (339*a*), and we have for customary wave excitations to deal with complex dielectric constant.

In diffraction problems, we deal with an obstacle that may be completely opaque to an arriving electromagnetic wave (specific conductivity $\sigma = \infty$), as in the case of high-conducting metal objects, in some way also for ocean water to quite high frequencies and customary earth crust if the frequency is not too high. Nevertheless, the obstacle is generally electromagnetically complex, since it offers a complex relative numeric dielectric constant $\kappa_e' = \kappa_r - j18 \times 10^3\sigma/f$, where σ is in mhos per meter, f is in megacycles per second, and $\kappa_r = \kappa/\kappa_0$ for κ of the obstacle in farads per meter.

When electromagnetic waves meet an obstacle, an additional wave disturbance also occurs. This disturbance depends upon the *primary* or arriving electromagnetic field $\mathcal{E}$, H as well as on the electromagnetic characteristics σ, μ, and κ of the obstacle *causing* a *secondary* field $\mathcal{E}'$, H'. We have then for the single vector C the corresponding identity $CC' \equiv C_xC_x' + C_yC_y' + C_zC_z' \equiv 0$; and at all space-time points (x, y, z, t), we deal with conjugate primary and secondary fields. This is, however, true only in some cases, and we have $C = \sqrt{C_x^2 + C_y^2 + C_z^2}$.

It is seen that for the function $F(x, y, z, t) = 0$ of Eq. (408), Eqs. (408) and (407) are identical, and we deal with a wave equation in each case with a solution that is a wave function. Inasmuch as w stands for

a scalar, it may also represent any rectangular component of a field vector with $F(x, y, z, t)$ as the volume concentration of some source function. Therefore, if w stands for the scalar potential V volts, we have in (408) the phase velocity $v = 1/\sqrt{\mu\kappa}$ meters per second. A vector function P in place of the scalar function w in wave equation (407) satisfies this equation only when each of its components is also a wave function, which means when

$$\nabla^2 P - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} \equiv \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0$$

P has nothing to do with symbol P used above.

For this reason, we can satisfy the electromagnetic field relations as well as the single complex vector C, for the $\mathcal{E}$ and H associations, by a complex scalar potential V_c and a complex vector potential Λ_c, given by

$$V_c = V \mp j\sqrt{\frac{\mu}{\kappa}}\, V_m; \qquad \Lambda_c = \Lambda \mp j\sqrt{\frac{\mu}{\kappa}}\, \Lambda_e \tag{415}$$

We have then

$$C = \text{curl } \Lambda_c \mp j\sqrt{\mu\kappa}\left(\nabla V_c + \frac{\partial \Lambda_c}{\partial t}\right) \tag{416}$$

which for the resultant current density J in amperes per square meter and the charge density ρ in coulombs per cubic meter also satisfies, for $v = 1/\sqrt{\mu\kappa}$, the relations

$$\nabla^2 V_c \mp \frac{\partial^2 V_c/\partial t^2}{v^2} + \frac{\rho}{\kappa} = 0; \qquad \nabla^2 \Lambda_c \mp \frac{\partial^2 \Lambda_c/\partial t^2}{v^2} + \mu J = 0 \tag{417}$$

Therefore, in case of atmospheric propagation $J = 0 = \rho$, and both relations are satisfied by

$$\nabla^2_w \mp \frac{\frac{\partial^2 w}{\partial t^2}}{v^2} = 0 \qquad \left\{\begin{matrix} \text{for } w = V_c \\ \text{for } w = \Lambda_c \end{matrix}\right\} \tag{418}$$

where v is then essentially equal to 3×10^8 m/sec. This outcome has reference *only to* unbounded (*unobstructed*) wave *propagation*. Relations (417) then read for harmonic waves

$$\nabla^2 V_c + \frac{\omega^2}{v^2} V_c = -\frac{\rho}{\kappa}; \qquad \nabla^2 \Lambda_c + \frac{\omega^2}{v^2}\Lambda_c = -\mu J \tag{419}$$

where J amperes per square meter also can be taken as the volume charge density of a *moving* charge, which is also expressed in coulombs

per cubic meter. For solenoidal or sourceless conditions, we arrive at a generalization of the Hertz vector solution, as given in Eq. (414) for $v = c = 3 \times 10^8$ m/sec, since $\sigma = 0$ and for air $\mu = \mu_0$ and $\kappa \cong \kappa_0$.

Figure 49 gives the formulas for the general medium with all three electromagnetic factors σ, μ, and κ effective, for the case of the true radiation field, since, for the indicated $\mathcal{E}_x$ and H_y polarizations of the electromagnetic field, we have in the Maxwellian curl equations on page 148

$$\underbrace{\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z}}_{\text{hence, } = 0} = -j\omega \underbrace{H_x}_{=0};$$

$$\frac{\partial \overbrace{H_z}^{=0}}{\partial y} - \frac{\partial H_y}{\partial z} = (\sigma + j\omega\kappa)\mathcal{E}_x$$

$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \overbrace{\mathcal{E}_z}^{=0}}{\partial x} = -j\omega\mu H_y;$$

$$\underbrace{\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}}_{\text{hence, } = 0} = (\sigma + j\omega\kappa)\underbrace{\mathcal{E}_y}_{=0}$$

$$\underbrace{\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}}_{\text{hence, } = 0} = -j\omega\mu \underbrace{H_z}_{=0};$$

$$\underbrace{\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}}_{\text{hence, } = 0} = (\sigma + j\omega\kappa)\underbrace{\mathcal{E}_z}_{=0}$$

leaves only the relations

$$\frac{\partial \mathcal{E}_x}{\partial z} = -j\omega\mu H_y; \quad -\frac{\partial H_y}{\partial z} = (\sigma + j\omega\kappa)\mathcal{E}_x$$

We have then only the electric and magnetic field gradients

$$\frac{\partial \mathcal{E}_x}{\partial z} = ZH_y; \; -\frac{\partial H_y}{\partial z} = Y\mathcal{E}_x; \text{ for } Z = -j\omega\mu$$

and $Y = -(\sigma + j\omega\kappa)$

Since for the distant radiation field, we have such field associations in space quadrature (TEM waves), the case of Fig. 49 refers to wave propagation along the Z direction. According to customary linear propagation theory, the propagation constant is $\Gamma = \sqrt{ZY}$ and the characteristic impedance is $Z_0 = \sqrt{Z/Y}$. This results in $\Gamma = j\omega\sqrt{\mu\kappa_e} = \alpha + j\beta$ complex radians per meter and $Z_0 = \sqrt{\mu/\kappa_e}$ complex ohms for $\kappa_e = \kappa - j\sigma/\omega$ farads per meter and $\omega/6.28$ in cycles per second. The XYZ system for unobstructed radiation space can then be rotated in any orientation to suit a particular TEM wave, and Γ and Z_0 can be taken as the *three-dimensional* characteristics for the *unbounded* radiation region. This assumes that we deal with an isotropic wave medium.

Since, according to the foregoing discussion, we have for plane waves, $\mathcal{E}/H = Z_0$ ohms, we also have $\mathcal{E} = Z_0 H$ volts per meter, where $\mathcal{E}$ and H are the resultant field associations. Both field vectors are perpendicular to each other, and we can combine them geometrically as long as they are expressed in the same units. Therefore, we can combine either $\mathcal{E}$

with $\pm jZ_0H$ volts per meter or H with $\pm j\mathcal{E}/Z_0$ amperes per meter. Doing the latter, we find

$$H \pm j\frac{\mathcal{E}}{Z_0} = \frac{\mu H \pm j\frac{\mu}{Z_0}\mathcal{E}}{\mu} = \frac{\mu H \pm j\sqrt{\mu\kappa_e}\,\mathcal{E}}{\mu} = \frac{C}{\mu} = \frac{\mu H \pm j\sqrt{\frac{\mu}{\kappa_e}}\,\kappa_e\mathcal{E}}{\mu} \tag{420}$$

where, for propagation in unobstructed air, $C = \mu H \pm j\sqrt{\mu_0\kappa_0}\,\mathcal{E}$ confirms Eq. (420) and the curl relation curl $C = \mp j\sqrt{\mu_0\kappa_0}\,(\partial C/\partial t)$, since

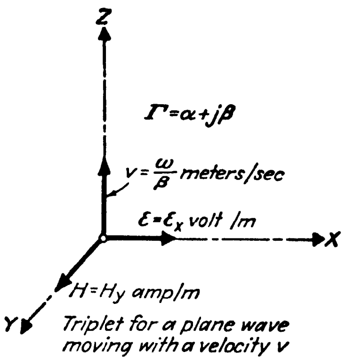


FIG. 49. Behavior of plane electromagnetic waves in the general and in the nondissipative medium. For *plane TEM* waves *in* the *general wave medium* of σ mhos/m, μ henrys/m; κ farads/m for the exciter frequency of $\omega/6.28$ cycles/sec

$$\frac{\partial \mathcal{E}_x}{\partial z} = \underbrace{-j\omega\mu}_{=z}\, H_y; \qquad \frac{\partial H_y}{\partial z} = \underbrace{-(\sigma + j\omega\kappa)}_{=y}\mathcal{E}_x$$

and three-dimensional medium propagation constant

$$\Gamma = \sqrt{YZ} = \sqrt{j\omega\mu(\sigma + j\omega\kappa)} = j\omega\sqrt{\mu\kappa_e} = \alpha + j\beta.$$

$$\kappa_e = \kappa - j\frac{\sigma}{\omega} \qquad \text{farads/m}$$

Characteristic wave impedance:

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{\sigma + j\omega\kappa}{j\omega\mu}} = \sqrt{\frac{\mu}{\kappa_e}} \qquad \text{ohms}$$

Admittance gradient:

$$Y = \frac{-\partial H_y/\partial z}{\mathcal{E}_x} \qquad \text{mhos/m}$$

Impedance gradient:

$$Z = \frac{-\partial\mathcal{E}/\partial z}{H_y} \qquad \text{ohms/m}$$

$$\Gamma = j\beta = j\frac{\omega}{v} = j\omega\sqrt{\mu\kappa}$$

for a nondissipative wave medium ($\sigma = 0$). $Z_0 = \sqrt{\mu/\kappa}$ resistive ohms for a nondissipative wave medium, and $\mathcal{E}_x$ and H_y are in time phase.

div $C = 0$ for such a condition. We also find, for propagation through unbounded air,

$$\left.\begin{aligned} \frac{\partial C}{\partial t} &= \frac{\partial(\mu_0 H)}{\partial t} \pm j\sqrt{\frac{\mu_0}{\kappa_0}}\frac{\partial(\kappa_0 \mathcal{E})}{\partial t} = -\operatorname{curl} \mathcal{E} \pm jZ_0 \operatorname{curl} H \\ \operatorname{curl} C &= \mp \frac{\partial C/\partial t}{c} = \frac{\pm Z_0 \operatorname{curl} H \pm j \operatorname{curl} \mathcal{E}}{3 \times 10^8} \\ \operatorname{curl} C &= \mp \frac{\partial C/\partial t}{c} = \pm(\mu_0 \operatorname{curl} H + j\sqrt{\mu_0\kappa_0} \operatorname{curl} \mathcal{E}) \end{aligned}\right\} \quad (421)$$

and note that the dimension of curl C is volt · seconds per cubic meter, while we find for the dimension of the common field vector C, volt · seconds per square meter. We also find from $C = B \pm jZ_0D$, where $B = \mu_0 H$, $D = \kappa_0 \mathcal{E}$, $Z_0 = \sqrt{\mu_0/\kappa_0}$, for propagation in air, and $c = 1/\sqrt{\mu_0\kappa_0}$

$$\frac{\partial C}{\partial t} = j\omega B \pm j\omega j Z_0 D; \qquad \operatorname{curl} C = \mp j\frac{\partial C/\partial t}{c} = \mp \frac{j}{c}(j\omega B \pm j\omega j Z_0 D)$$
$$= \mp j \underbrace{j\omega\sqrt{\mu_0\kappa_0}}_{\Gamma}\,\underbrace{(B \pm jZ_0 D)}_{C}$$

Therefore,

$$\operatorname{curl} C = \mp jC\Gamma \qquad \text{for div } C = 0 \tag{422}$$

The relation for the equivalent common field vector C must, therefore, satisfy the curl $C = \mp jC\Gamma$ and the div $C = 0$ differential equations. We have then essentially the free-space propagation constant $\Gamma = j\beta_0 = j\omega/c$, and (422) satisfies the general wave equation $\nabla^2 w + w\Gamma^2 = 0$, where

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

When dealing with a general medium (σ in addition to μ and κ), we have then $\Gamma = j\omega\sqrt{\mu\kappa_c}$ instead of $j\omega\sqrt{\mu_0\kappa_0}$. We have, therefore, a means of accounting for what happens when an electromagnetic wave meets an obstacle composed of a general medium. It is not only the μ, κ, and σ values of an obstacle but also the dimensions expressed in wavelength and the shape that play a part. For instance, a very highly conducting metal sheet of relatively large dimensions acts as a good reflector of microwaves; but around edges and wedges, we *may have pronounced diffraction effects into the geometrical shadow region* as well as into the actual radiation region. That the dimensions in relationship to the operating wavelength play an important part can be seen from the following: We use the phase constant $\beta = 2\pi/\lambda$ as a reference, where λ is the particular *operating* wavelength in a certain medium, and find from the imaginary part of the propagation constant $\Gamma = \alpha + j\beta = j\omega\sqrt{\mu\kappa_c}$. When an obstacle has dimensions smaller than $1/\beta$, *i.e.*, $<\lambda/6.28$ meters, the obstacle, at best, can act only as an elementary reradiator (like a Hertzian doublet) with very weak reradiations in all directions and no appreciable effect on the incident electromagnetic wave. We have then the case of a "stick" in a lake. The water waves will then diffract

without any difficulties around the stick. But when the diameter of the stick becomes larger, some water waves will be reflected and interfere with other arriving water waves. In the same way, as the obstacle becomes larger, the reradiation from it is no longer a weak elementary dipole effect but diffraction action occurs with maxima and minima effects in different directions. When, however, the dimensions of the obstacle become much larger than $\lambda/6.28$ meters, the interference effect is caused more by means of the electro-optical ray effect.

It must be realized that for inhomogenous mediums, we must, in many cases, use the general wave equation and apply the diffraction theory with proper boundary conditions for the electric and magnetic vectors. This is not taken care of by the classic Kirchhoff theory and wave equation. There are, however, many cases where the ray theory is applicable and geometrical optics can be used. There are also cases where neither the diffraction nor the electro-optical ray theory is satisfactory, since we deal with atmospheric wave-guide effects that partially leak out field power over the boundaries of atmospheric wave ducts. The laws of geometric optics then apply generally to regions of the atmosphere for which the change of the index of refraction $n = \sqrt{\kappa_e'}$ is small within 1 wavelength. It is also assumed that neighboring rays do not cross or approach each other; cata- and diacaustics (envelopes of rays after reflection and refraction) are excluded. The ray theory does not account also for many field effects in the geometrical shadow region, except the effects due to refraction. If the electric field vector $\mathcal{E}$ is normal to ground (along Z axis) and the ground is in the XY plane, we have, for regions of no free charge and current densities, the general wave equation

$$\nabla^2\mathcal{E} + \beta^2 n(z)\mathcal{E} = 0 \tag{423}$$

where $n(z)$ expresses that the relative index of atmospheric refraction is taken along the Z direction. For good conducting ground (holding for ocean surface to quite high frequencies and for customary earth crust into the standard broadcast range of frequencies), we have then $z = 0$ and $\mathcal{E} \cong 0$. The phase constant $\beta = 2\pi/\lambda$ is then all that is left in the propagation constant $\Gamma = \alpha + j\beta$, since $\alpha \cong 0$, and λ is, in this case, essentially the free-space wavelength in meters found from $300/f$, where f is in megacycles per second. With such assumptions, the index of refraction is a function only of the altitude z above the earth surface. The variables of the wave equation for $\mathcal{E}$ can be separated if we assume a solution like $\exp[j\beta(ax + by)]P(z)$, where

$$\frac{d^2P}{dz^2} + \beta^2A^2P = 0; \qquad A^2 = n^2 - a^2 - b^2$$

Since the index n of refraction is often close to unity, we have $A^2 = 1 - a^2 - b^2$. We have also a simplified relation

$$\frac{d^2P}{dz^2} + \beta^2(n^2 + A^2 - 1)P = 0$$

where P must satisfy the same boundary conditions as the electric vector $\mathcal{E}$. We have, therefore, a system of the Sturmian type, with solutions that cannot be identically zero but become identically zero for an infinite discrete set of values of the parameter. Hence, we have discrete values A_j with $j = 1, 2, 3, 4, \cdots$. We have then complex values and complex characteristic functions. Due to boundary conditions, P goes from an upward-traveling wave for large positive values of z to a stationary wave as z approaches ground or zero level.

Diffraction effects will occur not only around sharp wedges and edges, but also on obstacles of curved shapes, like a sphere, for instance. They will also occur when the sphere is composed of a general medium, since only a portion of the arriving electromagnetic field energy will be absorbed.

The discussion on the general wave equation shows that this equation $\nabla^2 w + w\Gamma^2 = 0$ can, for certain conditions, be used for solving problems by means of Maxwell's equations, since both $\nabla^2 V_c$ and $\nabla^2 \Lambda_c$ give, for $J = 0 = \rho$, expressions of the form $\nabla^2 w - (1/c^2)(\partial^2 w/\partial t^2) = 0$. Such expressions are satisfied by wave functions of the form

$$w = \Phi(x, y, z)\epsilon^{\pm jct\Gamma} \tag{424}$$

Inasmuch as a wave function depends on the coordinates and the time, we have, for instance, for the potential V at a point x

$$V(x, t) = V_0\epsilon^{kt-x\Gamma}$$

with $V_0 = A\epsilon^{j\psi} = A(\cos\psi + j\sin\psi)$, $k = \delta + j\omega$, and $\Gamma = \alpha + j\beta$. This is the general form, since all quantities are complex. The real part δ of k vanishes if we have sustained oscillations of fixed amplitude as regards time. For damped oscillations, δ accounts for the exponential shrinkage of the vector as it rotates with the angular velocity ω at a certain space point. In alternating-current theory, it is customary to deal with the real part for the instantaneous values, and we have then $V_{\text{Re}} = A\epsilon^{\delta t-\alpha x}\cos\varphi$, where $\varphi = \omega t + \psi - \beta x$. Since φ stands for the phase of the general wave, successive equiphase space points along the X direction must be 1 wavelength apart. The three-dimensional wave equation is of concern in the above discussion. It is

$$V(x, y, z, t) = V_0(x, y, z)\epsilon^{j[\omega t-\varphi(x,y,z)]}$$

for sustained wave excitations. For equal-phase locations, *i.e.*, $\varphi(x, y, z)$ = constant or fixed, we have now equal-phase *surfaces*. We can then distinguish among spherical, cylindrical, and plane equiphase surfaces with the phase velocity $c'(x, y, z) = \omega/\beta = \omega/\text{grad}\,\varphi$ normal to such surfaces.

Inasmuch as the general wave equation applied to a scalar potential V volts reads $\nabla^2 V \pm V\Gamma^2 = 0$, we have, for exponential waves traveling outward,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - V\Gamma^2 = 0 \tag{425}$$

in Cartesian coordinates. This relation is satisfied by the wave function

$$\sqrt{\Gamma_x^2 + \Gamma_y^2 + \Gamma_z^2} = \Gamma = j\omega\sqrt{\mu\kappa} \tag{426}$$

where $\omega/6.28$ is in cycles per second. Since we relate these propagation constant components to the wave front, which is the seat of the advancing electromagnetic energy, we can often simplify this expression. The wave front depends on the source of the electromagnetic wave and the distance from the source. Therefore, we must know the radiation pattern of the source as well as its frequency. For the distant radiation field (depending on frequency) of a vertical radiator along the Z direction, we have, for the plane wave front in the YZ plane, only one component and $\Gamma = \Gamma_x = j\omega\sqrt{\mu\kappa_v} \cong j\omega/c = j\beta$ in atmosphere.

As early as in 1732, D. Bernoulli solved the wave equation $\Delta^2 w \pm w\Gamma^2 = 0$ [where w is as in Eq. (424)], by means of elementary functions such as $\Phi_1(a)$, $\Phi_2(b)$, $\Phi_3(d)$, and $\Phi_4(e)$, where $w = \Phi_1(a)\Phi_2(b)\Phi_3(d)\Phi_4(e)$. We have then the expression

$$w = \epsilon^{mx+py+qz\pm st} \qquad \text{for } a = x,\ b = y,\ d = z,\ e = t$$
$$\text{for } s = c\sqrt{m^2 + p^2 + q^2} \qquad (427)$$

where m, p, q, and s are generally complex but may be real or purely imaginary. For plane electromagnetic waves, m, p, and q are imaginary with fixed magnitude as well as phase in planes normal to the direction of propagation. Hence, when Z is the axis of wave propagation, we have $\mathcal{E}_z = 0$ and $H_z = 0$. When the electric vector is along the Y axis, then the magnetic vector is along the X axis, and we have only $\mathcal{E}_y$ and H_x field associations. This necessitates that $m = 0$ and $p = 0$, since no propagation exists along the X and Y directions. Therefore, $q = -s/c$ and

$$\mathcal{E}_y = \mathcal{E}_0 \cos \frac{s}{c}(ct - z) = \mathcal{E}_0 \cos (st - \beta z) \qquad \text{volts/m;}$$
$$H_x = Z_0 \mathcal{E}_0 \cos (st - \beta z) \qquad \text{amp/m}$$
$$\beta = \frac{s}{c}; \qquad \frac{s}{6.28} \text{ in cycles/sec;} \qquad Z_0 \cong 377\Omega$$

Since the spherical coordinates r, φ, and θ are interrelated with the Cartesian coordinates x, y, and z by $r = \sqrt{x^2 + y^2 + z^2}$, $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, and $z = r \cos \theta$, we have

$$\nabla^2 w \pm w\Gamma^2 = \overbrace{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}}^{\nabla^2 w} \pm w\Gamma^2 = 0 \qquad \text{Cartesian coordinates}$$

$$\nabla^2 w \pm w\Gamma^2 = \frac{1}{r^2}\frac{\partial\left(r^2 \frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial\left(\sin\theta \frac{\partial w}{\partial \theta}\right)}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2} \pm w\Gamma^2 = 0$$

$$\nabla^2 w \pm w\Gamma^2 = \underbrace{\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial\left(\sin\theta \frac{\partial w}{\partial \theta}\right)}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 w}{\partial \varphi^2}}_{\nabla^2 w} \pm w\Gamma^2 = 0 \qquad \text{spherical coordinates} \qquad (428)$$

For *cylindrical coordinates* r, φ, and z, we have the interrelations $r = \sqrt{x^2 + y^2}$, $x = r\cos\varphi$, $y = r\sin\varphi$, and $z = z$, and

$$\nabla^2 w \pm w\Gamma^2 = \frac{1}{r}\frac{\partial\left(r\frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{\partial^2 w}{\partial z^2} \pm w\Gamma^2 = 0 \qquad (429)$$

The wave equation of any of the three coordinate systems is, according to the above discussion, separable. For instance, for spherical coordinates the wave function w can be satisfied by the elementary functions $\Phi_1(r)$, $\Phi_2(\theta)$, and $\Phi_3(\varphi)$, since we have the solution condition

$$w = \Phi_1(r)\Phi_2(\theta)\Phi_3(\varphi) \qquad (430)$$

The separation constants p and q, which hold for a single-valued field anywhere in space, together with (430) yield the separated relations, for $\nabla^2 w + w\Gamma^2 = 0$, as

$$\left.\begin{aligned} &\frac{d^2\Phi_3}{d\varphi^2} + p^2\Phi_3 = 0 \\ &\frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{d\Phi_2}{d\theta}\right)}{d\theta} + \left[q(q+1) - \frac{p^2}{\sin^2\theta}\right]\Phi_2 = 0 \\ &\frac{d^2\Phi_1}{dr^2} + \frac{2}{r}\frac{d\Phi_1}{dr}\left[\frac{\Gamma^2}{r^2} - q(q+1)\Phi_1\right] = 0 \end{aligned}\right\} \qquad (431)$$

The Φ_3 differential equation is well known and has the solution $\Phi_3 = \Phi_0\epsilon^{jp\varphi}$ or $\Phi_3 = \Phi'\epsilon^{-pj\varphi} + \Phi''\epsilon^{pj\varphi}$ leading to a form $\Phi_3 = \cos(p\varphi + \delta)$. The second relation is satisfied by the associated Legendre functions $P_q^p(\cos\theta)$ and $Q_q^p(\cos\theta)$. The last expression can be put in the form

$$\frac{d^2(\Phi_1\sqrt{r})}{dr^2} + \frac{1}{r}\frac{d(\Phi_1\sqrt{r})}{dr} + \left[\frac{\Gamma^2}{r^2} - \left(q + \frac{1}{2}\right)^2\right]\Phi_1\sqrt{r} = 0 \qquad (432)$$

This is a Bessel equation and expressible by $J_{q+\frac{1}{2}}(r\Gamma)$ and $J_{-(q+\frac{1}{2})}(r\Gamma)$. The separation constants may have any fixed values. For conditions that render $q + \frac{1}{2}$ an integer, the Bessel functions mentioned above are no longer independent and the companion or Neumann solutions $N_{q+\frac{1}{2}}(r\Gamma)$ hold.

At this stage of the discussion, it should be realized that for any value $n = q + \frac{1}{2}$, there are two independent solutions of the Bessel equation, namely, $J_n(x)$ and $N_n(x)$, with a phase relation similar to sine and cosine curves, except that the Bessel and Neumann curves decrease with $\sqrt{x}$, where in the above application $x = r\Gamma$, *i.e.*, the propagation constant $\Gamma = j\omega\sqrt{\mu\kappa}$ multiplied by the distance r from the radiation center in case of an electromagnetic wave in a pure dielectric medium. The general

solution represents an arbitrary linear combination of $J_n(x)$ and $N_n(x)$ *depending on the boundary conditions* of an obstructing medium. Two linear combinations of $J_n(x)$ and $N_n(x)$ are known as Hankel functions of the forms $J_n(x) + jN_n(x)$ or $J_n(x) - jN_n(x)$. It is also to be noted that $N_n(x)$ becomes logarithmically infinite for the argument $x = 0$, while $J_n(0)$ is finite for all values of n and equal to unity for $J_0(0)$, *i.e.*, for $n = 0$ and $x = 0$. But $J_n(0)$ is equal to zero for values of n other than zero. For orders of $n = q + \frac{1}{2}$ of Bessel functions, as occurring in the above expressions, the functions reduce to elementary functions and the related functions.

We have $K_q(x) = \sqrt{2x/\pi}\; K_{q+\frac{1}{2}}(x)$, $J_q(x) = \sqrt{\frac{1}{2}\pi x}\, J_{q+\frac{1}{2}}(x)$,

$$I_q(x) = \sqrt{\tfrac{1}{2}\pi x}\, I_{q+\frac{1}{2}}(x),\; N_q(x) = \sqrt{\tfrac{1}{2}\pi x}\, N_{q+\frac{1}{2}}(x)$$

The following are the details of the general wave equation for Cartesian, cylindrical, and spherical coordinates (428) and (429). They are needed not only for three-dimensional space radiation but also in expressions dealing with wave guides and wave ducts.

For Cartesian coordinates, we have, for the scalar wave function w,

$$\nabla^2 w \pm w\Gamma^2 = 0; \qquad \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

This relation can be solved by elementary functions $\Phi_1(x)$, $\Phi_2(y)$, and $\Phi_3(z)$ in the form $w = \Phi_1(x)\Phi_2(y)\Phi_3(z)$, which, inserted in the wave equation, yields

$$\Phi_2\Phi_3 \frac{\partial^2\Phi_1}{\partial x^2} + \Phi_1\Phi_3 \frac{\partial^2\Phi_2}{\partial y^2} + \Phi_1\Phi_2 \frac{\partial^2\Phi_3}{\partial z^2} = \mp \Phi_1\Phi_2\Phi_3\Gamma^2$$

A division by $\Phi_1\Phi_2\Phi_3$, yields

$$\underbrace{\underbrace{\frac{1}{\Phi_1}\frac{\partial^2\Phi_1}{\partial x^2}}_{\text{varies with } x} + \underbrace{\frac{1}{\Phi_2}\frac{\partial^2\Phi_2}{\partial y^2}}_{\text{varies with } y} + \underbrace{\frac{1}{\Phi_3}\frac{\partial^2\Phi_3}{\partial z^2}}_{\text{varies with } z}}_{\text{must also add up to a constant}} = \underbrace{\mp\Gamma^2}_{\text{is a constant}} = \underbrace{\pm\omega^2\mu\kappa}_{\text{for electromagnetic waves}}$$

This requires that each term to be added must be constant. Inasmuch as the variations are along the X, Y, and Z directions, the corresponding constants are also with respect to such directions, and we have

$$\frac{1}{\Phi_1}\frac{\partial^2\Phi_1}{\partial x^2} = \Gamma_x^2; \qquad \frac{1}{\Phi_2}\frac{\partial^2\Phi_2}{\partial y^2} = \Gamma_y^2; \qquad \frac{1}{\Phi_3}\frac{\partial^2\Phi_3}{\partial z^2} = \Gamma_z^2; \qquad \Gamma = \sqrt{\Gamma_x^2 + \Gamma_y^2 + \Gamma_z^2}$$

The choice of the coordinate system depends on the physical shape of a medium. If we deal, for instance, with a cylindrical wave guide, where the medium through which the waves are guided is bounded by cylindrical metal walls or walls of some other medium (in case of dielectric guides), the wave equation in cylindrical coordinates is to be used. Cartesian coordinates fit plane boundaries between two different mediums as in rectangular guides. For atmospheric wave ducts, we deal with spherical shells and spherical coordinates are in order. The double $\pm$ sign in front of the $w\Gamma^2$ term of the wave equation $\nabla^2 w \pm w\Gamma^2 = 0$ is carried along only for the sake of generality.

The relation $\nabla^2 w - w\Gamma^2 = 0$ is the case for unobstructed outward electromagnetic wave propagation. In *cylindrical* coordinates, we have, according to (429), the form

$$\underbrace{\frac{1}{wr}\frac{\partial\left(r\frac{dw}{dr}\right)}{\partial r} + \frac{1}{wr^2}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{w}\frac{\partial^2 w}{\partial z^2}}_{\nabla^2 w/w} = \mp\Gamma^2$$

if the wave equation is divided by the wave function w. Assuming again elementary functions $\Phi_1(r)$, $\Phi_2(\varphi)$, and $\Phi_3(z)$ for the three cylindrical variations $\partial w/\partial r$, $\partial^2 w/\partial\varphi^2$, and $\partial^2 w/\partial z^2$, for $w = \Phi_1(r)\Phi_2(\varphi)\Phi_3(z)$, we find

$$\frac{1}{r\Phi_1\Phi_2\Phi_3}\frac{\partial\left(r\frac{\partial\Phi_1}{\partial r}\right)}{\partial r}\Phi_2\Phi_3 + \frac{1}{r^2\Phi_1\Phi_2\Phi_3}\frac{\partial^2\Phi_2}{\partial\varphi^2}\Phi_1\Phi_3 + \frac{1}{\Phi_1\Phi_2\Phi_3}\frac{\partial^2\Phi_3}{\partial z^2}\Phi_1\Phi_2 = \mp\Gamma^2$$

which reduces to

$$\underbrace{\frac{1}{r\Phi_1}\frac{\partial\left(r\frac{\partial\Phi_1}{\partial r}\right)}{\partial r}}_{\text{does not vary with } z} + \underbrace{\frac{1}{r^2\Phi_2}\frac{\partial^2\Phi_2}{\partial\varphi^2}}_{\text{does not vary with } z} + \underbrace{\frac{1}{\Phi_3}\frac{\partial^2\Phi_3}{\partial z^2}}_{\Gamma_z^2} = \underbrace{\mp\Gamma^2}_{\text{constant}} = \underbrace{\pm\omega^2\mu\kappa}_{\text{for electromagnetic waves}} \tag{433}$$

We now have cylindrical surfaces that everywhere have the *same* radius r, no matter on which portion of the Z (or cylinder) axis. The angular measure φ around the cylinder axis is likewise the *same* all along the Z axis. This means that neither the r variation ($\partial/\partial r$) nor the φ variation ($\partial/\partial\varphi$) in the above expression can be affected by the z variation ($\partial/\partial z$). The right side of the equation is again fixed, *i.e.*, constant, for a certain medium (μ and κ) and frequency. It is, therefore, also not a function of z. The only remaining term $(\partial^2\Phi_3/\partial z^2)/\Phi_3$ cannot then be function of z either, and we have as the first step

$$\frac{\partial^2\Phi_3}{\partial z^2} = \Phi_3\Gamma_z^2 \tag{434}$$

which introduced in (433) results in

$$\frac{1}{r\Phi_1}\frac{\partial\left(r\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} + \frac{1}{r^2\Phi_2}\frac{\partial^2\Phi_2}{\partial\varphi^2} = \mp\Gamma^2 - \Gamma_z^2$$

Putting also the fixed factor r^2 to the right, we find

$$\underbrace{\frac{r}{\Phi_1}\frac{\partial\left(r\frac{\partial\Phi_1}{\partial r}\right)}{\partial r}}_{\text{does not vary with } \varphi} + \underbrace{\frac{1}{\Phi_2}\frac{\partial^2\Phi_2}{\partial\varphi^2}}_{\Gamma_\varphi^2} = \underbrace{(\Gamma^2 - \Gamma_z^2)r^2}_{\text{is a constant}} \tag{435}$$

if only the positive sign in front of Γ^2 is retained (holds for an outgoing wave propaga-

tion). Inasmuch as the first term does not vary with φ and the entire right-hand expression is constant, the remaining second term to the left also cannot depend upon φ and may be put equal to the constant Γ_φ^2. We have then the differential equations

$$\frac{r}{\Phi_1}\frac{d\left(r\frac{\partial\Phi_1}{\partial r}\right)}{dr} + \Gamma_\varphi^2 - (\Gamma^2 - \Gamma_z^2)r^2 = 0 = r\underbrace{\frac{d\left(r\frac{d\Phi_1}{dr}\right)}{dr}}_{r^2\frac{d^2\Phi_1}{dr^2} + r\frac{d\Phi_1}{dr}} + \Phi_1[\Gamma_\varphi^2 - (\Gamma^2 - \Gamma_z^2)r^2]$$

We arrive, therefore, at the Bessel equation

$$r^2\frac{d^2\Phi_1}{dr^2} + r\frac{d\Phi_1}{dr} + \Phi_1[\Gamma_\varphi^2 - r^2(\Gamma^2 - \Gamma_z^2)] = 0 \tag{436}$$

where Γ_z and Γ_φ may be taken as separation constants. Since for a nondissipative medium ($\sigma = 0$), we have $\Gamma = j\omega\sqrt{\mu\kappa} = j\omega/v$, where v is the phase velocity and ω/v denotes the phase constant β, we find $\Gamma^2 = -\omega^2/v^2 = -\beta^2$. For the same reason, we also have $(-\Gamma_z^2) = +\beta_z^2$ and $(\Gamma_\varphi^2) = -\beta_\varphi^2$. Hence,

$$r^2\frac{d^2\Phi_1}{dr^2} + r\frac{d\Phi_1}{dr} + [(\beta^2 - \beta_z^2)r^2 - \beta_\varphi^2]\Phi_1 = 0 \tag{437}$$

The standard Bessel equation reads

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

with the Bessel function $J_n(x)$, where n is the order and x the argument. Therefore, the phase constant β_φ along the φ cylinder coordinate ($\beta_\varphi = \omega/v_\varphi$) gives the order of the circular cylinder function, and $r\sqrt{\beta^2 - \beta_z^2}$ is the argument. We have then the particular solution $J_{\beta_\varphi}(r\sqrt{\beta^2 - \beta_z^2})$, since it is finite on the axis ($r = 0$) of the cylinder.

With respect to *spherical* coordinates, we can apply to the wave equation

$$\frac{1}{wr^2}\frac{\partial\left(r^2\frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{wr^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial w}{\partial\theta}\right)}{\partial\theta} + \frac{1}{wr^2\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} - \Gamma^2 = 0$$

the elementary solution $\Phi_1(r)$, and the combined elementary solution

$$\Phi'(\varphi, \theta) = \Phi_2(\theta)\Phi_3(\varphi) \tag{438}$$

and find, for $w = \Phi_1\Phi'$ as the abbreviated form,

$$\frac{\Phi'}{r^2\Phi_1\Phi'}\frac{\partial\left(r^2\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} + \frac{\Phi_1}{r^2\Phi_1\Phi'\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial\Phi'}{\partial\theta}\right)}{\partial\theta} + \frac{\Phi_1}{r^2\Phi_1\Phi'\sin^2\theta}\frac{\partial^2\Phi'}{\partial\varphi^2} - \Gamma^2 = 0$$

or

$$\frac{1}{\Phi_1}\frac{\partial\left(r^2\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} - r^2\Gamma^2 + \frac{1}{\Phi'\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial\Phi'}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\Phi'\sin^2\theta}\frac{\partial^2\Phi'}{\partial\varphi^2} = 0$$

Therefore,

$$\underbrace{\frac{1}{\Phi_1}\frac{\partial\left(r^2\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} - r^2\Gamma^2}_{\substack{\text{depends only on } r \text{ but}\\ \text{is } not \text{ a function of } \theta\\ \text{and } \varphi \text{ (may be put equal}\\ \text{to a constant } k^2)}} = \underbrace{-\frac{1}{\Phi'\sin\theta}\left[\frac{\partial\left(\sin\theta\frac{\partial\Phi'}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2\Phi'}{\partial\varphi^2}\right]}_{\substack{\text{depends only on } \theta \text{ and } \varphi \text{ but is } not \text{ a function of}\\ r \text{ (may be put equal to a constant } k^2)}}$$

The left side of this expression is, therefore, independent of the right side and vice versa. The equality sign requires that for possible r values, there must be possible respective φ and θ values that satisfy the equation. This is possible only if the left side as well as the right side of the equation are equal to a *common* constant k^2, which, for the sake of generality, is taken multivalued. We have, therefore, the first separation constant k, since, by means of it, we obtain *two separate* expressions

$$\frac{1}{\Phi_1}\frac{\partial\left(r^2\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} - r^2\Gamma^2 = k^2; \qquad -\frac{1}{\Phi'\sin\theta}\left[\frac{\partial\left(\sin\theta\frac{\partial\Phi'}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2\Phi'}{\partial\varphi^2}\right] = k^2$$

which also read

$$\left.\begin{aligned}
&\frac{\partial\left(r^2\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} - (r^2\Gamma^2 + k^2)\Phi_1 = 0 \qquad \text{is reducible to a Bessel equation}\\
&\sin\theta\frac{\partial\left(\sin\theta\frac{\partial\Phi'}{\partial\theta}\right)}{\partial\theta} + \frac{\partial^2\Phi'}{\partial\varphi^2} + k^2\Phi'\sin^2\theta = 0 \qquad \text{is a spherical harmonics equation, requiring further separation}
\end{aligned}\right\} \tag{439}$$

The separation of the second partial equation can be brought about by means of the elementary solutions $\Phi_2(\theta)$ and $\Phi_3(\varphi)$, since the nonabridged function $\Phi'(\varphi, \theta)$ is as given in (438). We then have

$$\Phi_3\sin\theta\frac{\partial\left(\sin\theta\frac{\partial\Phi_2}{\partial\theta}\right)}{\partial\theta} + \Phi_2\frac{\partial^2\Phi_3}{\partial\varphi^2} + k^2\Phi_2\Phi_3\sin^2\theta = 0$$

A division by $\Phi_2\Phi_3$ yields

$$\frac{\sin\theta}{\Phi_2}\frac{\partial\left(\sin\theta\frac{\partial\Phi_2}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\Phi_3}\frac{\partial^2\Phi_3}{\partial\varphi^2} + k^2\sin^2\theta = 0$$

or

$$\underbrace{\frac{\sin\theta}{\Phi_2}\frac{\partial\left(\sin\theta\frac{\partial\Phi_2}{\partial\theta}\right)}{\partial\theta} + k^2\sin^2\theta}_{\substack{= p^2\\ \text{(since only dependent upon } \theta \text{ and}\\ not \text{ a function of } \varphi)}} = \underbrace{-\frac{1}{\Phi_3}\frac{\partial^2\Phi_3}{\partial\varphi^2}}_{\substack{= p^2\\ \text{(since } not \text{ a}\\ \text{function of } \theta)}}$$

Hence, left and right sides are again mutually independent. We have again θ values that correspond to φ values in such a manner as to render each side of the equation equal to a common constant p^2, which can then be also multivalued. Therefore, p is the remaining separation constant, and we obtain the separate equations

$$\frac{\sin\theta}{\Phi_2}\frac{d\left(\sin\theta\dfrac{d\Phi_2}{d\theta}\right)}{d\theta} + k^2\sin^2\theta = p^2; \qquad -\frac{1}{\Phi_3}\frac{d^2\Phi_3}{d\varphi^2} = p^2$$

or

$$\frac{d^2\Phi_3}{d\varphi^2} + p^2\Phi_3 = 0 \qquad \text{well-known differential equation}$$

$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\dfrac{d\Phi_2}{d\theta}\right)}{d\theta} + \left(k^2 - \frac{p^2}{\sin^2\theta}\right)\Phi_2 \qquad \text{associated Legendre equation} \tag{440}$$

The separation constant k may be complex but is usually real and of the form $k^2 = q(q + 1)$ with q an integer. This is made use of in formulas (431). This can be seen from the following transformations and conditions

$$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\dfrac{d\Phi_2}{d\theta}\right)}{d\theta} + \left[q(q+1) - \frac{p^2}{\sin^2\theta}\right]\Phi_2 = 0 \tag{441}$$

where we have two separate and independent solutions: $P_q^p(\cos\theta)$ and $P_q^p(-\cos\theta)$, where q has *non*integral values. We then have

$$P_q^p(\cos\theta) = 2^p\cos\frac{(q+p)\pi}{2}\,\frac{\left(\dfrac{q+p-1}{2}\right)!\,\sin^p\theta}{\left(-\dfrac{1}{2}\right)!\left(\dfrac{q-p}{2}\right)!}\,\Phi(a)$$
$$+ 2^{p+1}\sin\frac{(p+q)\pi}{2}\,\frac{\left(\dfrac{p+q}{2}\right)!\,\sin^p\theta\cos\theta}{\left(\dfrac{p-q-1}{2}\right)!\left(-\dfrac{1}{2}\right)!}\,\Phi(b)$$

$$a = \underbrace{\frac{p+q+1}{2}}_{\alpha};\ \underbrace{\frac{p-q}{2}}_{\beta};\ \underbrace{\frac{1}{2}}_{\gamma};\ \underbrace{\cos^2\theta}_{} \quad\text{and}\quad b = \underbrace{\frac{p+q+2}{2}}_{\alpha};\ \underbrace{\frac{p-q+1}{2}}_{\beta};\ \underbrace{\frac{3}{2}}_{\gamma};\ \underbrace{\cos^2\theta}_{}$$

for the hypergeometric[1] series

$$\Phi(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1\cdot\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}z^2$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\gamma(\gamma+1)(\gamma+2)}z^3 - \cdots$$

[1] An excellent treatise is one by Chester Snow, Hypergeometric Functions and Legendre Functions with Applications to Integral Equations of the Potential Theory, *Nat. Bur. Stand. Pub.*, 1942.

for the four elements α, β, γ, and z, with only z variable. This series converges for all real values of z such that $-1 < z < 1$ but diverges if $z > 1$ and if $z < -1$. When $z = 1$, the series will converge if $\gamma > \alpha + \beta$ and will diverge if $\gamma \leqq \alpha + \beta$. For $z = -1$, the series converges for $\gamma + 1 > \alpha + \beta$, and it diverges for $\gamma + 1 \leqq \alpha + \beta$. It is also possible to interchange α and β; and if either of them is a negative integer, we have a finite number of terms. If γ is not a negative integer, we have no infinite terms. The hypergeometric function converges for $x < 1$. The above series for $P_q^p(\cos\theta)$ holds for real values of the argument (< 1).

Putting $\cos\theta = x$, we find $dx/d\theta = -\sin\theta$ or $d\theta = -dx/\sin\theta$, $\sin^2\theta = 1 - x^2$. Assuming only integer values $0, \pm 1, \pm 2, \pm 3, \cdots$, for p in (440), we find

$$(1 - x^2)\frac{d^2\Phi_2}{dx^2} - 2x\frac{d\Phi_2}{dx} + \left(k^2 - \frac{p^2}{1 - x^2}\right)\Phi_2 = 0 \tag{442}$$

upon such substitutions into the associated Legendre equations (440). Hypergeometric functions can then be used for solving the equation. For $p = 0$, Eq. (442) results in a Legendre equation; and for $k^2 = q(q + 1)$, we obtain the result of Eq. (441). We have then $q = 0, 1, 2, 3, \cdots$, and obtain Legendre polynomials $P_q(x)$. For integral values of p, we have the Legendre polynomials $P_q^p(x)$

$$P_q^p(x) = (-1)^p \sin^p\theta \frac{d^pP_q(x)}{dx^p} = (-1)^p(1 - x^2)^{0.5p}\frac{d^pP_q(x)}{dx^p}$$

and the function

$$\left.\begin{aligned} \Phi_2(x) &= (-1)^pP_q^p(x) = (1 - x^2)^{0.5p}\frac{d^pP_q(x)}{dx^p} \\ P_q(x) &= \sum_{r=0}^{\infty}\frac{(-1)^r(q + r)!}{(q - r)!(r!)^2}\sin^{2r}(0.5\theta) \end{aligned}\right\} \tag{443}$$

If q is an integer, $P_q(x)$ is a polynominal in x, and $P_q(-x)$ is distinguished by the factor $(-1)^q$. Therefore, the second solution of the Legendre equation has a function $Q_q(x)$ leading to

$$\left.\begin{aligned} Q_q^p(x) &= (-1)^p(1 - x^2)^{0.5p}\frac{d^pQ_q(x)}{dx^p} \\ Q_q(x) &= P_q(x)\log\left(\cot\frac{\theta}{2}\right) - \left(P_{q-1}P_0 + \frac{1}{2}P_{q-2}P_1 + \frac{1}{3}P_{q-3}P_2 \right. \\ &\qquad \left. + \cdots + \frac{1}{q}P_0P_{q-1}\right) \end{aligned}\right\} \tag{444}$$

for real values of θ.

As already mentioned, the factor p may be complex. If p is real and an integer, the function Φ_3 of (440) is periodic with period 2π. The $\Phi'(\varphi, \theta)$ differential equation (439), becomes, for $k = 0$,

$$\sin\theta \frac{\partial\left(\sin\theta \frac{\partial\Phi'}{\partial\theta}\right)}{\partial\theta} + \frac{\partial^2\Phi'}{\partial\varphi^2} = 0$$

$$\left.\begin{array}{ll}
\Phi'(\varphi, \theta) = F_1\left(\epsilon^{j\varphi}\cot\dfrac{\theta}{2}\right) + F_2\left(\epsilon^{-j\varphi}\cot\dfrac{\theta}{2}\right) & \text{general solution with arbitrary functions } F_1 \text{ and } F_2 \\
\Phi'(\varphi, \theta) = \cos(p\varphi)\cot^p\left(\dfrac{\theta}{2}\right); \quad \cos(p\varphi)\tan^p\left(\dfrac{\theta}{2}\right) & \\
\Phi'(\varphi, \theta) = \sin(p\varphi)\cot^p\left(\dfrac{\theta}{2}\right); \quad \sin(p\varphi)\tan^p\left(\dfrac{\theta}{2}\right) & \text{solutions, when periodic in } \varphi \\
\Phi'(\varphi, \theta) = \Phi'(\theta) = \log(\cot \tfrac{1}{2}\theta) & \text{solution independent of } \varphi
\end{array}\right\} \quad (445)$$

For the $\Phi_1(r)$ differential equation (439), we have, on account of

$$\Gamma^2 = (j\omega\sqrt{\mu\kappa})^2 \cong \underbrace{(j\omega\sqrt{\mu_0\kappa_0})^2}_{\text{for air}} = -\frac{\omega^2}{c^2} = -\beta_0^2 \qquad \text{for air;}$$

$$\frac{\partial\left(r^2\frac{\partial\Phi_1}{\partial r}\right)}{\partial r} = r^2\frac{\partial^2\Phi_1}{\partial r^2} + 2r\frac{\partial\Phi_1}{\partial r}$$

the differential equation for the $\Phi_1(r)$ function

$$r^2\frac{d^2\Phi_1}{dr^2} + 2r\frac{d\Phi_1}{dr} + (r^2\beta_0^2 - k^2)\Phi_1 = 0 \qquad (446)$$

If it were not for the factor 2, we should have a Bessel equation, the solution of which could be written directly, as was done in connection with Eq. (437). Putting $\Phi_1 = \Phi/\sqrt{r}$, we find the following derivation for the new function Φ and the abbreviation $a = r^2\beta_0^2 - k^2$.

$$\frac{d\Phi_1}{dr} = -\frac{1}{2}r^{-3/2}\Phi + r^{-1/2}\frac{d\Phi}{dr}$$

$$\frac{d^2\Phi_1}{dr^2} = -\frac{1}{2}\left(-\frac{3}{2}r^{-5/2}\Phi + r^{-3/2}\frac{d\Phi}{dr}\right) - \frac{1}{2}r^{-3/2}\frac{d\Phi}{dr} + r^{-1/2}\frac{d^2\Phi}{dr^2}$$
$$= \frac{3}{4}r^{-5/2}\Phi - r^{-3/2}\frac{d\Phi}{dr} + r^{-1/2}\frac{d^2\Phi}{dr^2}$$

Using these substitutions in (446), we find

$$r^2r^{-1/2}\left(\frac{3}{4}r^{-2}\Phi - r^{-1}\frac{d\Phi}{dr} + \frac{d^2\Phi}{dr^2}\right) + 2rr^{-1/2}\left(-\frac{1}{2}r^{-1}\Phi + \frac{d\Phi}{dr}\right) + ar^{-1/2}\Phi = 0$$

The common factor $r^{-1/2}$ cancels out, and we have

$$\frac{3}{4}\Phi - r\frac{d\Phi}{dr} + r^2\frac{d^2\Phi}{dr^2} - \Phi + 2r\frac{d\Phi}{dr} + a\Phi = 0 \qquad \text{or}$$

$$r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr} + \left(a - \frac{1}{4}\right)\Phi = 0$$

Introducing the value for the quantity a, we find

$$\left.\begin{aligned} &r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr} + \left(\beta_0^2r^2 - k^2 - \frac{1}{4}\right)\Phi = 0 \qquad \text{our result is a Bessel equation} \\ &x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{standard comparison Bessel equation} \end{aligned}\right\} \tag{447}$$

The standard Bessel equation can also be written as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) = 0$$

where n is the order of the functions playing a part and x is the argument. Since k is a separation constant, it *cannot* contain the quantity r, which may be varied, and $n = \sqrt{k^2 + 1/4}$ is the order and $x = \beta_0 r$ is the argument. Inasmuch as the Bessel equation is a second-order differential equation, we must generally expect two linear independent solutions. If $n = 1$, *i.e.*, $k^2 + 1/4 = 1$, we have first-order functions with a solution

$$\left.\begin{aligned} y &= AJ_1(x) + BN_1(x) \qquad \text{for standard form} \\ \Phi &= AJ_1(\beta_0 r) + BN_1(\beta_0 r) \qquad \text{for our result} \end{aligned}\right\} \tag{448}$$

where J_1 is the Bessel function and N_1 the Neumann function of the first order, with values given in appropriate tables. We may have other values for the order n. According as n is or is not an integer, we have

$$\left.\begin{aligned} y &= AJ_n(x) + BK_n(x) \qquad \text{for } n \text{ an integer } (1, 2, 3, \cdots) \\ y &= AJ_n(x) + BJ_{-n}(x) \qquad \text{for } n \textit{ not} \text{ an integer} \end{aligned}\right\} \tag{449}$$

where, for the Gauss function $\prod(n)$ which is the same as the Gamma function $\Gamma(n+1)$,

$$
\left.
\begin{aligned}
J_n(x) &= \frac{x^n}{2^n\prod(n)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right] \\
J_{-n}(x) &= \frac{x^{-n}}{2^{-n}\prod(-n)}\left[1 - \frac{x^2}{2(-2n+2)} \right. \\
&\qquad\qquad \left. + \frac{x^4}{2\cdot 4(-2n+2)(-2n+4)} - \cdots\right] \\
J_n(x) &= \sum_{\nu=0}^{\nu=\infty} \frac{(-1)^\nu}{\prod(n+\nu)\prod(\nu)}\left(\frac{x}{2}\right)^{n+2\nu} \\
J_{-n}(x) &= \sum_{\nu=0}^{\nu=\infty} \frac{(-1)^\nu}{\prod(-n+\nu)\prod(\nu)}\left(\frac{x}{2}\right)^{-n+2\nu} = \frac{n!}{\Gamma(n+1)}\sum_0^\infty \frac{(-1)^\nu x^{n+2\nu}}{2^{n+2\nu}\cdot\nu!(n+\nu)!} \\
K_n(x) &= J_n(x)\log x - \frac{x^{-n}}{2^{1-n}}\sum_0^{n-1}\frac{(n-\nu-1)!x^{2\nu}}{2^{2\nu}\nu!} \\
&\qquad - \frac{x^n}{2^{1+n}}\sum_0^\infty \frac{(-1)^\nu}{(n+\nu)!\nu!}\left[P_\nu + P_{\nu+n}\left(\frac{x}{2}\right)^{2\nu}\right] \\
P_\nu &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \quad + \frac{1}{\nu}
\end{aligned}
\right\} \tag{450}
$$

Tables[1] and curves for the Gauss and Gamma functions are given in the Appendix, and extended tables for the $J_n(x)$ functions are available today as well as for the other functions specified above.

Choosing $k^2 = q(q+1)$ as applied in (441) and utilized in (431), the result of (447) yields, for the parenthesis value,

$$
\begin{aligned}
\beta_0^2 r^2 - k^2 - \tfrac{1}{4} = \beta_0^2 r^2 - (k^2 + \tfrac{1}{4}) = \beta_0^2 r^2 - (q^2 + q + \tfrac{1}{4}) \\
= (\beta_0 r)^2 - (q + \tfrac{1}{2})^2
\end{aligned}
$$

and the result of (447) changes to the Bessel equation

$$
r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr} + \left[(\beta_0 r)^2 - \left(q + \frac{1}{2}\right)^2\right]\Phi = 0 \tag{451}
$$

expressing that our functions are of the order $n = q + \frac{1}{2}$ and argument

[1] Consult tables by E. Jahnke and F. Emde, tables issued by the Bureau of Standards, tables by N. W. McLachlan and tables in A. Hund, "Frequency Modulation," Appendix, McGraw-Hill Book Company, Inc., New York, 1942.

$\beta_0 r$ with a Bessel factor $J_{q+\frac{1}{2}}(\beta_0 r)$, and accounts for the solution mentioned in connection with (432).

The secondary wave $\mathcal{E}'$, H', due to the arrival of a primary wave $\mathcal{E}$, H at an obstructing medium, diverges at a great distance from the obstacle, and Eq. (446) yields a solution, for $q = 0$,

$$\Phi_1 = \frac{1}{r}\,\epsilon^{-jr\Gamma}$$

The divergent wave then has an amplitude decay proportional to $1/r$, where r is the distance from the radiation center. None of the solutions for Φ_1 has the correct form, since $J_{\frac{1}{2}}(r\Gamma) = A \sin (r\Gamma)$ and $J_{-\frac{1}{2}}(r\Gamma) = A \cos (r\Gamma)$ for $A = \sqrt{0.636/(r\Gamma)}$. For electromagnetic fields in air, we have then practically $(1/r) \cos [\beta_0(r - ct)]$ for progressive divergent electromagnetic fields and $(1/r) \cos [\beta_0(r + ct)]$ for progressive convergent waves. With the source at the origin, forced stationary waves are signified by $(1/r) \cos (\beta_0 r) \cos (\beta_0 ct)$, and free stationary waves by $(1/r) \sin (\beta_0 r) \cos (\beta_0 ct)$. According to Stokes,[1] for any value of the separation constant q, we have to use Bessel $b_q(x)$ and Neumann $n_q(x)$ functions for the divergent waves; and for $x = r\Gamma$, we have

$$b_q(x) = \sqrt{\frac{\pi x}{2}}\,J_{q+\frac{1}{2}}(x); \qquad n_q(x) = -\sqrt{\frac{\pi x}{2}}\,N_{q+\frac{1}{2}}(x);$$
$$\Gamma = j\omega\sqrt{\mu\kappa} = j\omega/c = j\beta \qquad (452)$$

which are connected with the spherical Hankel functions $h_q^{(1)}(x)$ and $h_q^{(2)}(x)$, as in

$$h_q^{(1)}(x) = b_q(x) - jn_q(x) = \sqrt{\frac{\pi x}{2}}\,H_{q+\frac{1}{2}}^{(1)}(x);$$
$$b_q(x) + jn_q(x) = \sqrt{\frac{\pi x}{2}}\,H_{q+\frac{1}{2}}^{(2)}(x) = h_q^{(2)}(x) \qquad (453)$$

When the real part of x is large and positive, the asymptotic series

$$h_q^{(1)}(x) = (-j)^{q+1}\epsilon^{jx}\left[1 + \frac{j}{2x}\,\frac{q(q+1)}{1!} - \frac{1}{(2x)^2}\,\frac{(q-1)q(q+1)(q+2)}{2!} + \cdots\right] \qquad (454)$$

holds. This expansion terminates when q is an integer, resulting in a true function. It is necessary only to change the sign for the $h_q^{(2)}(x)$ expansion. These formulas are all based on the cylinder function of

[1] *Phil. Trans.*, **158**, 447 (1868); "Collected Papers," Vol. 4, p. 321; and Rayleigh's "Sound," Vol. 2, p. 304.

half order that resulted in the expression (451) and is also given by the Φ_1 relation of (431). The solution of (451) is then generally given by $\Phi(r) = (\beta_0 r)^{-\frac{1}{2}} Z_{q+\frac{1}{2}}(\beta_0 r)$, since, for air, we have $r\Gamma = j\beta \cong j\beta_0$. The elementary wave function w_e, which is finite all over the surface of a sphere and of one value only, is then

$$\left.\begin{aligned} w_e &= \frac{1}{\sqrt{\beta_0 r}} Z_{q+\frac{1}{2}}(\beta_0 r) P_q^p \cos\theta \cos(p\varphi) \\ w_e &= \frac{1}{\sqrt{\beta_0 r}} Z_{q+\frac{1}{2}}(\beta_0 r) P_q^p \cos\theta \sin(p\varphi) \end{aligned}\right\} \tag{455}$$

Moreover, we have the formulas

$$\begin{aligned} b_q(x) &= \frac{x^{q+1}}{1\cdot 2\cdot 3 \cdots (2q+1)}\left[1 - \frac{x^2}{2(2q+3)} + \frac{x^4}{2\cdot 4(2q+3)(2q+5)} - \cdots\right] \qquad (456) \\ h_q^{(2)}(x) &= j^{q+1}\epsilon^{-jx}\left[1 - \frac{j}{2x}\frac{q(q+1)}{1} - \frac{1}{(2x)^2}\frac{(q-1)q(q+1)(q+2)}{2!} + \cdots \underbrace{(-j)^q\frac{(2q)!}{(2x)^q q!}}_{\text{final term for } q \text{ an integer}}\right] \end{aligned}$$

When q is an integer, which is generally the case,

$$b_q(x) = \tfrac{1}{2}[j^{q+1}\epsilon^{-jx} + (-j)^{q+1}\epsilon^{jx}], \qquad \text{if } r \text{ is large} \tag{457}$$

Therefore,

$$\left.\begin{aligned} b_q(x) &\to \frac{1}{x}\cos\left(x - \frac{1+q}{2}\pi\right) \\ n_q(x) &\to \frac{1}{x}\sin\left(x - \frac{1+q}{2}\pi\right) \end{aligned}\right\} \qquad \text{for large values of } x$$

which leads to the Hankel functions

$$\left.\begin{aligned} b_q(x) - jn_q(x) &\to \frac{1}{x}\epsilon^{-ja} \qquad \text{forward traveling wave} \\ b^q(x) + jn_q(x) &\to \frac{1}{x}\epsilon^{jd} \qquad \text{backward traveling wave} \end{aligned}\right\} \tag{458}$$

$$a = x - \frac{q+1}{2}\pi; \qquad d = x + \frac{q+1}{2}\pi$$

Since, at large distances, the leading terms are inversely proportional to

x, we have also an inverse distance r law. It shows why the transverse field associations in the distant radiation region are

$$\frac{1}{r}\epsilon^{-j\left(\beta_0 r - \frac{q-1}{2}\pi\right)} \qquad \text{functions}$$

The diffraction effects of a sphere made up of a general medium of σ mhos per meter, μ henrys per meter, and κ farads per meter refer to the expression of the equivalent *common field vector* of Eq. (420), which reads

$$C = B \pm jD\sqrt{\frac{\mu}{\kappa_e}} = \mu H + j\sqrt{\mu\kappa_e}\,\mathcal{E};$$
$$\kappa_e = \kappa - j\sigma/\omega \qquad \text{farads/m}$$

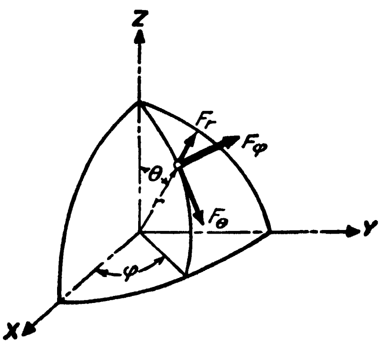


FIG. 50. Significant field vectors F. F may represent the electric as well as the magnetic field vector.

For respective field vectors F_r, F_φ, and F_θ in Fig. 50, which may stand for either the electric $\mathcal{E}$ or the magnetic H fields, we have the relations (373). We have found that generally the electromagnetic field depends on Bessel functions. All three equations of (431) play a part, since, as in (440), we also have a customary differential equation of form $d^2y/dx^2 + kx = 0$ and an associated Legendre equation. For respective field vectors F of Fig. 50, we deal with the wave equation

$$\frac{\partial\left(r^2\frac{\partial w}{\partial r}\right)}{\partial r} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial w}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} + \omega^2\mu\kappa_e r^2 w = 0 \qquad (459)$$

where the wave function $w = \Phi_1(r)\Phi_2(\theta)\Phi_3(\varphi)$ is

$$w = rH_r \text{ for } F_r = H_r; \quad F_\varphi = \mathcal{E}_\varphi; \quad F_\theta = H_\theta \qquad \text{TE waves} \qquad (460)$$
$$w = r\mathcal{E}_r \text{ for } \mathcal{E}_\theta,\ \mathcal{E}_r \text{ and } H_\varphi \text{ fields} \qquad \text{TM waves} \qquad (461)$$

and $\kappa_e = \kappa \cong \kappa_0$ for propagation in air. However, the solution of this wave equation gives *about* an *exponential law with* the *distance* r if we are in the distant radiation field (large $\beta r = \omega r/c$ values or large r values). Table 11 gives the comparisons for the field vectors indicated as F_r, F_φ, and F_θ in Fig. 50, which may represent $\mathcal{E}$ or H values, respectively.

Since we have the common field vector $C\epsilon^{j\omega t} = \mu H \pm j\sqrt{\mu\kappa_e}\,\mathcal{E}$ for the general medium, this field vector will satisfy the relations, for $\Gamma = j\omega\sqrt{\mu\kappa_e}$,

$$\pm C_r\Gamma = \frac{1}{r^2 \sin\theta}\left[\frac{\partial(rC_\varphi \sin\theta)}{\partial\theta} - \frac{\partial(rC_\theta)}{\partial\varphi}\right];$$

$$\pm C_\varphi\Gamma = \frac{1}{r}\left[\frac{\partial(rC_\theta)}{\partial r} - \frac{\partial C_r}{\partial\theta}\right];$$

$$\pm C_\theta\Gamma = \frac{1}{r\sin\theta}\left[\frac{\partial C_r}{\partial\varphi} - \frac{\partial(rC_\varphi \sin\theta)}{\partial r}\right] \quad (462)$$

for the spherical components along the r, φ, and θ coordinates. If $F = u \pm jv$ is a solution, we find

$$C_r = \frac{\partial^2(rF)}{\partial r^2} + rF\Gamma^2; \qquad C_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2(rF)}{\partial r\,\partial\varphi} \mp \frac{\Gamma}{r}\frac{\partial(rF)}{\partial\theta},$$

$$C_\theta = \frac{1}{r}\frac{\partial^2(rF)}{\partial r\,\partial\theta} \pm \frac{\Gamma}{r\sin\theta}\frac{\partial(rF)}{\partial\varphi} \quad (463)$$

TABLE 11

Longitudinal H_r or TE waves	Longitudinal $\mathcal{E}_r$ or TM waves	Remarks	
$H_r = \frac{w}{r}$	$\mathcal{E}_r = \frac{w}{r}$	$\sigma = 0$; κ and μ medium	
$H_\varphi = \frac{1}{ab}\frac{\partial^2(rw)}{\partial r\,\partial\varphi}$	$\mathcal{E}_\varphi = \frac{1}{ab}\frac{\partial^2(rw)}{\partial r\,\partial\varphi}$	$\frac{d\left(r^2\frac{d\Phi_1}{dr}\right)}{dr} + [(\beta r)^2 - a]\Phi_1 = 0$	(431)
$H_\theta = \frac{1}{ar}\frac{\partial^2(rw)}{\partial r\,\partial\theta}$	$\mathcal{E}_\theta = \frac{1}{ar}\frac{\partial^2(rw)}{\partial r\,\partial\theta}$	$\frac{1}{\sin\theta}\frac{d\left(\sin\theta\frac{d\Phi_2}{d\theta}\right)}{d\theta} + \left(a - \frac{p^2}{\sin^2\theta}\right)\Phi_2 = 0$	
$\mathcal{E}_r = 0$	$H_r = 0$		
$\mathcal{E}_\varphi = j\frac{\omega\mu}{ar}\frac{\partial(rw)}{\partial\theta}$	$H_\varphi = -j\frac{\omega\kappa}{ar}\frac{\partial(rw)}{\partial\theta}$		
$\mathcal{E}_\theta = -j\frac{\omega\mu}{ab}\frac{\partial(rw)}{\partial\varphi}$	$H_\theta = j\frac{\omega\kappa}{ab}\frac{\partial(rw)}{\partial\varphi}$	$\frac{d^2\Phi_3}{d\varphi^2} + p^2\Phi_3 = 0$	
		$w = \Phi_1(r)\Phi_2(\theta)\Phi_3(\varphi)$	
		$a = q(q+1)$; $b = r\sin\theta$; $\beta = \omega\sqrt{\mu\kappa} = \frac{\omega}{v}$	
		$v \cong c$ for propagation in air	

The electric and magnetic vectors can be derived from these relations by equating ambiguous and nonambiguous terms. In the relation $C = \mu H \pm j\sqrt{\mu\kappa_e}\,\mathcal{E}$, the term $j = \sqrt{-1}$ and the ambiguous sign $\pm$ do not depend upon the ambiguity, which happens when we determine $\sqrt{-1}$. Hence, a pair of vectors $\mathcal{E}$ and H can be found by equating coefficients of ambiguous sign.

58. Relations for Electromagnetic Radiation along a Spherical Earth Surface. Reference is made to Fig. 47 where, in connection with TE, TM, and TEM waves, all possible spherical components are analyzed. If $\mathcal{E}_\theta$, $\mathcal{E}_r$, and H_φ are the significant components, we find from Eq. (373), for the general medium,

$$(\sigma + j\omega\kappa)\mathcal{E}_r = \frac{1}{r \sin\theta}\frac{\partial(H_\varphi \sin\theta)}{\partial\theta}; \qquad (\sigma + j\omega\kappa)\mathcal{E}_\theta = -\frac{1}{r}\frac{\partial(rH_\varphi)}{\partial r};$$

$$-j\omega\mu H_\varphi = \frac{1}{r}\left[\frac{\partial(r\mathcal{E}_\theta)}{\partial r} - \frac{\partial\mathcal{E}_r}{\partial\theta}\right] \qquad (464)$$

Since $H_r = 0$ for TM waves, no magnetic displacement current is possible across the spherical surface (r = constant). For such surfaces, we can then derive the $\mathcal{E}$ field from the *scalar* potential V and connect it with the Hertz vector Π. Hence,

$$\mathcal{E}_\theta = -\frac{\partial V/\partial\theta}{r} \text{ volts/m}; \qquad V = -\frac{\partial(r\Pi)}{\partial r} \text{ volts} \qquad (465)$$

since V depends on the distance r. We have then

$$\mathcal{E}_\theta = \frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r\,\partial\theta} \qquad \text{volts/m} \qquad (466)$$

We must deal with complex quantities, since we have the time function $\epsilon^{j\omega t} = \cos\omega t + j\sin\omega t$. Therefore, Π, $\mathcal{E}$, H, V are all complex time functions. Introducing (466) into the first two expressions of (464) yields

$$\mathcal{E}_r = -\frac{1}{r\sin\theta}\frac{\partial\left(\frac{\partial\Pi}{\partial\theta}\sin\theta\right)}{\partial\theta}; \qquad H_\varphi = -j\omega\kappa_e\frac{\partial\Pi}{\partial\theta} \qquad (467)$$

Since for the complex dielectric constant κ_e of the medium we have the propagation constant $\Gamma = j\omega\sqrt{\mu\kappa_e} = \alpha + j\beta$ and a phase velocity $c' = v = \omega/\beta$ meters per second, as well as $\Gamma^2 = -\omega^2\mu\kappa_e = -(\omega/v)^2 = -\beta^2$, we find from the third relation of (464)

$$\frac{\partial\Pi}{\partial\theta}\Gamma^2 = \frac{1}{r}\left\{\frac{\partial^3(r\Pi)}{\partial^2 r\,\partial\theta} + \frac{\partial\left[\frac{1}{r\sin\theta}\frac{\partial\left(\frac{\partial\Pi}{\partial\theta}\cos\theta\right)}{\partial\theta}\right]}{\partial\theta}\right\}$$

or
$$\underbrace{\frac{1}{r}\left[\frac{\partial^2(r\Pi)}{\partial r^2} + \frac{1}{r\sin\theta}\frac{\partial\left(\frac{\partial\Pi}{\partial\theta}\sin\theta\right)}{\partial\theta}\right]}_{\nabla^2 w} + \underbrace{\beta^2\Pi}_{w\Gamma^2} = 0 \qquad (468)$$

which is again the *general wave equation for spherical coordinates.*

Suppose that the earth is a perfect sphere and acts like a perfect conductor ($\sigma = \infty$). The latter condition holds only *roughly* up to the standard broadcast range, since the effect of the dielectric constant (5 to 25)κ_0 farads per meter is negligible at such frequencies. The conductivity is by no means perfect, even for ocean water (only about 5 mhos/m), but good enough to account for approximate diffraction formulas. For quite a wide range of frequency, the earth crust behaves then like an

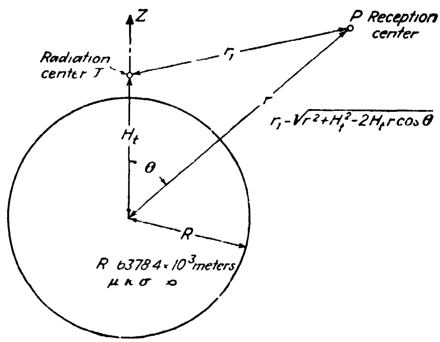


FIG. 51. Conditions for a perfectly conducting and spherical earth surface. For the neighborhood region the area at $r = H_t$ m is essentially plane.

opaque sphere to electromagnetic waves and, so to speak, as an *imperfect* reflector with a *blurred* image effect (if the ray theory were used). Along the earth's surface, electromagnetic diffraction then takes place. Assuming that the effect due to conduction currents plays a role mostly in comparison with the electric displacement current, the electromagnetic field essentially vanishes below the earth surface, and

$$(\mathcal{E}_\theta)_{r=R} \cong 0 = \left[\frac{1}{r}\frac{\partial^2(r\Pi)}{\partial r\,\partial\theta}\right]_{r=R} \tag{469}$$

With a transmitter T (Fig. 51) at location H_t, $\varphi = 0$, $\theta = 0$, we assume a radiation from an elementary electric dipole. In the vicinity of the radiator, we have, at a distance H_t from the center of the spherical obstacle, essentially a plane surface; and at a distance $r_1 = d$ from the primary radiation center, we find the Hertz vector

$$\Pi = \frac{1}{d}\left[F\left(t - \frac{d}{c}\right) + G\left(t + \frac{d}{c}\right)\right]$$

where $F(t - d/c)$ is an *outgoing* and $F(t + d/c)$ a returning wave motion function and, of course, in the above general solution, *only the outgoing* function $F(t - d/c)$ is of concern and $c = 3 \times 10^8$ m/sec, since so far we deal only with propagation in air. We have then for the primary

radiation coming from T

$$\prod_p = \frac{A\epsilon^{-r_1\Gamma}}{r_1} = \frac{A\epsilon^{-j\beta r_1}}{r_1} \qquad \text{m} \cdot \text{volts} \tag{470}$$

at the receiving center P, where $\beta = \omega\mu_0\kappa_0 = \omega/c$. Hence, when such radiation arrives at points P on the surface of the earth, it must cause a *forced* oscillation in the medium of the earth surface due to free and bound charges oscillating in step with the incident $\mathcal{E}$, H field. For the general medium, we have $\Gamma = j\omega\sqrt{\mu\kappa_c}$, since σ also is present. Secondary fields $\mathcal{E}'$, H' and $\mathcal{E}''$, H'' are then produced on either side of the earth surface. It is the secondary field $\mathcal{E}'$, H' between the earth surface and the primary source at T that is of concern with a corresponding Hertz vector $\prod_s$, and we have

$$\prod = \prod_p + \prod_s \tag{471}$$

as the *total* Hertz vector, which satisfies the *outer* boundary condition of the surface of the earth. The singularity of $\prod_p$ at T, has then the form as given by (470), while $\prod_s$ can have no singularity. The wave equation (468) determines the characteristic structure of the secondary radiation. We can then operate with $\prod_s = \prod_\theta \prod_r$ for solving (468), where $\prod_\theta$ depends *only on* θ and $\prod_r$ *only on* r. As has been shown by means of the separation constant of the form $q(q + 1)$, we can split up the partial differential equation (468) into two total differential equations

$$\left.\begin{aligned} &\frac{1}{\sin\theta}\frac{d\left(\frac{d\prod_\theta}{d\theta}\sin\theta\right)}{d\theta} + q(q+1)\prod_\theta = 0 \qquad \text{associated Legendre equation} \\ &\frac{d^2(r\prod_r)}{dr^2} + \left[\frac{\omega^2}{v^2} - \frac{q(q+1)}{r^2}\right]\left(r\prod_r\right) = 0 \qquad \text{Bessel equation} \end{aligned}\right\} \tag{472}$$

Such a choice of separation constant has the property that

$$q(q+1) \equiv \left(q + \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{4(q + \frac{1}{2})^2 - 1}{4}$$

and

$$\left.\begin{aligned} &\frac{d^2(r\prod_r)}{dr^2} + \left[\beta^2 - \frac{4(q + \frac{1}{2})^2 - 1}{4r^2}\right]\left(r\prod_r\right) = 0 \qquad \text{our result} \\ &\frac{d^2y}{dx^2} + \left(\beta^2 - \frac{4n^2 - 1}{4x^2}\right)y = 0 \qquad \text{standard Bessel form with solution } y = \sqrt{x}\,Z_n(\beta x) \end{aligned}\right\} \tag{473}$$

This is an equation of the Bessel family, dependent on *any cylinder functions* $Z_n(x)$ of the order $n = q + \frac{1}{2}$ and argument $x = \beta r$, where $\beta = \omega/v$, which is ω/c for propagation in atmosphere. The function $Z_n(x)$ is an abbreviation for $k_1 J_n(x) + k_2 N_n(x)$, where k_1 and k_2 are arbitrary (real or complex) constants. The solution of (473) is then

$$r\prod_r = \sqrt{r}\, Z_{q+\frac{1}{2}}(\beta r) \tag{474}$$

Inasmuch as the electromagnetic field must vanish at infinity, we have the Hankel[1] function of the second kind, which is of form $H^{(2)}_{q+\frac{1}{2}}(\beta r)$ in addition to such factors, explained on page 332 and given by (453), as

$$r\prod_r = \sqrt{\frac{\pi}{2}\beta r}\, H^{(2)}_{q+\frac{1}{2}}(\beta r) \equiv h^{(2)}_q(\beta r) \tag{475}$$

where $h^{(2)}_q(x) \equiv pJ_{q+\frac{1}{2}}(x) - jpN_{q+\frac{1}{2}}(x)$, according[2] to (453).

For the solution of the Legendre equation of (472), we apply the same procedure as employed in connection with Eqs. (440) and (442). The Legendre equation is

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + q(q+1)y = 0 \qquad \text{or}$$

$$\frac{d\left[(1 - x^2)\dfrac{dy}{dx}\right]}{dx} + q(q+1)y = 0 \tag{476}$$

[1] The Bessel differential equation $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} + \left(1 - \dfrac{n^2}{x^2}\right)y = 0$, has the general integral $Z_n(x) = k_1 J_n(x) + k_2 N_n(x)$, where $J_n(x)$ is the Bessel function of the first kind of order n and $N_n(x)$ is the Neumann function, which is a Bessel function of the second kind and order n, both functions having the common argument x. This means that we have either $y = J_n(x)$ or $y = N_n(x)$ in the above differential equation. The $J_n(x)$ solution is everywhere finite, while the second solution $N_n(x)$ is infinite at the origin ($x = 0$). The functions $H^{(1)}_n(x)$ and $H^{(2)}_n(x)$ are Bessel functions of the third kind and order n and known as "Hankel functions." For zero-order Hankel functions, we have

$$H^{(1)}_0(x) = J_0(x) + jN_0(x); \qquad H^{(2)}_0(x) = J_0(x) - jN_0(x)$$

and one of the Hankel functions always becomes zero with infinite complex arguments. For other details see pp. 295, 322, and 497.

[2] Since in our applications $q = 0, 1, 2, 3$ or any other integer, we have $Z_{\frac{1}{2}}(x)$, $Z_{1.5}(x)$, $Z_{2.5}(x)$, $Z_{3.5}(x)$, $\cdots$, and the portions $pJ_{q+\frac{1}{2}}(x)$ with $p = \sqrt{\pi x/2}$, then give simple expressions, such as $pJ_{\frac{1}{2}}(x) = \sin x$,

$$pJ_{1.5}(x) = \frac{\sin x}{x} - \cos x; \qquad pJ_{2.5}(x) = \left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x$$

$$pJ_{3.5}(x) = \left(\frac{15}{x^3} - \frac{6}{x}\right)\sin x - \left(\frac{15}{x^2} - 1\right)\cos x \cdots$$

in which the quantity q is constant. If we restrict q to a positive integer, try a solution in the form $y = 1 + ax + bx^2 + cx^3 + \cdots$, and call $y_q(x)$ the answers, we have $y_0(x) = 1$ for $q = 0$, $y_1(x) = x$ for $q = 1$, $y_2(x) = (3x^2/2) - \frac{1}{2}$ for $q = 2$, $y_3(x) = (5x^3/2) - 3x/2$ for $q = 3$, $y_4(x) = (35x^4/8) - (30x^2/8) + 3$ for $q = 4$, and for $q = n$ and for $x = \cos\theta$

$$\begin{aligned} y_n(x) &= 2\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}\left\{\cos(n\theta) + \frac{1}{1}\frac{n}{2n-1}\cos[(n-2)\theta]\right. \\ &\quad + \frac{1\cdot 3}{1\cdot 2}\frac{n(n-1)}{(2n-1)(2n-3)}\cos[(n-4)\theta] \\ &\quad \left.+ \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\cos[(n-6)\theta] + \cdots\right\} = P_n(\cos\theta) \\ &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2}\right. \\ &\quad \left. + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right] \end{aligned} \tag{477}$$

where $P_n(\cos\theta)$ is the general form that applies to our $P_q(\cos\theta)$ cases. The quantities $P_0(x)$, $P_1(x)$, $P_2(x)$, $\cdots$ are the zonal harmonic spherical or Legendre functions of the zeroth, first, second, etc., order. The solid zonal harmonic of the qth degree, $P_q(\cos\theta)$, is also called the surface zonal harmonic of the qth degree. There are also functions of the type P_q^p as already brought out in connection with Eqs. (431) and (441). We have then, for instance, $P_1^1(x) = \sqrt{1 - x^2} = \sin\theta$, for $x = \cos\theta$ and $q = 1 = p$; $P_2^1(x) = 3x\sqrt{1 - x^2} = {}^3\!/_2 \sin 2\theta$, for $q = 2$, $p = 1$ and $x = \cos\theta$; similarly, we find $P_2^2(x) = 3(1 - x^2) = {}^3\!/_2(1 - \cos 2\theta)$,

$$P_3^1(x) = {}^3\!/_2\sqrt{1 - x^2}\,(5x^2 - 1) = {}^3\!/_8(\sin\theta - 5\sin 3\theta),$$

etc., where

$$P_q^p(\cos\theta) = \frac{(p+q)!}{2\pi q!}j^{-p}\int_0^{2\pi}(\cos\theta - j\sin\theta\cos\varphi')^q\cos(p\varphi')\,d\varphi' \tag{478}$$

since we have to account for another axis of reference. The customary location of a space point in spherical coordinates is r, θ, and φ, where r is the distance from the origin O of a Cartesian coordinate system and given by $r = \sqrt{x^2 + y^2 + z^2}$, θ is the angle of r with the Z axis making the height at the point P above the XY plane, $z = r\cos\theta$, and φ is the angle that the projection of r on the XY plane makes with the X axis. Another space point P', also a distance r from the origin (on the same spherical surface), is oriented at r, θ', and φ'. Calling the first point on the sphere P and the other point on the sphere P', we have $OP = r$, the polar axis of the first point, and $OP' = r$, the other polar axis for P'. The zonal harmonics at space point P with respect to OP' are expressed by $P_q(\cos\psi)$, where ψ is the angle that OP' makes with OP. Only the angle φ' enters relation (478).

The Laplace equation $\nabla^2 V = 0$, where V is the scalar potential at point $P(x, y, z)$ in *free* space, can be written as

$$0 = \nabla^2 V = \underbrace{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}_{\text{in Cartesian coordinates}} = \underbrace{\frac{\partial\left(r^2\frac{\partial V}{\partial r}\right)}{\partial r} + \frac{1}{\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial V}{\partial\theta}\right)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}}_{\text{in spherical coordinates}} \tag{479}$$

and can be solved like the scalar w instead of the scalar V by a spherical harmonic of the nth degree. This can be seen also from the following: If w or V is expanded by Taylor's theorem as a series of positive integral powers of x, y, and z, where $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, and $z = r \cos \theta$, we find

$$w = a_0 + a_1x + b_1y + c_1z + a_2x^2 + b_2y^2 + c_2z^2 + d_2yz + e_2xz + f_2xy + a_3x^3 + b_3y^3 + \cdots$$

which series can also be written as

$$\begin{aligned} w &= S_0(x, y, z) + S_1(x, y, z) + S_2(x, y, z) + \cdots + S_n(x, y, z) \\ S_0(x, y, z) &= a_0; \qquad S_1(x, y, z) = a_1x + b_1y + c_1z; \\ S_2(x, y, z) &= a_2x^2 + b_2y^2 + c_2z^2 + d_2yz + e_2xz + f_2xy \\ S_n(x, y, z) &= a_nx^n + b_ny^n + c_nz^n + d_n(yz)^{n-1} + e_n(xz)^{n-1} + f_n(xy)^{n-1} \end{aligned}$$

where the S_0 function stands for the term of zeroth degree, the S_1 function for the group of terms of the first degree, and generally $S_n(x, y, z)$ for the group of terms of the nth degree and the function S_n is a spherical harmonic of the degree n. Using the Laplace coordinate transformation from spherical to Cartesian coordinates, namely, $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, and $z = r \cos \theta$, then the spherical harmonic $S_n(x, y, z)$ of the nth degree can be written as $r^n\Phi_n(\theta, \varphi)$, where the double function $\Phi_n(\theta, \varphi)$ does not depend upon r. The double function is then called a "surface harmonic." If in the relation of (479), the expression in spherical coordinates, the scalar V is assumed to be the product of a function of r *alone* and a function of θ *alone* (as shown on page 326), we have the solution

$$V = \sum_{n=0}^{n=q} \left(A_q r^q + \frac{B_q}{r^{q+1}} \right) P_q (\cos \theta) \tag{480}$$

where q is a positive integer like 0, 1, 2, 3, $\cdots$, A_q and B_q are arbitrary constants, and $P_q(\cos \theta)$ are surface zonal harmonics given by (477). It can be shown that $S_n(x, y, z)/r^{2n+1}$ is a solution of $\nabla^2 V = 0$, since it is a spherical harmonic of the $(n + 1)$th degree with $S_n(x, y, z)$ as solid harmonics, which confirms (480).

If we assume $\cos \theta = x$ in Eq. (472), we find $dx/d\theta = -\sin \theta$, and

$$\sin \theta \frac{d\Pi_\theta}{d\theta} = \sin \theta \left(\frac{d\Pi_v}{-\dfrac{dx}{\sin \theta}} \right) = -\sin^2 \theta \frac{d\Pi_v}{dx} = -(1 - x^2) \frac{d\Pi_v}{dx};$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} = \frac{1}{\sin \theta} \frac{d}{-\dfrac{dx}{\sin \theta}} = -\frac{d}{dx}$$

Using these substitutions in (472), we find

$$\frac{d[(1 - x^2)\, d\Pi_v/dx]}{dx} + q(q + 1)\Pi_v = 0 \tag{481}$$

which is the Legendre equation (176) for $y = \Pi_v$. According to the discussion of this equation just given, we have the solution

$$\Pi_v = P_q(x) = P_q(\cos \theta) \tag{482}$$

since (481) is satisfied by a spherical function of the qth order. Since (472) is the outcome of (468) and the separation was accomplished by means of the operation $\Pi_s = \Pi_r \Pi_\theta$, we find the secondary Hertz vector

$$\Pi_s = \sum_{q=0}^{q=\infty} A_q \frac{h_q^{(2)}(\beta r)}{r} P_q(\cos\theta) \qquad \text{m} \cdot \text{volts} \qquad (483)$$

We deal with a linear wave equation (468), and an *infinite* number of partial terms make up the secondary radiation effect due to the spherical obstacle. We can also write the primary radiation effect as the sum of an *infinite* number of terms of spherical functions and find for (470)

$$\left.\begin{aligned} \Pi_p &= \frac{A\epsilon^{-j\beta r_1}}{r_1} = A \sum_{q=0}^{q=\infty} B_q \frac{h_q^{(2)}(\beta r)}{r} P_q(\cos\theta) \\ &\qquad\qquad \text{for } r > H_t \\ \Pi_p &= \frac{A\epsilon^{-j\beta r_1}}{r_1} = A \sum_{q=0}^{q=\infty} B'_q \frac{b_q(\beta r)}{r} P_q(\cos\theta) \\ &\qquad\qquad \text{for } r < H_t \end{aligned}\right\} \text{m} \cdot \text{volts} \qquad (484)$$

and where, according to (452) and (453),

$$B_q = -2j\left(q + \frac{1}{2}\right)\frac{b_q(\beta H_t)}{\beta H_t}; \qquad B'_q = -2j\left(q + \frac{1}{2}\right)\frac{h_q^{(2)}(\beta H_t)}{\beta H_t};$$

$$b_q(\beta r) = \sqrt{0.5\pi\beta r}\, J_{q+\frac{1}{2}}(\beta r)$$

$$h_q^{(2)}(\beta r) = \sqrt{0.5\pi\beta r}\, H_{q+\frac{1}{2}}^{(2)}(\beta r) \qquad (485)$$

We must add the effects of the primary and secondary radiation and find for $H_t > r$

$$\Pi = \Pi_p + \Pi_s = \sum_{q=0}^{q=\infty}\left[A_q \frac{h_q^{(2)}(\beta r)}{r} + AB'_q \frac{b_q(\beta r)}{r}\right] P_q(\cos\theta)$$

$$\text{m} \cdot \text{volts} \qquad (486)$$

The factor A_q is found from the boundary condition (469), holding at the surface of the spherical earth ($r = R$), and

$$A_q \left\{\frac{d[h_q^{(2)}(\beta r)]}{dr}\right\}_{r=R} + AB'_q \left\{\frac{d[b_q(\beta r)]}{dr}\right\}_{r=R} = 0 \qquad (487)$$

Hence,

$$A_q = -AB'_q \frac{\dfrac{d[b_q(\beta R)]}{dr}}{\dfrac{d[h_q^{(2)}(\beta R)]}{dr}} \qquad (488)$$

We find then the total Hertz vector for the space outside the sphere

$$\Pi = A \sum_{q=0}^{q=\infty} \left\{ B_q - B'_q \frac{\dfrac{d[b_q(\beta R)]}{dr}}{\dfrac{d[h_q^{(2)}(\beta R)]}{dr}} \right\} \frac{h_q^{(2)}(\beta r)}{r} P_q(\cos\theta)$$

$$= -\frac{2A}{\beta r R} \sum_{q=0}^{q=\infty} \left(q + \frac{1}{2}\right) j \frac{b_q(\beta R)\dfrac{d[h_q^{(2)}(\beta R)]}{dr} - \dfrac{d[b_q(\beta R)]}{dr} h_q^{(2)}(\beta R)}{\dfrac{d[h_q^{(2)}(\beta R)]}{dr}}$$

$$\times\, h_q^{(2)}(\beta R) P_q(\cos\theta)$$

$$= -\frac{2A}{\beta r R} \sum_{q=0}^{q=\infty} \left(q + \frac{1}{2}\right) \frac{h_q^{(2)}(\beta r)}{\dfrac{d[h_q^{(2)}(\beta R)]}{dr}} P_q(\cos\vartheta) \qquad (489)$$

since

$$j\left\{ b_q(\beta R) - \frac{d[h_q^{(2)}(\beta R)]}{dr} - \frac{d[b_q(\beta R)]}{dr} h_q^{(2)}(\beta R) \right\} \equiv 1$$

according to the elementary $k_1 J_n(x) + k_2 N_n(x) = Z_n(x)$ relations and differential formulas such as, for argument x in the abbreviated form $Z_n(x) = Z_n$,

$$Z_{n-1} + Z_{n+1} = \frac{2n}{x} Z_n; \qquad N_{n-1}J_n - N_n J_{n-1} = \frac{2}{\pi x};$$

$$\frac{dZ_n}{dx} = \frac{1}{2}(Z_{n-1} - Z_{n+1})$$

$$\frac{d[x^m Z_n(px)]}{dx} = p x^m Z_{n-1}(px)$$

where k_1 and k_2 are real or complex constants.

By means of the residue theorem, the series of (489) can be changed to

$$\Pi = -\frac{A}{j\beta r R} \int_{\text{contour}} \frac{h_{t-\frac{1}{2}}^{(2)}(\beta r)}{\dfrac{d[h_{t-\frac{1}{2}}^{(2)}(\beta R)]}{dr}} P_{t-\frac{1}{2}}(-\cos\theta) \frac{t\,dt}{\cos(\pi t)} \qquad (490)$$

where the integration contour surrounds the negative half plane below the real axis, since the integrand is an odd function of t and because $h_{-t-\frac{1}{2}}^{(2)} = \epsilon^{-j\pi t} h_{t-\frac{1}{2}}^{(2)}$ and $P_{-t-\frac{1}{2}}(\cos\theta) = P_{t-\frac{1}{2}}(\cos\theta)$. For large values of q, we have the asymptotic formula

$$P_q(\cos\theta) = \sqrt{\frac{2}{\pi q \sin\theta}}\sin\left[\left(q + \frac{1}{2}\right)\theta + \frac{\pi}{4}\right];$$

$$\frac{P_{t-\frac{1}{2}}(-\cos\theta)}{\cos(\pi t)} = \sqrt{\frac{2}{\pi(t - \frac{1}{2})\sin\theta}} = \frac{\sin\left[t(\pi - \theta) + \frac{\pi}{4}\right]}{\cos(\pi t)}$$

$$\cong \sqrt{\frac{2}{\pi t \sin\theta}}\,\epsilon^{-j\left(\theta t + \frac{\pi}{4}\right)}$$

which, introduced in (490), results in

$$\Pi = -\frac{A}{j\beta r R}\epsilon^{-j\frac{\pi}{4}}\sqrt{\frac{0.637}{\sin\theta}}\int_{\text{contour}} \frac{h^{(2)}_{t-\frac{1}{2}}(\beta r)}{\frac{d[h^{(2)}_{t-\frac{1}{2}}(\beta R)]}{dr}}\epsilon^{jt\theta}\sqrt{t}\,dt \tag{491}$$

This relation is now applicable to any space point outside the spherical earth surface. At the surface of the earth, we have $r = R$ and

$$\Pi = j\frac{A}{\beta R^2}\epsilon^{-j\frac{\pi}{4}}\sqrt{\frac{0.637}{\sin\theta}}\int_{\text{contour}} \frac{h^{(2)}_{t-\frac{1}{2}}(\beta R)}{\frac{d[h^{(2)}_{t-\frac{1}{2}}(\beta R)]}{dr}}\epsilon^{-jt\theta}\sqrt{t}\,dt \tag{492}$$

For large values of $\beta R = (\omega/c)R$, we find

$$\left.\begin{aligned}
h^{(2)}_{t-\frac{1}{2}}(\beta R) &\cong \frac{j}{\sqrt{\sin\delta}}\cos\underbrace{\left[\frac{\pi}{4} + \frac{\omega}{c}R(\sin\delta - \delta\cos\delta)\right]}_{\alpha} \\
&= j\frac{\cos\alpha}{\sqrt{\sin\delta}} \\
\frac{d[h^{(2)}_{t-\frac{1}{2}}(\beta R)]}{dr} &\cong j\sqrt{\sin\delta}\,\sin\alpha \\
\cos\delta = \frac{t}{\beta R} = \frac{tc}{\omega R};\qquad &\sin\delta = \frac{c}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{t}{R}\right)^2}
\end{aligned}\right\} \tag{493}$$

For a distance TG between the radiation center T and the surface point G on the earth's surface, we have for the total Hertz vector

$$\Pi = j\frac{A}{R}\epsilon^{-j\frac{\pi}{4}}\sqrt{\frac{0.637}{\sin\theta}}\int_{\text{contour}} \frac{\sqrt{t}\,\epsilon^{-jt\theta}\,dt}{\sqrt{t^2 - \left(\frac{\omega R}{c}\right)^2}}$$

$$= \frac{2A\epsilon^{-j\frac{\omega}{c}R\theta}}{R\sqrt{\theta\sin\theta}} = \frac{2A}{\sqrt{d}}\,\frac{\epsilon^{-j\frac{\omega}{c}R\theta}}{\sqrt{R\sin\theta}} \tag{494}$$

where $d = R\theta$ is the distance of the surface point G from the radiation center T and $R \sin \theta$ is the radius of the parallel latitude through G. This expression holds only for distances TG that are not too large, and Eq. (493) has to be evaluated for larger distances. The factor $\sin \delta$ is regular, and we have zero positions when

$$\frac{\pi}{4} + \frac{\omega}{c} R(\sin \delta - \delta \cos \delta) = p\pi \text{ for } p = 0, 1, 2, 3, \cdots$$

with values

$$t_p = \frac{\omega}{c} R + \rho_p \sqrt[3]{\frac{\omega}{c} R}\, \epsilon^{-j\frac{\pi}{3}}; \qquad \rho_p = \frac{1}{2}\left(\frac{3\pi}{4}\right)^{2/3} \sqrt{1 + 4p}$$

giving $\rho_0 = 0.81$, $\rho_1 = 2.37$, $\rho_2 = 3.83$, $\cdots$, and

$$\prod = -\frac{A}{R} \sqrt{\frac{2\pi}{\frac{\omega}{c} R}}\, \epsilon^{-j\left(\frac{\pi}{4} - \theta\frac{\omega}{c}R\right)} \left(\frac{\omega}{c}\right)^{2/3} \frac{1}{\sqrt{\sin \theta}}\, \epsilon^{j\frac{\pi}{3}} \sum_{p=0}^{p=\infty} \frac{1}{\rho_p} \epsilon^{-j\rho_p \sqrt[3]{\frac{\omega}{c} R}\, \epsilon^{-j\frac{\pi\theta}{3}}} \tag{495}$$

If we use only the first term of the series, we multiply only by

$$\frac{1}{\rho_0} \epsilon^{-j\rho_0 \sqrt[3]{\frac{\omega}{c} R}\, \epsilon^{-j\frac{\pi\theta}{3}}}$$

with a corresponding maximum value

$$\prod_{\max} = \frac{A}{R} \sqrt{\frac{2\pi}{\frac{\omega}{c} R}} \left(\frac{\omega}{c} R\right)^{2/3} \frac{1}{\rho_0} \frac{1}{\sqrt{\sin \theta}}\, \epsilon^{-\rho_0 \sqrt[3]{\frac{\omega}{c}R}\, 0.865\theta}$$

The decrease in amplitude can be found from the ratio $\prod_2/\prod_1$ (ratio of maximum Hertz vectors) of two surface points θ_1 and θ_2, where $\theta_2 > \theta_1$. We have then

$$\frac{\prod_2}{\prod_1} = \sqrt{\frac{\sin \theta_1}{\sin \theta_2}}\, \underbrace{\epsilon^{-0.865(\theta_2 - \theta_1)\rho_0 \sqrt[3]{\frac{\omega}{c} R}}}_{\text{accounts for diffraction effect}} \tag{496}$$

Since the earth radius is $R = 6378$ km, $\rho_0 = 0.81$, and $\omega/c = \beta = 2\pi/\lambda$, we find the exponent of

$$-\rho_0 \sqrt[3]{2\pi} \sqrt[3]{\frac{R}{\lambda}}\, 0.865(\theta_2 - \theta_1) = -0.81 \sqrt[3]{2\pi} \sqrt[3]{6378}\, 0.865 \frac{\theta_2 - \theta_1}{\sqrt[3]{\lambda}}$$

for the wavelength in kilometers. Hence,

$$\frac{\prod_2}{\prod_1} = \sqrt{\frac{\sin \theta_1}{\sin \theta_2}}\, \epsilon^{-23.92\frac{\theta_2 - \theta_1}{\sqrt[3]{\lambda}}} \tag{497}$$

for $\theta_2 > \theta_1$. An application is then as follows: Suppose we have the case of electromagnetic diffraction along the earth's surface over a distance of 3,000 km. Since, for the great circle, 6.28 radians correspond to $6.28 \times 6378 \cong 40{,}076$ km, the range of 3,000 km corresponds to $6.28 \times 3{,}000/40{,}076 = 0.464$ radians, corresponding to $57.296 \times 0.464 = 26.6$ deg. We have, therefore, $\theta_2 - \theta_1 = 0.464$; and for a frequency of 37.5 kc/sec or 0.0375 Mc/sec, we have a free-space wavelength $\lambda = 300/0.0375 = 8{,}000$ m, which is 8 km. This gives an exponent $23.92 \times 0.464/2 = 5.549$. Hence, we have a decay factor $\epsilon^{-5.549} = 0.00389$ or 1/257.24, which is decidedly more decay than experiment would show. For an exponent equal to 100, we have $\epsilon^{-100} = 3.72 \times 10^{-44}$, which is an immense amplitude reduction. Such a case would correspond to an exponent equation $100 = 23.92 \times 0.464/\sqrt[3]{\lambda}$, rendering $\sqrt[3]{\lambda} = 0.2392 \times 0.464 = 0.111$ with a wavelength $\lambda = 12{,}321 \times 10^{-9}$ km $= 1.2321$ cm corresponding to $300/0.012321 \cong 24{,}400$ Mc/sec.

59. Remarks on the Van der Pol-Bremmer Formula for Wave Propagation (Due to Diffraction). B. van der Pol and H. Bremmer[1] have also advanced a theory along the curved earth and given the expression

$$\mathcal{E}^{(\text{volts/m})} = 2\mathcal{E}_0 \sqrt{2\pi\zeta_a} \sum_{s=0}^{s=\infty} f_s(h_1) f_s(h_2) \frac{\epsilon^{-j\tau_s\zeta_a}}{\delta + 2\tau_s} \tag{198}$$

for electromagnetic propagation along spherical earth in the geometric shadow region (below the line of sight). The quantity τ_s is a function of the medium constants μ, κ, and σ of the earth, and the respective height functions $f_s(h)$ do not depend on the distance between radiation and reception centers. The quantity $\mathcal{E}_0$ is the free-space field intensity. An asymptotic expansion for respective transmitter and receiver height functions $f_s(h_1)$ and $f_s(h_2)$ for elevated antennas is

$$f_s(h) = \frac{3}{\sqrt{2\pi}} \frac{(2\chi_a)^{-1/4}}{\sqrt{-2\tau_s}} \frac{\epsilon^{-j\tau_s\sqrt{2\chi_a}}}{J_{1/3}(z_s) + J_{-1/3}(z_s)} \tag{499}$$

for

$$z_s = \tfrac{1}{3}(-2\tau_s)^{3/2}\epsilon^{-j\pi} = \tfrac{1}{3}(2r_s)\epsilon^{j(\pi - 3\theta_s/2)}; \qquad \tau_s = r_s\epsilon^{-j\theta_s} \tag{500}$$

C. R. Burrows and M. C. Gray have used the Van der Pol-Bremmer relations in forms more suitable for technical applications and introduced a universal shadow factor F_s that gives the actual field over a spherical

[1] *Phil. Mag.*, Ser. 7, **24,** 141–175, July, 1937; 825–864, November, 1937; **25,** 817–834, June, 1938; **27,** 261–275, March, 1939 BURROWS, C. R., and M. C. GRAY, *Proc. IRE*, **29,** 16–24, January, 1941. NORTON, K. A, FCC Mimeograph 39920. References by C. R. Burrows and M. C. Gray, are given in Sec. 206. Consult also K. A. Norton, *Proc. IRE*, **24,** 1367, October, 1936, 1203, September, 1937; 1366, November, 1937, and other IRE papers by K. A. Norton in the Appendix. Many of the notations are due to Burrows and Gray.

earth to the field over plane earth and same electromagnetic constants of the earth. Applying Eq. (465) to (464) yields the series

$$\frac{\mathcal{E}}{2\mathcal{E}_0} = G_1 \sqrt[4]{\frac{2\chi_0}{2\chi_a}} \sum_{s=0}^{s=\infty} \frac{\epsilon^{-j\tau_s(\sqrt{2\chi_0}-\sqrt{2\chi_a})}}{\frac{1}{3}\sqrt{-2\tau_s}\,(\delta + 2\tau_s)[J_{1/3}(z_s) + J_{-1/3}(z_s)]}$$
$$= A_1 F_s G_1 G_2 \quad (501)$$

The final form represents the *Burrows-Gray formulas,* where G_1 is the height factor for the transmitter antenna, G_2 is the factor of the pickup antenna, A_1 is the plane earth attenuation factor, and F_s is the shadow factor. Expressing f in megacycles per second, λ in meters, the range d along the spherical earth in meters; using relative values $\mu_r = \mu/\mu_0$ and $\kappa_r = \kappa/\kappa_0$ of ground permeability and dielectric constant with respect to empty space; and σ in mhos per meter, we have the relative equivalent ground dielectric constant

$$\kappa_e' = \kappa_r - j18 \times 10^3 \frac{\sigma}{f} = \kappa_r - j60\lambda\sigma \qquad \text{numeric} \qquad (502)$$

The quality of the ground then is

$$Q = \frac{f\kappa_r}{18 \times 10^3\sigma} = \frac{\kappa_r}{60\sigma\lambda} \qquad (503)$$

Due to atmospheric refraction we can, in equivalence, assume an equivalent earth radius $R_e = kR$ for the actual earth radius in meters. The factor δ is then

$$\delta = \underbrace{\left(\frac{2\pi R_e}{\lambda}\right)^{4/3} \frac{1}{K_1}}_{\text{for vertical polarization}} = \underbrace{\left(\frac{2\pi R_e}{\lambda}\right)^{2/3} K_2}_{\text{for horizontal polarization}} \qquad (504)$$

where

$$K_1 = \frac{(\kappa_e')^2}{\kappa_e' - 1}; \qquad K_2 = \kappa_e' - 1 \qquad (505)$$

The distance parameter ζ becomes, for $300/f = \lambda$ meters,

$$\zeta = \underbrace{\frac{2\pi d}{K_1\lambda}}_{\text{vertical polarization}} = \underbrace{\frac{2\pi d}{K_2\lambda}}_{\text{horizontal polarization}} \qquad (506)$$

The spherical earth distance parameter ζ_a is

$$\zeta_a = \frac{2\pi \dfrac{d}{\lambda}}{\left(2\pi \dfrac{R_e}{\lambda}\right)^{2/3}} \qquad (507)$$

We have $\zeta_a = 4.43 \times 10^{-5} d/\sqrt[3]{\lambda}$ if $R_e = 4R/3$. The shadow factor F_s is a function of ζ_a. For moderate heights, the effect of radiation and reception center elevation with respect to ground and to the separation of radiation and reception locations, as well as to the curvature of the earth, does not enter. The antenna height factors are then given by

$$G + j\chi \tag{508}$$

where for an elevation of h meters of respective antenna centers

$$\left.\begin{aligned} \chi &= \frac{2\pi h}{\lambda\sqrt{K_1}} \quad \text{for vertical polarization} \\ \chi &= \frac{2\pi h\sqrt{K_2}}{\lambda} \quad \text{for horizontal polarization} \end{aligned}\right\} \tag{509}$$

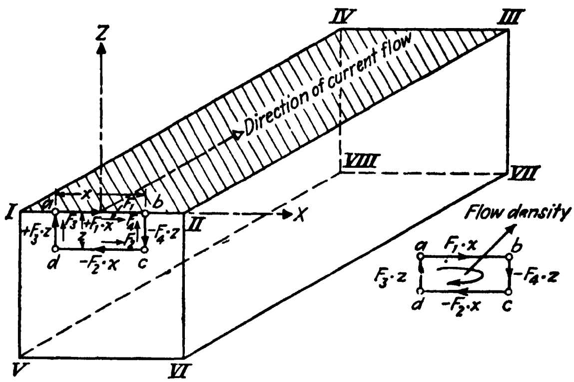
These relations hold for short distances, at heights for which $2\pi h_1 h_2/(\lambda d)$ is somewhat less than unity. For long distances, we have

$$\chi_a = \frac{\chi_{\text{vertical}}}{\sqrt{\delta}} = \frac{\dfrac{2\pi h}{\lambda}}{\sqrt[3]{2\pi R_e/\lambda}} \tag{510}$$

for values less than about 0.5. For $R_e/R = 4/3$, we have $\chi_a = 0.0167 h\lambda^{-2/3}$. The quantity χ_0 is the value of χ_a at the line of sight from the ground below the other antenna. The universal shadow factor F_s denotes the ratio of the actual field received to the field if the ground were plane and had the same μ, κ, and σ constants.

60. Notes on Fields and Current Sheets. At UHF's, the current flows essentially only at the surface of a conductor next to the active fields. Thus, for a linear radiator, it is the outside surface; while for a concentric line, it is the outside surface of the inner conductor and the inside surface of the outside conductor for a properly designed and terminated line. Such surface currents are then taken as current films or current sheets of linear density J amperes per meter. In Fig. 52, a solid medium of cross section I-II-VI-V-I and length I-IV is shown. The quantity F denotes any field vector per meter length for polarizations along the $+X$ and $+Z$ directions. We then generally have electric and magnetic current sheets, since F may be a magnetic H or an electric $\mathcal{E}$ vector, respectively. The purpose of this section is to show that the H field can be replaced by an electric current sheet and the $\mathcal{E}$ field by a magnetic current sheet, even though magnetic current sheets have never been found experimentally. This should not matter, however, since such a concept is a mere mathematical expedient and we have already solutions ahead of a possible experimental result. Hence, when the respective

fields are known or assumed over a certain surface, such as the XZ surface of Fig. 52, we are in a position to substitute the actual currents and charges of a radiator by such current sheets over the surface, where the electromagnetic field is known. We know that the F circulation around a closed path such as *abcda* is equivalent to a flow effect normal to the area xz enclosed by this contour. Let us take up the case where the



I-II-III-IV-I is a current sheet of very small thickness in the Z direction. F_1 and F_2 are fields along the $+X$ direction; F_3 and F_4 are fields along the $+Z$ direction. F is either H amp/m or $\mathcal{E}$ volt/m. For the unit vector n, the elementary circulation per unit area is $n \times F$ = flow density. The line integral $\oint_1 F\,dl$ around the *closed* contour *abcda* is $\oint_{l=x+z+x+z \text{ meters}} F\,dl = F_1x - F_4z - F_2x + F_3z$ = total flow normal to area xz sq m. F circulation per unit area $= \dfrac{\oint_1 F\,dl}{xz} = \dfrac{x(F_1 - F_2) + z(F_3 + F_4)}{xz} = \dfrac{F_1 - F_2}{z} + \dfrac{F_3 - F_4}{x}$

FIG. 52. Shaded area is a portion of the surrounding current sheet.

solid homogeneous medium is a perfect conductor or at least approximately so, by taking a medium of copper with a conductivity of $\sigma = 5.8 \times 10^7$ mhos/m. We know then that the tangential $\mathcal{E}$ components must vanish, *i.e.*, $F_1 = \mathcal{E}_1 = 0 = \mathcal{E}_2 = F_2$, and only $F_3 = \mathcal{E}_3$ and $F_4 = \mathcal{E}_4$ components can exist at the surface of such conductors. If F denotes the H field, we have for such conductor surfaces only $F_1 = H_1$ and $F_2 = H_2$ components, since the normal components $F_3 = H_3 = 0$ and $F_4 = H_4 = 0$.

Applying these conditions to H fields, we find for the H circulation

$$\oint_l H\,dl = H_1x - \underbrace{H_4z}_{=0} - H_2x + \underbrace{H_3z}_{=0} = x(H_1 - H_2) \qquad \text{amps}$$

since the normal H_3 and H_4 components vanish at the surface. The magnetomotive forces H_4z and H_2x in amperes are negative, because the

indicated circulation in Fig. 52 is in opposition with H_4 and H_2 amperes per meter. The circulation per unit area is then

$$\frac{\oint_l H\,dl}{xz} = \frac{x(H_1 - H_2)}{xz} = \frac{H_1 - H_2}{z} \qquad \text{amp/m}$$

When taking the circulation for an infinitesimal circulation, we have an elementary area $dS = dx\,dz = xz \rightarrow 0$; and for the tangential H_1 and H_2 fields of the shaded electric current sheet of Fig. 52, we have

$$\left.\begin{aligned} \oint_{dl} H\,dl = (H_1 - H_2)\,dS = J\,dS + \frac{\partial(\kappa\mathcal{E})}{\partial t}\,dS \qquad \text{amp;} \\ \frac{\partial(\kappa\mathcal{E})}{\partial t} = j\omega\kappa\mathcal{E}\,dS \qquad \text{negligible for a good conductor} \\ J \cong H_1 - H_2 \qquad \text{amp/m} \end{aligned}\right\} \qquad (511)$$

Therefore, J denotes the linear electric current density $H_1 - H_2$ in amperes per meter along the Y direction (along the shaded current sheet). This result assumes that the thickness z of the depth to which a current is still below the surface of the conductor is infinitely small. Computations given on pages 41–43, 45–49 show that the skin effect at VHF's and UHF's is so pronounced that there is practically no penetration depth and only a surface current. The shaded surface I-II-III-IV is, in this case, an *actual* electric current sheet, as is the entire surrounding surface of the solid copper line of cross section I-II-VI-V-I square meters. For the tangential $\mathcal{E}$ components at the electric current sheet of Fig. 52, we have $F_1 = \mathcal{E}_1$ and $F_2 = \mathcal{E}_2$ volts per meter and $\mathcal{E}_1 - \mathcal{E}_2 = 0$.

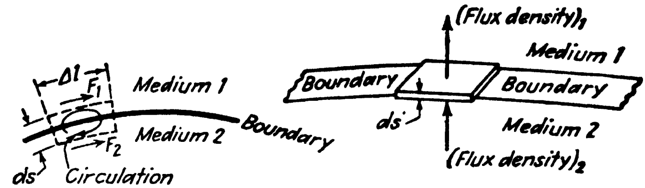
In a similar way, we can assume a magnetic current sheet and find, for the tangential field components,

$$\mathcal{E}_1 - \mathcal{E}_2 = -J_m \quad \text{volts/m;} \qquad H_1 - H_2 = 0 \quad \text{amp/m} \qquad (512)$$

where J_m denotes now the linear *magnetic* current density $-(\mathcal{E}_1 - \mathcal{E}_2)$ volts per meter.

61. Notes on Boundary Conditions in Connection with Current Sheets. The advantage of current sheets can be seen in connection with "equivalent" current sheets. The current sheet of Fig. 52 is an "actual" current sheet, since the copper bar represents an HF conductor that carries the current and causes the fields. But if we deal with wave-front radiators (pages 261, 1028), such as the opening of a wave guide, the aperture of a parabolic radiator with a primary source at the focus, the respective fields at the opening, and the aperture, respectively, can be replaced by equivalent current sheets. Generally, a moving wave front in the true radiation field is formed by orthogonal $\mathcal{E}$ and H vectors, which are tan-

gential to the wave front and move with it along the normal to the $\mathcal{E}$, H surface. Behind the wave front, we have electromagnetic field energy, and ahead of the wave front is no field energy. For air as a medium (perfect dielectric), there can be no surface charge ($\rho = 0$) at the wave



(Flux density)$_1$ $= \kappa_1 \mathcal{E}_{n_1}$ (Flux density)$_2$ $= \kappa_2 \mathcal{E}_{n_2}$
(Flux density)$_1$ $= \mu_1 H_{n_1}$ (Flux density)$_2$ $= \mu_2 H_{n_2}$

FIG. 53. Boundary conditions between two media μ_1, κ_1 and μ_2, κ_2. Thickness ds is infinitesimal.

front, excluding, therefore, normal components of $\mathcal{E}$ and H. The boundary conditions at the wave front are then expressed by normal and tangential field components.

At a boundary that has perfect conductivity, the tangential electric field must vanish $(\mathcal{E}_1 - \mathcal{E}_2) = 0$ (Fig. 52); while for the boundary separating two dielectric mediums, the tangential electric as well as the tangential magnetic fields must be continuous. This can be realized in connection with Fig. 53 and Eqs. (1) on page 11, which read

$$\text{curl } \mathcal{E} = -\frac{\partial(\mu H)}{\partial t}; \qquad \text{curl } H = J + \frac{\partial(\kappa \mathcal{E})}{\partial t} \qquad \text{differential forms}$$

$$\oint_l \mathcal{E}\, dl = -\int\int_S \frac{\partial(\mu H)}{\partial t}\, dS; \quad \oint_l H\, dl = \int\int_S \left(J + \frac{\partial(\kappa\mathcal{E})}{\partial t}\right) dS \text{ integral forms}$$

In Fig. 53, tangential forces F_1 and F_2 are on either side of the surface separating any two mediums and therefore *infinitesimally close*, so that the area S enclosed by the closed circulation l shrinks to the boundary line of length Δl of vanishing area, and we have

$$\oint_l F\, dl = (F_1 - F_2)\, \Delta l$$

$$\oint_l \mathcal{E}\, dl = (\mathcal{E}_1 - \mathcal{E}_2)\, \Delta l = -\int\int_S \frac{\partial(\mu H)}{\partial t} \underbrace{dS}_{=0} = 0$$

$$\oint_l H\, dl = (H_1 - H_2)\, \Delta l = \int\int_S \left[J + \frac{\partial(\kappa\mathcal{E})}{\partial t}\right] \underbrace{dS}_{=0} = 0$$

since for an area degenerating into a boundary line, the entire surface integral must vanish. Therefore, $\mathcal{E}_1 = \mathcal{E}_2$ and $H_1 = H_2$, and the respective tangential field components must be equal on either side of the surface of separation. For the normal components to the area of separation, we have to deal with respective flux densities

$\kappa\mathcal{E}$ coulombs per square meter and μH webers per square meter. This means that we have to take a small surface element of ΔS square meters of the boundary into consideration and utilize the Gauss law

$$\text{div } (\kappa\mathcal{E}) = \rho \qquad \text{coulombs/m}^3 \qquad \text{differential form}$$

$$\int\int_S \kappa\mathcal{E}_n\, dS = \int\int\int_\tau \rho\, d\tau \qquad \text{coulombs} \qquad \text{integral form}$$

For a *surface* charge density of ρ' coulombs per square meter acting on ΔS square meters of the boundary area, we have

$$\int\int_{\Delta S} \kappa\mathcal{E}_n\, dS = (\kappa_1\mathcal{E}_{n_1} - \kappa_2\mathcal{E}_{n_2})\,\Delta S = \rho'\,\Delta S \qquad \text{or}$$

$$\rho' = \kappa_1\mathcal{E}_{n_1} - \kappa_2\mathcal{E}_{n_2} \qquad \text{coulombs/m}^2 \qquad (513)$$

while for a surface of separation with *no* surface charge ($\rho = 0$), we have $\kappa_1\mathcal{E}_{n_1} = \kappa_2\mathcal{E}_{n_2}$. Since we have never, as yet, isolated magnetic charges, we have div $(\mu H_n) = 0$ and $\mu_1 H_{n_1} = \mu_2 H_{n_2}$.

Now equivalent current sheets at surfaces where electromagnetic fields are known or assumed refer generally to normal and tangential field components of H and $\mathcal{E}$. However, at boundaries between any two mediums, a knowledge of the tangential components is sufficient, since Eqs. (1) on page 11 automatically give the normal components. Equations (1) are the Maxwellian equations to which can be added the divergence relations div $(\kappa\mathcal{E}) = \rho$ and div $(\mu H) = 0$, yielding at the surface of separation at any two mediums 1 and 2

$$\left.\begin{aligned} \text{curl } \mathcal{E}_1 &= -\frac{\partial(\mu_1 H_{n_1})}{\partial t}; \qquad \text{curl } \mathcal{E}_2 = -\frac{\partial(\mu_2 H_{n_2})}{\partial t} \\ \text{curl } H_1 &= J_{n_1} + \frac{\partial(\kappa_1\mathcal{E}_{n_1})}{\partial t}; \qquad \text{curl } H_2 = J_{n_2} + \frac{\partial(\kappa_2\mathcal{E}_{n_2})}{\partial t} \end{aligned}\right\} \qquad (514)$$

where F_1 and F_2, for $F = H$ or equal to $\mathcal{E}$ as in Fig. 53, denote the tangential components on respective sides of the boundary, *i.e.*, *at* the boundary, and the subscripts n refer to field and current components *normal* to the boundary, *i.e.*, to the surface of mediums' separation. For the separation of two pure dielectric mediums, μ_1, κ_1 and μ_2, κ_2, respectively (with *no* conductive effects), we have the normal conduction current densities $J_{n_1} = 0 = J_{n_2}$. That Eqs. (514) apply at the boundary is obvious, since the curl of a vector F denotes an elementary F circulation per unit area (area infinitesimally small) and is equivalent to a new vector that is *normal* to the vanishingly small area $dS = S \rightarrow 0$. The new vector then follows the corkscrew rule, where for a clockwise F circulation the new vector enters normal to the area formed by the circulation and for a counterclockwise F circulation withdraws normally from the area. This is applied in the following section.

62. Current-sheet Concept Applied to Wave-front Radiators. The front face of a paraboloid with a radiator at the focus can be taken as a wave-front radiator, since, according to optics, the ray paths from the

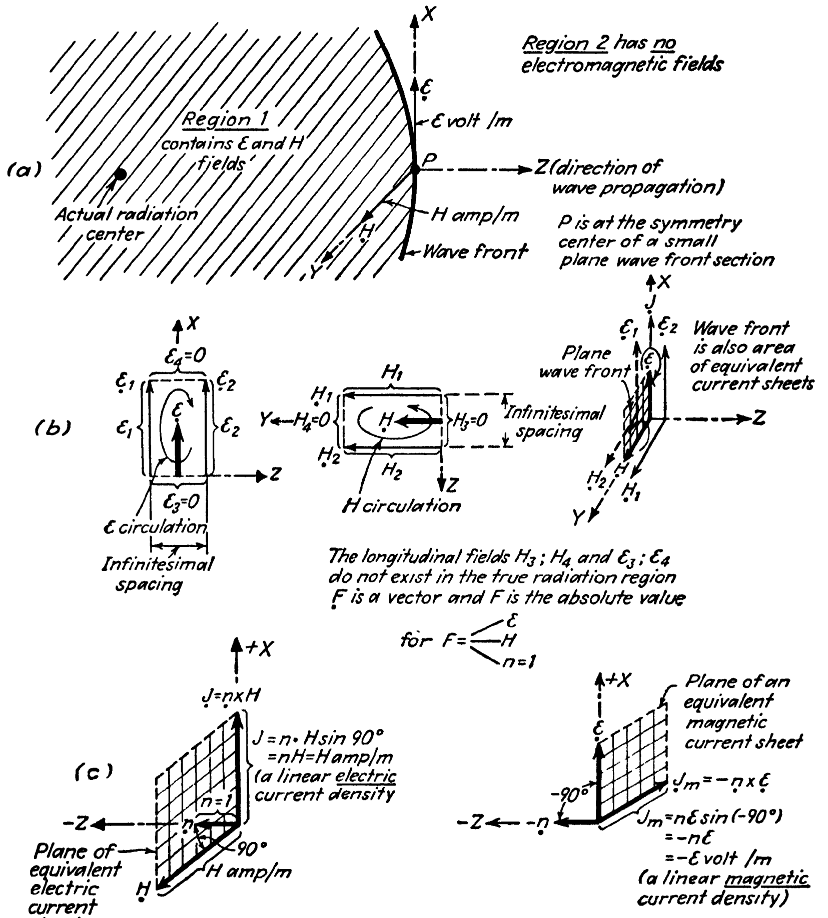


FIG. 54. Equivalent electric and magnetic current sheets in amp/m and volts/m in a forward-moving electromagnetic wave front.

focal source to any point of the front plane of the paraboloid are all of equal lengths if we exclude the paths that do not experience reflection at the parabolic surface. This can be avoided by screening off direct rays with a screen in front of the actual radiator. The fields in the front plane are then all in phase, and we deal with a plane wave front, even though the path length is short. Another form of wave-front radiator is a slot radiator. Such a radiator uses an aperture in an enclosed metal

box with a radiating source within the box. For a suitable shape with appropriate dimensions, effective wave-front radiation will take place at such openings with comparatively small aperture sizes at UHF's. Suppose we consider such a wave front as is indicated in Fig. 54. Let us assume that we have, for the dimensions chosen, essentially a plane wave front (such an assumption is taken for simplicity's sake, since a spherical wave front can also be treated as under diffraction on pp. 1020, 1022) in the true radiation region and that provision is made that the opaque metal walls, due to wave reflections on them, do not appreciably affect the wave-front condition of the aperture (this can be done by screening and materials on the walls inside the metal box). For such conditions, we then have an undisturbed wave front, just as though the material forming the aperture were not present. Figure 54*a* then gives the general condition for a large spherical wave front in an unobstructed radiation space. At field point P, we may imagine a small area in the XY plane, which is practically a plane wave front, and assume field associations that are as in Fig. 54*b*. We have then the orthogonal field vectors $\mathcal{E}$ and H, which are in time phase and form the plane front of limited area. This wave front moves, in the case of atmosphere, essentially with 3×10^8 m/sec along the $+Z$ direction. Even though on either side of the $\mathcal{E}$, H plane, we have the same mediums, *i.e.*, essentially $\mu_1 = \mu_2 = \mu \cong \mu_0$ and $\kappa_1 = \kappa_2 = \kappa \cong \kappa_0$, and no conductivity on either region, we must realize that in the region to the left of the wave front, we have electromagnetic field energy while, in the space ahead of the wave front, we have a field-*free* condition. This means that the *wave front is a plane of discontinuity*. First, the transverse $\mathcal{E}$ and H field associations in the wave front are the effect of the source that radiates. Therefore, $\mathcal{E}$ and H are equivalent to the radiation portion of the source. Since the magnitudes and the directions of $\mathcal{E}$ and H in the XY plane are responsible for any field effects beyond this plane, *i.e.*, in region 2, we can replace these fields by corresponding current sheets, which must also take the discontinuity at the wave front into account. Thus, on one side of the wave front, we have the tangential magnetic field component H_1; and on the other side (ahead of the XY plane), we have the tangential H_2 component, and the difference, according to (511), must account for the *linear* current density J in amperes per meter, which is normal to the respective H circulations on either side of the advancing wave front. Since we have

$$J^{(\text{amp/m})} = H_1 - H_2 \qquad \textit{magnitude relation;}$$

$$\underset{.}{J} = \underset{.}{n} \times \underset{.}{H} \qquad \textit{vector relation} \qquad (515)$$

$H_1 - H_2$ is no longer equal to zero, but equal to the linear current density.

Therefore, the difference between respective tangential magnetic fields on either side of the wave front accounts for the actual H field and for the *discontinuity* at the wave front. The discontinuity is taken care of by the linear electric current density J in amperes per meter, and an electric current sheet in the wave-front plane will take care of the H effect of the electromagnetic wave. The vector relation of (515) as well as Fig. 54c show that the unit vector $\underset{\cdot}{n}$ is *normal* to the wave front which is replaced by the "equivalent" electric current sheet. According to (512), we can also replace the $\mathcal{E}$ effect on the wave front by means of the tangential electric field components on either side of the wave front. We find a discontinuity, since $\mathcal{E}_1 - \mathcal{E}_2$ does not vanish but is equal to $-J_m$. The $\mathcal{E}$ effect can be accounted for by replacing the wave front by a magnetic current sheet. This is also indicated in Fig. 54c and by

$$-J_m^{(\text{volts/m})} = \mathcal{E}_1 - \mathcal{E}_2 \quad \textit{magnitude relation;}$$

$$\underset{\cdot}{J}_m = -\underset{\cdot}{n} \times \underset{\cdot}{\mathcal{E}} \quad \textit{vector relation} \qquad (516)$$

These relations assume that there is such a hypothetic possibility as a magnetic conduction current in volts. By means of (515) and (516), we can imagine such respective current sheets in the plane wave front, with the actual source removed, and account from such sheets for the transverse, *i.e.*, tangential, magnetic and electric field components. The details of (515) and (516) and Figs. 54b and 54c are then as follows:

Since the indicated $\mathcal{E}$ and H circulations in Fig. 54b account for the flow density normal to the plane formed by respective circulations, and since the direction of the flow density is according to the right-hand screw rule, $\underset{\cdot}{J}$ and $\underset{\cdot}{J}_m$ are as indicated in Fig. 54b, and the magnitude relations of (515) and (516) are the outcome. The corresponding vector relations are shown in Fig. 54c. We deal then with a vector product $\underset{\cdot}{n} \times \underset{\cdot}{H}$ and $\underset{\cdot}{n} \times \mathcal{E}$, where $\underset{\cdot}{n}$ is a unit vector normal to the wave front with a magnitude $n = 1$. Generally, for a vector product, we have to consider also the way the vector multiplication is carried out, since, for a field vector $\underset{\cdot}{F}$, we have $\underset{\cdot}{n} \times \underset{\cdot}{F} = -\underset{\cdot}{F} \times \underset{\cdot}{n}$, because the right-hand screw rule is basic. In our case, the smaller angle between $\underset{\cdot}{n}$ and $\underset{\cdot}{F}$ is 90 deg; and if $\underset{\cdot}{n}$ is turned into the $\underset{\cdot}{F}$ direction, we have a new vector $\underset{\cdot}{n} \times \underset{\cdot}{F}$, which is in the direction that a right-hand screw advances and of magnitude $nF \sin (90 \text{ deg}) = F$. The flow density is, therefore, always normal to the $\underset{\cdot}{F}$ vector and normal to the unit vector $\underset{\cdot}{n}$. We note from Fig. 54c that both current sheets are in the plane wave front. If there should be a mental reservation, with respect to using a magnetic current sheet, we could always proceed with the electric current sheet, which is a physical reality according to experimental evidence, and then compute from (515) the associated $\mathcal{E}$ condition also by means of the three-dimensional space impedance, which is 377 ohms for atmosphere. Nevertheless, no harm is done when a hypothetical magnetic current sheet is also used. We should have then in Maxwell's equations, besides div $(\kappa\mathcal{E}) = \rho$ coulombs per cubic meter, also div $(\mu H) = \rho_m$, where ρ_m would stand for the magnetic charge density per unit volume and ρ_m' would represent the magnetic surface density per square meter. We should then also have an additional *electric* vector potential Λ_e in ampere · seconds

per meter in addition to the magnetic vector potential $\Lambda_m = \Lambda$ in volt · seconds per meter. This has already been discussed on pages 223–224 and in formulas (241) and (242), for the general case, which for the above case simplifies to

$$\Lambda_e = \frac{\kappa}{4\pi} \int\int_S \frac{J_m \epsilon^{-j\beta r}}{r_0} dS \qquad \text{amp} \cdot \text{sec/m};$$

$$\Lambda_m = \Lambda = \frac{\mu}{4\pi} \int\int_S \frac{J \epsilon^{-j\beta r}}{r_0} dS \qquad \text{volt} \cdot \text{sec/m} \tag{517}$$

$$\text{curl } \mathcal{E} = -J_m - \frac{\partial(\mu H)}{\partial t}; \qquad \text{curl } H = J + \frac{\partial(\kappa \mathcal{E})}{\partial t}$$

since current sheets are current surfaces and refer to surface integrals instead of the general volume integral, and respective linear current densities are in volts per meter for J_m and amperes per meter for J, yielding, of course, the above-mentioned units for the electric and magnetic vector potentials.

63. The Equivalence Principle by A. E. H. Love and Its Generalization (Induction Theorem of S. A. Schelkunoff).[1] The equivalence theorem refers to *current sheets*, which are capable of causing a transmitted field, while the induction theorem refers to electric and magnetic current sheets, which cause a reflected as well as a transmitted $\mathcal{E}$, H field.

In the equivalence theorem, a field in a source-free region can be produced by a distribution of the electric and magnetic currents in the boundary region (Sec. 62). For instance, the area of the mouth (opening) of a wave guide, which radiates into free space, is such a boundary region. The actual source distribution on the other side of a boundary region can, therefore, be replaced by an "equivalent" distribution of such currents.

In the induction theorem, an induced electromagnetic field can, in equivalence, be produced by electric and magnetic current sheets with densities $J = n \times H_i$ amperes per meter and $J_m = n \times \mathcal{E}_i$ volts per meter, where n stands for a unit normal to a surface within which the actual sources reside. The quantities $\mathcal{E}_i$ and H_i are the impressed field associations over the surface of the aperture if the wave guide were continued to infinity. The induction theorem is more general than the equivalence principle, and we can consider certain portions of an electromagnetic field as due to fictitious electric and magnetic current sheets of respective densities J and J_m.

At the open end of a wave-guide radiator—at the inner side of the opening—we have a wave impedance dependent upon the dimension of

[1] LOVE, A. E. H., *Phil. Trans. Roy. Soc.*, **A197**, 1–45 (1901). MACDONALD, H. M., *Proc. Math. Soc.* (*London*), Ser. E, **10**, 91–95 (1911). SCHELKUNOFF, S. A., *Bell Sys. Tech. Jour.*, 92–112, January, 1936; *Phys. Rev*, **56**, 308, August, 1939; "Electromagnetic Waves," p. 158, D. Van Nostrand Company, Inc., New York, 1943.

the guide and the operating frequency; while at the outer side of the guide opening, the wave impedance would be about 377 ohms to plane TEM waves. The principal TEM wave cannot be produced by a hollow guide, since there is no current return. Within the guide, we then deal with either TE or TM modes, which, in addition, also have a longitudinal H or a longitudinal $\mathcal{E}$ field component, respectively. Hence, reflection must take place at the guide opening, besides wave emergence into open space.

Assuming $\mathcal{E}_i$ and H_i the electric and magnetic fields that could be replaced by their tangential components, these fields would occur at the plane of the opening if the guide were extended to infinity. The reflected fields $\mathcal{E}_r$ and H_r are added to the impressed fields $\mathcal{E}_i$ and H_i in order to account for the actual fields $\mathcal{E}_i + \mathcal{E}_r$ and $H_i + H_r$ within the guide—next to the aperture—under actual conditions. The actual transmitted fields $\mathcal{E}_t$ and H_t, just outside the opening, due to the continuity condition, are then

$$\mathcal{E}_t = \mathcal{E}_i + \mathcal{E}_r \qquad \text{and} \qquad H_t = H_i + H_r \tag{518}$$

According to Schelkunoff, the electromagnetic field can, in equivalence, be produced by electric and magnetic current sheets over the opening of the guide with densities $J_i = n \times H_i$ amperes per meter and $J_{m_i} = -n \times \mathcal{E}_i$ volts per meter if the unit normal n points out of the opening.

If we assumed equivalent electric and magnetic current sheets, which account for the transmitted portions $\mathcal{E}_t$ and H_t beyond the opening, and assumed zero field instead of the reflected fields $\mathcal{E}_r$ and H_r, the respective electric and magnetic current sheets over the guide opening would be specified by

$$\dot{J}_t = \dot{n} \times \dot{H}_t \qquad \text{and} \qquad \dot{J}_{m_t} = -\dot{n} \times \dot{\mathcal{E}}_t \tag{519}$$

where the absolute value of J_t is in amperes per meter and of J_{m_t} is in volts per meter.

According to Schelkunoff, the closed surface over the opening and the outer surface of a radiating horn contain respective current sheets defined by (519), accounting for the radiated field $\mathcal{E}_t$, H_t outside and zero field inside said surface. This is then the space-free equivalence theorem, for which the horn is ignored in the calculations, and the effect is computed from the (unobstructed space) retarded vector potentials by means of

$$\left.\begin{aligned} H^{(\text{amp/m})} &= \frac{\text{curl}\,\Lambda_m}{\mu} - j\omega\Lambda_e + \frac{\text{grad}\,(\text{div}\,\Lambda_e)}{j\omega\mu\kappa} \\ \mathcal{E}^{(\text{volts/m})} &= -\frac{\text{curl}\,\Lambda_e}{\kappa} - j\omega\Lambda_m + \frac{\text{grad}\,(\text{div}\,\Lambda_m)}{j\omega\mu\kappa} \end{aligned}\right\} \tag{520}$$

where the magnetic vector potential

$$\Lambda_m = \frac{\mu}{4\pi} \int\int_S \frac{J_t \epsilon^{-j\beta r}}{r}\, dS \qquad \text{volt} \cdot \text{sec/m} \tag{521}$$

and the electric vector potential

$$\Lambda_e = \frac{\kappa}{4\pi} \int\int_S \frac{J_{mt} \epsilon^{-j\beta r}}{r}\, dS \qquad \text{amp} \cdot \text{sec/m} \tag{522}$$

with the following details:

Equations (520) to (522) are found from Eqs. (239), (241), and (242), and

$$\left.\begin{aligned} &\mu H = \text{curl}\ \Lambda_m - \mu \frac{\partial \Lambda_e}{\partial t} - \mu \nabla U; \qquad \mu \nabla U = \mu\ \text{grad}\ U; \\ &\qquad\qquad \mathcal{E} = -\nabla V - \frac{\partial \Lambda_m}{\partial t} - \frac{\text{curl}\ \Lambda_e}{\kappa} \\ &\nabla V = \text{grad}\ V; \qquad \frac{\partial \Lambda}{\partial t} = j\omega\Lambda; \qquad \Lambda = \Lambda_e; \qquad \Lambda = \Lambda_m \end{aligned}\right\} \tag{520a}$$

From (239), (241), and (242), we have, for the electric space-charge concentration ρ coulombs per cubic meter and an assumed magnetic charge concentration ρ_m coulombs per cubic meter,

$$V = -\frac{1}{4\pi\kappa} \int\int\int_\tau \frac{\rho \epsilon^{-r\Gamma}}{r}\, d\tau = \underbrace{-\int\int\int_\tau \frac{\text{div}\ (J\epsilon^{-r\Gamma})}{4\pi(\sigma + j\omega\kappa)r}\, d\tau}_{\text{for a dissipative medium}}$$

$$= \underbrace{-\frac{1}{j4\omega\kappa} \int\int\int_\tau \frac{\text{div}\ (J\epsilon^{-j\beta r})}{r}\, d\tau}_{\text{for a nondissipative medium } (\sigma = 0)} \qquad \text{volts}$$

$$U = -\frac{1}{4\pi\mu} \int\int\int_\tau \frac{\rho_m \epsilon^{-r\Gamma}}{r}\, d\tau = -\int\int\int_\tau \frac{\text{div}\ (J_m \epsilon^{-r\Gamma})}{4\pi j\omega\mu r}\, d\tau$$

$$= \underbrace{-\frac{1}{j4\pi\omega\mu} \int\int\int_\tau \frac{\text{div}\ (J_m \epsilon^{-j\beta r})}{r}\, d\tau}_{\text{for a nondissipative medium } (\sigma = 0)} \qquad \text{amp}$$

$$\underbrace{\Lambda_e = \frac{\kappa}{4\pi} \int\int\int \frac{J_m \epsilon^{-j\beta r}}{r}\, d\tau \quad \text{amp} \cdot \text{sec/m}; \qquad \Lambda_m = \frac{\mu}{4\pi} \int\int\int \frac{J\epsilon^{-j\beta r}}{r}\, d\tau \quad \text{volt} \cdot \text{sec/m}}_{\text{for } \sigma = 0 \text{ mhos/m and } \Gamma = \alpha + j\beta = j\beta}$$

Therefore,

$$\left.\begin{aligned}
&-j4\pi\omega\kappa V = \frac{4\pi}{\mu}\,\mathrm{div}\,(\mathbf{\Lambda}_m); \qquad -j4\pi\omega\mu U = \frac{4\pi}{\kappa}\,\mathrm{div}\,(\mathbf{\Lambda}_e) \\
&\text{or} \\
&V = -\frac{\mathrm{div}\,\mathbf{\Lambda}_m}{j\omega\mu\kappa} \quad \text{volts}; \qquad U = -\frac{\mathrm{div}\,\mathbf{\Lambda}_e}{j\omega\mu\kappa} \quad \text{amp} \\
&\text{and} \\
&-\mathrm{grad}\,V = -\nabla V = +\frac{\mathrm{grad}\,[\mathrm{div}\,\mathbf{\Lambda}_m]}{j\omega\mu\kappa} \quad \text{volts/m}; \\
&-\mu\,\mathrm{grad}\,U = -\mu\nabla U = +\frac{\mathrm{grad}\,[\mathrm{div}\,\mathbf{\Lambda}_e]}{j\omega\kappa} \quad \text{volt}\cdot\text{sec/m}^2 \\
&-\mathrm{grad}\,U = -\nabla U = +\frac{\mathrm{grad}\,[\mathrm{div}\,\mathbf{\Lambda}_e]}{j\omega\mu\kappa} \quad \text{amp/m}
\end{aligned}\right\} \qquad (520b)$$

where grad U is in amperes per meter, which, when used in (520*a*), confirms (520). Since we deal again with current sheets (surfaces), the vector potentials simplify from volume integrals to the corresponding surface integrals of (521) and (522). In relations (520) to (522), r is the distance to the field point where $\mathcal{E}$ and H act.

For large guide and horn openings, for which the cutoff frequency is appreciably below the operating frequency, essentially no reflections

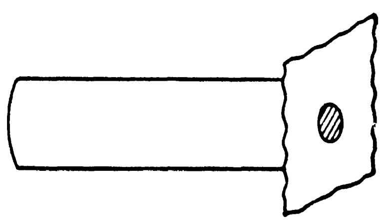


FIG. 55. Aperture is taken as the plane of electric and magnetic current sheets.

occur at the mouth (or the opening) and $J_t \cong J_i$ and $J_{m_t} \cong J_{m_i}$. It is to be noted that the surface S in (521) and (522) over which the integrations are to be made over the respective current sheets surround the horn or the pipe in case of a guide.

Another procedure replaces the sources within a guide or a horn by electric and magnetic current sheets of densities given by (519), but extending only over the opening of the guide. Hence, the presence of the guide or of the horn is no longer ignored, and respective current sheets act as partial reflectors. The procedure applies simply if a guide has an infinitely conducting backup plane at the opening as in Fig. 55. We have then electric and magnetic current sheets over a perfect conductor, since the hole can be closed, in effect, by a metallic sheet because the field due to the respective current sheets is zero on the side of the

pipe. Now the image of the electric current sheet in the plane is negative, thus canceling the effect of the sheet. The image of the magnetic current sheet is positive, thus doubling the effect of the sheet. Hence,

$$\Lambda_m = 0 \qquad \text{and} \qquad \Lambda_e = \frac{2\kappa}{4\pi}\int\int_{S_0} \frac{J_{m_t}\epsilon^{-j\beta r}}{r}\,dS = \frac{\kappa}{2\pi}\int\int_{S_0}\frac{J_{m_t}\epsilon^{-j\beta r}}{r}\,dS \quad \text{amp}\cdot\text{sec/m} \qquad (523)$$

where the integration is to be made only over the area of S_0 square meters of the guide opening.

64. Remarks on Pulsed Radio Waves. The ideal electromagnetic pulse in unobstructed radiation space is within a rectangular contour (envelope). This is not always the case, and we have the comparisons for the pulse energy W_t

$$\text{For any pulse} \qquad W_t = \int_0^t p\,dt \qquad \text{joules}$$

where p is in instantaneous watts and t in seconds.

$$\text{For ideal pulse} \qquad W_t = \int_0^t p_{\max}\,dt = tp_{\max} \qquad \text{joules}$$

where $p_{\max}$ denotes the maximum instantaneous peak power of the pulse. The equivalent pulse width t_e in seconds for a pulse shape of any form is then

$$t_e = \frac{1}{p_{\max}}\int_0^t p\,dt \qquad \text{sec}$$

where t seconds denotes the actual pulse width. If N is the number of pulses per second, the average power is equal to $p_{\max}t_eN$ watts. For $t_e = \frac{1}{2}\mu\text{sec}$, we have for a 100-Mc wave, $\frac{1}{2} \times 10^{-6} \times 100 \times 10^6 =$ 50 cycles per pulse. We should have 500 complete cycles for a 1,000-Mc wave excitation. We can, therefore, say that for frequencies of this order and higher, an electromagnetic pulse is quite a long train of waves; just as in customary radio code transmissions, even a dot contains many carrier-frequency cycles. Nevertheless, the pulse width of t_e seconds can be made very narrow on the screen of a cathode-ray-tube indicator (by using a suitable horizontal sweep frequency), and a pronounced vertical deflection caused by each pulse picked up. It takes, therefore, only a small average power in watts to produce good indications.

65. Final Remarks on Mathematical Procedure Employed in Electromagnetic Field Relations. The foregoing sections of this chapter dealt with exact and reasonably accurate mathematical procedures used in radiation phenomena. The limitations of flow functions, of the Hertz functions as well as their features, were brought out in direct application

to practical problems. It was also shown that vector point functions, known as "vector potentials," are of value especially when we deal with radiation problems and how equivalent current sheets can be used to account for the radiated field. It is true that with the knowledge of current and charge distributions on radiators, we could always find the electromagnetic field with Eqs. (1), even though a metal screen with an opening for radiation into the space beyond the screen were used, if the boundary conditions on the partially opaque screen are observed. The mathematics would be difficult, however, and is greatly facilitated by the current sheet concept and the method employing vector potentials. Chapter III is only a compilation of essential procedures, and many more applications and methods will be found in the following chapters, together with numerical applications. The procedure of finding the radiated field based on field assumptions can also be applied to conductors. We also noted that by means of a discontinuous field, we may have a method of estimating the field if the discontinuous field is assumed. We have also discontinuous variations, where other procedures are needed for the solution. There are also short cuts based on a general radiation formula[1] based on the radiation in pure dielectric mediums and a known distribution of electric and magnetic currents. The hypothetical magnetic currents yield more generality in the radiation formula and apply to slot radiators based on the diffraction theory. They also apply to radiations from electromagnetic horns and the like. Use is then made of the radiated power and electric as well as magnetic radiation vectors.

[1] SCHELKUNOFF, S. A., *Proc. IRE*, **10**, 660, October, 1939. "Electromagnetic Waves," p. 333, D. Van Nostrand Company, Inc., New York, 1943.

CHAPTER IV

PROPAGATION CHARACTERISTICS

In unbounded space as well as guided space, we deal with a propagation characteristic that defines both the phase constant and the dissipation, generally known as the "attenuation constant." The guided space plays a part if we deal with transmission systems, such as Lecher wires, coaxial feeders, and wave guides in particular. The propagation constant contains for a harmonic wave excitation, in many practical cases, a real part α, which can be expressed in nepers per meter, and an imaginary part $j\beta$, which is in circular radians per meter. In either case, we deal with something that is "numerical" with respect to 1 m of propagation travel, since the neper unit is based on a logarithmic decay and the circular-radian unit is based on any circular arc equal to its radius with a value of 360 deg/6.28 = 57.2958 deg, if measured at the center. The attenuation may also follow another law. By means of the attenuation constant, we can explain the losses that account for the amplitude decay of both the electric and magnetic field vectors. By use of the phase constant, we have a means of accounting for the phase (wave) distribution, *i.e.*, for the wavelength. This is *by no means*, fixed, even though the frequency may be perfectly stabilized. It depends upon the phase velocity, which in turn depends upon the phase constant for a constant frequency. The phase distribution along an antenna has a great deal to do with the meter · ampere output if we deal with a linear antenna less than a half-wavelength distribution. We then usually employ the term "equivalent" or "effective" length, which, when multiplied by the current at the point of maximum current, will give the same product as the length and uniform current of an elementary electric dipole at the same location, yielding identical radio field intensity in the direction of maximum field effect.

66. General Remarks on the Propagation Constant. When we compare the three-dimensional propagation constant $\Gamma = \alpha + j\beta$ generalized hyperbolic radians per meter with the propagation constant holding for customary transmission lines, we find by means of $\Gamma = \sqrt{ZY}$ complex radians per meter propagation

$$\left.\begin{array}{ll}\Gamma = \alpha + j\beta = \sqrt{\underbrace{(j\omega\mu)}_{Z}\underbrace{(\sigma + j\omega\kappa)}_{Y}} & \text{from Fig. 49 for } \mathcal{E},\ H \text{ fields} \\ \Gamma = \sqrt{\underbrace{(R + j\omega L)}_{Z}\underbrace{(G + j\omega C)}_{Y}} & \text{from page 407 for voltages and currents per meter propagation}\end{array}\right\} \quad (524)$$

yielding

$$\left.\begin{array}{l}\alpha = \sqrt{\dfrac{\omega\mu}{2}\left[\sqrt{\sigma^2 + \omega^2\kappa^2} - \omega\kappa\right]} \quad \text{for fields} \\ \alpha = \sqrt{\tfrac{1}{2}\left[\sqrt{(R^2 + \omega^2L^2)(G^2 + \omega^2C^2)} + (GR - \omega^2CL)\right]} \\ \qquad \text{for voltages and currents} \quad \text{nepers/m}\end{array}\right\} \quad (525)$$

$$\left.\begin{array}{l}\beta = \sqrt{\dfrac{\omega\mu}{2}\left[\sqrt{\sigma^2 + (\omega\kappa)^2} + \omega\kappa\right]} \quad \text{for fields} \\ \beta = \sqrt{\tfrac{1}{2}\left[\sqrt{(R^2 + \omega^2L^2)(G^2 + \omega^2C^2)} - (GR - \omega^2CL)\right]} \\ \qquad \text{for voltages and currents} \quad \text{circular radians/m}\end{array}\right\} \quad (526)$$

for the attenuation constant α and phase constant β. It is assumed that the respective transmission-line parameters R ohms per meter, G mhos per meter, L henrys per meter, and C farads per meter are constant, just as are the field medium parameters σ mhos per meter, μ henrys per meter, and κ farads per meter. It does not matter how the transmission line is oriented in space as long as we have the same insulation between the conductors of fixed size and spacing. It also does not matter in which direction an electromagnetic wave passes in an isotropic medium.

The *propagation constant* Γ expresses what happens owing to energy transfer. A harmonic current is subject to the time factor $\epsilon^{j\omega t}$ (a mere phase factor). The electromagnetic field is likewise subject to a phase factor. For a complex propagation constant, whether referring to currents along a transmission line or to field associations (in the case of a progressive wave), we have for traveling waves along the $+X$ axis, in either case,

$$\left.\begin{array}{l}I_x = I_0\epsilon^{j\omega t}\epsilon^{-x\Gamma} = I_0\epsilon^{j\omega t}\epsilon^{-\alpha x}\epsilon^{-j\beta x} = I_0\epsilon^{-\alpha x}\epsilon^{j(\omega t - \beta x)} \quad \text{amp} \\ \mathcal{E}_z = \mathcal{E}_0\epsilon^{-\alpha x}\epsilon^{j(\omega t - \beta x)} \quad \text{volts/m}; \qquad H_y = H_0\epsilon^{-\alpha x}\epsilon^{j(\omega t - \beta x)} \quad \text{amp/m}\end{array}\right\} \quad (527)$$

where I_x, $\mathcal{E}_z$, and H_y are the respective values a distance x meters from the reference values I_0, $\mathcal{E}_0$, and H_0.

Now for a line: The medium of the lines as well as the medium between go and return conductors affects the propagation constant. For electromagnetic waves in *unbounded* space, *only* the *space medium affects* the *propagation* constant. For TE and TM waves in wave guides

(bounded medium) and TEM waves along a double line, the medium of the boundary also affects the propagation constant in addition to the medium within the boundary (within the guide). This is evident, since, for a customary Lecher wire line, the ohmic resistance must affect the propagation properties; and when the line is submerged in a dissipative medium such as salt water, ordinary ground, etc., the medium losses between the conductors also affect the attenuation constant as well as the phase constant. For a wave guide the customary theory assumes perfect conductivity of the boundary (the guide envelope). Therefore, there can be no tangential electric field component along the metallic envelope, and the guide behaves like a high-pass network for which we have only a pass region (above the cutoff frequency) and an attenuation region (below this frequency). Any attenuation for such an ideal conductivity can be due only to medium losses in the wave medium (above cutoff). The attenuation region of the ideal guide, as a matter of fact of any guide, gives a ready means for a useful attenuator.

The propagation constant Γ is defined by the logarithm to the base $\epsilon = 2.7182818 \ldots$ of a complex current ratio or a complex field ratio, respectively. Hence,

$$\Gamma = \alpha + j\beta = \log_\epsilon \left(\frac{\mathcal{E}_0}{\mathcal{E}_z}\right) = \log_\epsilon \left(\frac{H_0}{H_y}\right);$$

$$\Gamma = \alpha + j\beta = \log_\epsilon \left(\frac{I_0}{I_x}\right) \quad \text{where all quantities are generally complex} \qquad (528)$$

and $\mathcal{E}$, H, and I are vectors. The vector notation with a dot below is usually omitted in this text, in order to avoid involved nomenclature. For space radiation, we deal mostly with an outgoing progressive wave (except when certain pulsed radiation with return echoes is used). In transmission lines that are not perfectly matched at the load end, we deal with an outgoing and a returning progressive wave.

67. General Remarks on the Attenuation Constant. For harmonic wave excitation, we deal with a periodic wave whose displacement has a periodic variation with time or distance or both. According to (527), we have

$$\frac{I_x}{I_0} = \epsilon^{j\omega t}\epsilon^{-x\Gamma} = \epsilon^{j\omega t - x\Gamma}; \qquad \frac{\mathcal{E}_z}{\mathcal{E}_0} = \epsilon^{j\omega t - x\Gamma}; \qquad \frac{H_y}{H_0} = \epsilon^{j\omega t - x\Gamma}$$

If we take the values at a distance x_1 to a point 1 and at a distance x_2 to a point 2, we have

$$F_1 = F_0\epsilon^{j\omega t}\epsilon^{-x_1\Gamma}; \qquad F_2 = F_0\epsilon^{j\omega t}\epsilon^{-x_2\Gamma}; \qquad F = \mathcal{E}; \qquad F = H; \qquad F = I$$

and obtain for the *propagation ratio*

$$\frac{F_2}{F_1} = \epsilon^{-x_2\Gamma}\epsilon^{-x_1\Gamma} = \epsilon^{-(x_2-x_1)\Gamma} = \epsilon^{-\alpha(x_2-x_1)}\epsilon^{-j\beta(x_2-x_1)}$$
$$= \epsilon^{-\alpha(x_2-x_1)}\{\cos[\beta(x_2 - x_1)] - j\sin[\beta(x_2 - x_1)]\} \quad (529)$$

This ratio F_2/F_1 has *magnitude* as well a *phase*, since the magnitude is given by $\epsilon^{-\alpha(x_2-x_1)}$ and the phase by a negative angle of $-360(x_2 - x_1)/\lambda$ degrees, because $\beta = 360$ deg/λ, and λ and $(x_2 - x_1)$ are in the same units, namely, in meters. It is the magnitude that is of concern when we deal with the attenuation constant α. The *attenuation ratio* $\epsilon^{-\alpha(x_2-x_1)}$ then represents the magnitude of the propagation constant ratio. Therefore, the propagation ratio is the ratio of the displacement at point 2 to that at point 1. By "displacement" is meant a change in the wave medium that is proportional to the square root of the stored energy, which for electromagnetic waves, is represented by the electric or magnetic flux density, *i.e.*, $\kappa\mathcal{E}$ and μH. Hence, in the true radiation region, we have the electric ($\kappa\mathcal{E}$) and the magnetic (μH) displacement transverse to the direction of propagation (to the X direction, which may be in any geographical direction).

68. General Remarks on the Phase or Wavelength Constant. The phase propagation ratio $\epsilon^{-j\beta(x_2-x_1)}$ is obtained by dividing the propagation ratio F_2/F_1 by its magnitude $\epsilon^{-\alpha(x_2-x_1)}$. It is the phase constant β that is the characteristic value, since $x_2 - x_1$ is only a distance in meters by which β is multiplied in order to obtain the total propagation phase delay $-\beta(x_2 - x_1) = 360(x_2 - x_1)/\lambda$ electrical degrees while the wave travels through a distance of $x_2 - x_1$ meters. The value of β is found from Eq (526), from which formula we find then either $6.28(x_2 - x_1)$ radians per meter or $360(x_2 - x_1)$ degrees per meter. The phase constant β gives a means of finding the phase velocity c' as well as the group velocity c'' in meters per second. We have to distinguish between such velocities, as is shown on page 50, since there may be a difference between the velocity with which a point of a certain phase of a periodic wave travels in the direction of propagation (phase velocity c') and the velocity c'' of the envelope of a wave occupying a frequency *band* over which the envelope delay is fixed. The envelope delay represents a time delay of the envelope of a wave and is equal to the slope of the phase against the angular frequency. It holds for a frequency band over which the slope is uniform.

69. General Remarks on the Wave Front. If a wavelike disturbance is, with a fixed phase, propagated with a fixed speed, we have a wave motion. The *surface of constant phase is* known as a "wave front." The normal distance between two successive surfaces of equal phases is

equal to 1 wavelength. Therefore, in a continuous wave, the surface over which the displacements are in the same phase at the same instant is known as the "wave front." Since, according to (527), the phase is determined by the phase factor $\epsilon^{j(\omega t - \beta r)}$, where r denotes the distance traveled, we find

$$\omega t - \beta r = \text{constant} \qquad \text{for the wave front} \tag{530}$$

The phase velocity c' is found from the time rate of the surface forming the wave front. Hence,

$$\frac{d(\omega t - \beta r)}{dt} = \omega - \beta \frac{dr}{dt} = \omega - \beta c' = 0 \qquad \text{or} \qquad c' = \frac{\omega}{\beta} \qquad \text{m/sec} \tag{531}$$

since $d(\text{constant})/dt = 0$. This velocity refers *only* to *steady* states and may be *greater* or smaller than $c = 3 \times 10^8$ m/sec. As we shall learn, the group velocity $c'' = d\omega/d\beta$ meters per second differs from the phase velocity only in dispersive mediums. For a normal dispersion, the group velocity c'' is smaller than the phase velocity c'. The phase velocity c' appears often in this text as v meters per second. For an anomalous dispersion, $c'' > c'$. Moreover, from (530), we note that at the beginning of a wave, we have $r = 0$ and $\omega t - 0 = \text{constant}$. The wave front is then the surface over which the displacement from zero starts at the same instant. It must be realized that there are cases where the plane of constant phase *i.e.*, the wave front, does *not* coincide with the plane of constant amplitude. This happens, for instance, when a plane electromagnetic wave enters a plane surface of the earth crust under an angle *different* from 90 deg. The earth crust is a general medium, which also exhibits conductivity σ. While the electromagnetic wave proceeds entirely in air, the plane of constant amplitude coincides with the wave front (plane of constant phase); but below the earth surface, the direction of propagation is first changed owing to refraction. Also the velocity of propagation is now different. The new wave front is, of course, again a plane of constant phase normal to the new direction of wave propagation. However, the plane of constant amplitude in the earth crust is now parallel to the earth surface.

With respect to the wave front, we must realize that a wavelike disturbance may also be due to very sharp pulses; and when we deal with mediums that also exhibit conduction effects ($\sigma \neq 0$), the group velocity becomes greater than the phase velocity c'. Even c'' may exceed the value of c, which gives the impression that energy is being moved faster than the velocity of light under such a condition. This means that the group velocity c'' cannot be taken as the velocity of wave power transfer for all conditions. We have similar conditions in ordinary

networks where a sudden alternating voltage is applied, causing first a transient effect that changes for a sustained supply into a fixed alternating effect. Therefore, the very first arrival of the pulse occurs with the velocity c. Even though this is the front of the wavelike disturbance, it does not correspond to what may be termed the wave front of the actual power flow, since we have a forced function and a natural function that damps out owing to medium dissipation. At the very first moment of arrival, the steady function cancels the natural term, thereafter building up the actual signal as the natural term dies out.

If we deal with a plane wave front as in Fig. 56, we can replace the wave front by electric and magnetic current sheets. The wave front moves with the velocity of v meters per second. Assume that the

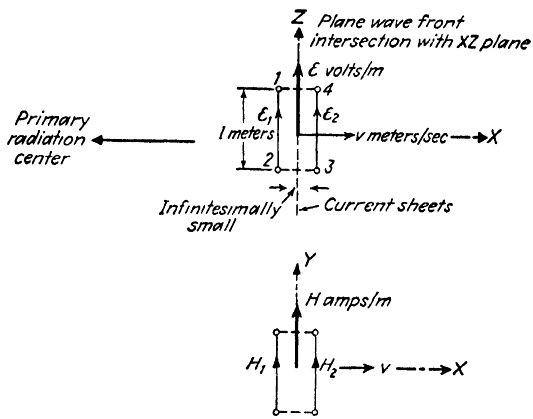


FIG. 56. Current sheets in an advancing wave front. (H amp/m is normal to paper and along Y axis.)

electric $\mathcal{E}$ vector is as indicated and the magnetic H vector normal to the paper. According to (516) on page 355, the actual electric field intensity of $\mathcal{E}$ volts per meter must be equal to the difference of the tangential forces, and $l\mathcal{E}$ volts must be the voltage around the rectangular path 1-2-3-4. But the magnetic vector H pointing normal to the rectangular area at P into the paper causes a magnetic displacement of μH per unit area. But each second the wave front sweeps through v meters. Therefore, the magnetic displacement increases at the rate of μHlv volts, which must be equal to $l\mathcal{E}$ volts, or

$$\mathcal{E} = \frac{\mu v H l}{l} = \mu v H = \mu \frac{H}{\sqrt{\mu \kappa}} = \sqrt{\frac{\mu}{\kappa}}\, H \cong 377H \qquad \text{volts/m}$$

This solution is, therefore, explained by the magnetic current sheet, since the left relation of (516) accounts for $\mathcal{E}$ volts per meter and the

linear density J_m volts per meter. The *electric* current sheet with J amperes per meter linear current density and the difference $H_1 - H_2$ [Eq. (515)] accounts for the actual H field of the wave front in amperes per meter. It then yields an H circulation with a magnetomotive force of Hl amperes, which must be equal to the rate at which the *electric* displacement $\kappa\mathcal{E}$ increases, *i.e.*, $\kappa\mathcal{E}lv$ amperes, yielding also the result $\mathcal{E}/H = \sqrt{\mu/\kappa}$ ohms.

70. Phase Velocity, Group Velocity, Signal Velocity, and Velocity of Electromagnetic Wave Propagation in Empty Space. For a uniform motion, the velocity expresses the distance moved through per unit time. When a body passes through a distance of s meters during t seconds, we have a velocity of s/t meters per second for a uniform or constant speed. If the speed is not uniform, we have, for the time interval dt during which the velocity can be assumed uniform, the value ds/dt for the velocity.

According to the theory of relativity, the velocity $c = 3 \times 10^8$ m/sec of light is the upper limit of propagation speed, which also happens to be the velocity of electromagnetic wave propagation in empty space.

In the theory of transmission lines and wave guides, in partially dissipative space, etc., we find that we must distinguish among phase velocity c', group velocity c'', and the velocity c of light. For instance, for wave propagation in the ionized region of the upper atmosphere, we find the simple relation

$$c = \sqrt{c'c''} \text{ m/sec} \tag{532}$$

which shows that the velocity of wave propagation in empty space is the geometrical mean of the phase velocity c' and the group velocity c''. According to this expression, the phase velocity c' can be greater than the velocity c of light without conflicting with the theory of relativity, since even for an infinite phase velocity c', relation (532) would require only that the group velocity c'' vanish altogether and that no power at all be propagated. For a phase velocity c' greater than c, this would mean that the wave energy coming by the way of the ionosphere may be considerably delayed. This would confirm the experimental long-time electromagnetic wave echoes.[1] As far as the group velocity c'' and the phase velocity c' are concerned, we have, for any power transfer of a wavelike electromagnetic disturbance, conditions as indicated in Fig. 57. We note that for AM (amplitude modulation), the amplitude contour may move faster or slower than the enclosed dotted carrier variation.

It should be realized that for general propagation of a wavelike electromagnetic disturbance, relation (532) does not always apply. For

[1] HUND, A., "Phenomena in High-frequency Systems," pp. 378 to 405, McGraw-Hill Book Company, Inc., New York, 1936.

instance, for transmission lines and wave guides, c' is equal to ω/β, where ω can be taken as the radian frequency or 2π times the frequency in cycles per second, and β is the phase constant $2\pi/\lambda$, where λ meters refers to the wavelength due to the phase distribution, *i.e.*, due to the phase velocity c'. The group velocity c'' is equal to $d\omega/d\beta$. For a properly designed HF feeder, such as a coaxial line or a parallel-wire system, we

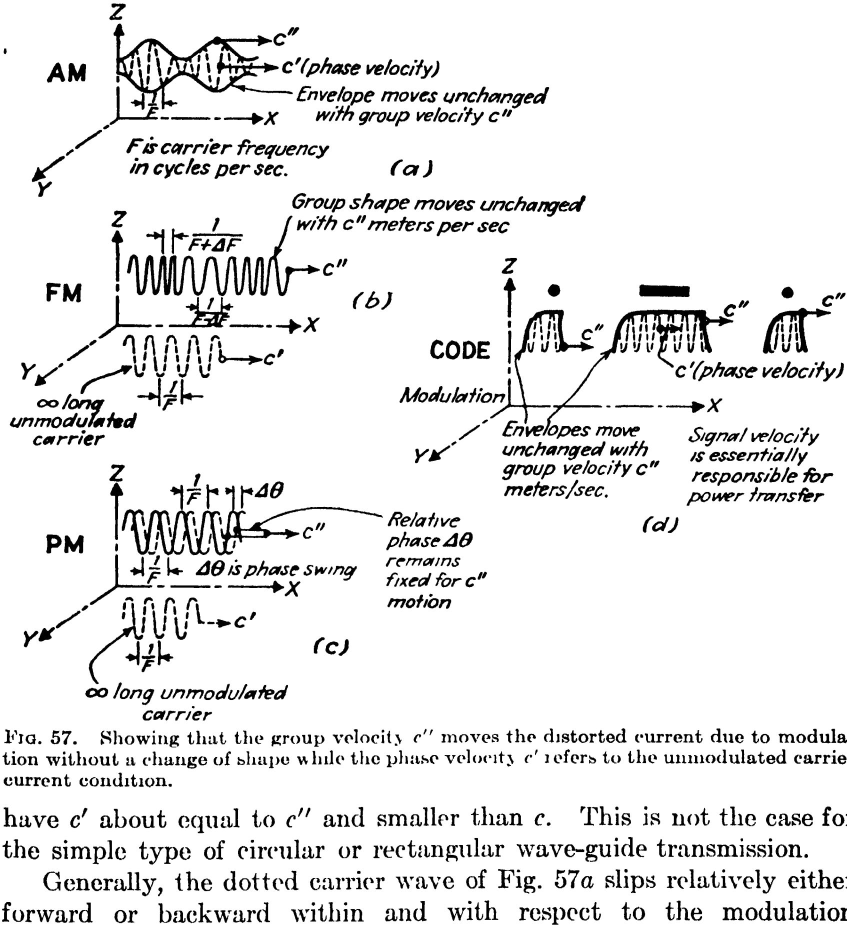


FIG. 57. Showing that the group velocity c'' moves the distorted current due to modulation without a change of shape while the phase velocity c' refers to the unmodulated carrier current condition.

have c' about equal to c'' and smaller than c. This is not the case for the simple type of circular or rectangular wave-guide transmission.

Generally, the dotted carrier wave of Fig. 57*a* slips relatively either forward or backward within and with respect to the modulation envelope, which, itself, moves with the group velocity c''. Because the envelope moves unchanged, so to speak, as a whole or as *a group*, the term "group velocity" is used. Since the shape of the envelope, in the case of amplitude modulation, is a measure[1] for the modulation energy,

[1] If we deal with pulsed electromagnetic field energy (a wave packet), the field power transferred occurs approximately with the group velocity c''. But for anom-

the group velocity c'' must also be the velocity with which energy is being propagated, while the phase velocity c' refers to the motion of an infinitely long sinusoidal wave train that carries no power.

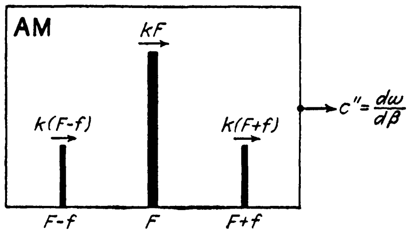
In the case of frequency modulation (Fig. 57*b*), the group shape moves as a whole with the velocity c'' and transfers the modulation power. For phase modulation (Fig. 57*c*) the phase-shifted carrier, with respect to the unmodulated carrier phase condition, moves with the fixed $\Delta\theta$ phase deviation with the group velocity c'', and transfers modulation power. By fixed $\Delta\theta$ is meant that for a $\Delta\theta$ phase deviation in the case of PM, the corresponding electromagnetic wave moves out with a phase distortion of $\Delta\theta$. For code modulation or keying (Fig. 57*d*), it is again the respective short and long envelopes due to keying that move with velocity c'', while the carrier wave within respective envelopes may glide relatively either forward or backward within respective envelopes, depending on whether c' is larger or smaller than c''. For the *general* wave medium (μ, κ, and σ present), it is the signal velocity with which power is transferred. Therefore, we see that irrespective of the mode of modulation, the phase velocity c' is the velocity with which a point of a certain phase travels in the direction of propagation. The group velocity c'' is the velocity of propagation of a wave occupying a frequency *band* over which the envelope delay is uniform. It differs, therefore, from the phase velocity only in a medium in which the phase velocity c' varies with the frequency or the direction of wave propagation. We have, therefore, conditions as indicated in Fig. 58 and can imagine that respective spectrum distributions, for a certain degree of modulation, move as a whole with the group velocity while the constituent components of carrier or center frequency F and side-band frequencies $F \pm mf$, as might be said, move each with their particular phase velocity c', depending on the medium through which the field energy is propagated. Conditions such as occur in the ionized upper atmosphere are hardly of concern for UHF wave propagation and are discussed in another publication.[1] But atmospheric refraction due to a gradient of the dielectric constant κ

alous dispersion (medium exhibits also conductivity effects), we have $c'' > c'$ and c'' may be even *greater* than $c = 3 \times 10^8$ m/sec. In this case, c'' can be only partially responsible for the energy transfer, and we have to consider an additional velocity, known as the "signal" velocity. A. Sommerfeld, *Ann. Physik*, **44**, 177–202 (1914). Brillouin, *Rept. Congr. Intern. Électricité*, Vol. II, Sec. 1, Paris, 1932. J. A. Stratton, in his "Electromagnetic Theory," McGraw-Hill Book Company, Inc., New York, 1941, gives a clear account on pp. 333–340 with significant illustrations in Figs. 61, 62*a*, 62*b*, and 63.

[1] HUND, A., "Phenomena in High-frequency Systems," pp. 367–405, McGraw-Hill Book Company, Inc., New York, 1936.

farads per meter in the lower atmosphere cannot be ignored at such frequencies, especially at times of decided humidity gradients.

71. Derivation of Phase- and Group-velocity Expressions for Different Systems of Propagation. For feeder lines, the formulas given in Fig. 58 likewise apply. If R, C, L, and G denote the resistance, capacitance, inductance, and transverse leakage conductance of a feeder per unit length (per 1 m) in practical units (ohms, farads, henrys, and mhos),



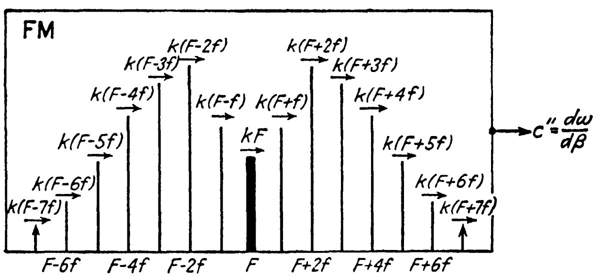


FIG. 58. The entire energy spectrum as a whole moves with the group velocity $c'' = d\omega/d\beta$, while the phase velocity $c' = \omega/\beta$ determines the space distribution of the field intensity and the space wavelength. $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2\pi/\lambda} = \lambda$, the magnitude of which depends on the medium. $\omega = 2\pi$ times frequency. F is carrier frequency, f is frequency of intelligence. For $m = 0, 1, 2, 3, \ldots$ we have component waves of frequencies $F \pm mf$.

we know, from the theory of transmission lines, that for $a = GR + \omega^2 CL$ and $b = GR - \omega^2 CL$, we have both phase and amplitude distortion, since

$$\left.\begin{aligned} \text{Phase constant } \beta &= \sqrt{\tfrac{1}{2}[\sqrt{a^2 + (GL - CR)^2\omega^2} - b]} \\ \text{Amplitude decay } \alpha &= \sqrt{\tfrac{1}{2}[\sqrt{a^2 + (GL - CR)^2\omega^2} + b]} \end{aligned}\right\} \tag{533}$$

Amplitude distortion occurs because the amplitude decay $\epsilon^{-\alpha}$ depends on α nepers per meter, and α depends on ω, *i.e.*, the frequency f in cycles per second. *Phase distortion* occurs, since the phase rotation β per meter travel is not proportional to ω, *i.e.*, proportional to the frequency f. Hence, $\omega/\beta = c'$ varies with the frequency, and the group velocity $c'' = d\omega/d\beta$ with which the power is transmitted along the feeder is no longer

equal to the phase velocity c' and to the velocity $c = 3 \times 10^8$ m/sec, which holds in free space. The group velocity c'' is then generally smaller than the phase velocity c' and always smaller than c. For no phase distortion, c'' must be equal to c'. This occurs when $GL - CR = 0$ and

$$\beta = \sqrt{\tfrac{1}{2}(a - b)} = \omega \sqrt{CL}; \qquad \alpha = \sqrt{\tfrac{1}{2}(a + b)} = \sqrt{GR} \quad (534)$$

Hence, we also have no amplitude distortion, since no frequency term appears in the α relation. What is of interest here is the β relation, and we note that β is now directly proportional to the frequency and

$$c' = \frac{\omega}{\beta} = \frac{1}{\sqrt{CL}} \quad (535)$$

For such a distortion-free feeder,

$$\frac{d\beta}{d\omega} = \sqrt{CL} = \frac{1}{c''} \quad (536)$$

proving that the group and phase velocities are now equal to each other and at best equal to the velocity c of light. Generally, the velocity c' is about 5 per cent smaller for well-designed HF feeders that use air as a dielectric (coaxial and parallel wires). This means that a quarter-wavelength impedance transformation feeder would be only $0.95 \times 0.25\lambda$ meters long, where λ is the wavelength in free space in meters and found from $300/f$, where f is in megacycles per second.

Moreover, $\Gamma = \alpha + j\beta$ is generally the propagation constant of a feeder. Then for each meter length along the feeder, we have the generalized line angle

$$\Gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad \text{generalized radians/m} \quad (537)$$

where α is in pure hyperbolic radians per meter and β in pure circular radians per meter. Assuming that at high frequencies G as well as R is negligible in comparison with the susceptance ωC and inductive reactance ωL, respectively, we find $\Gamma \cong j\beta = j\omega \sqrt{CL}$. This is again the same result as for the distortionless feeder discussed above, and again $c' = c''$.

At VHF's and UHF's, neglecting the transverse leakage conductance G mhos in comparison with the capacitive susceptance ωC mhos seems justified but *not* R in comparison with ωL if we are interested in distinguishing among c', c'', and c. The first of Eqs. (533) can also be written in the form

$$\beta = \sqrt{0.5[\sqrt{R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} - (GR - \omega^2 CL)]} \quad (538)$$

Neglecting G, we find

$$\beta \cong \sqrt{0.5[\sqrt{\omega^2C^2(R^2 + \omega^2L^2)} + \omega^2CL]} = \sqrt{0.5\omega^2CL\left(\sqrt{\frac{R^2}{\omega^2L^2} + 1}\right)} \tag{539}$$

But

$$\sqrt{\Delta^2 + 1} \cong \frac{1}{2}\Delta^2 + 1 \qquad \text{if } \Delta = \frac{R^2}{\omega^2L^2} \ll 1$$

and

$$\beta \cong \omega\sqrt{CL}\sqrt{\frac{1}{2}\left(\frac{R^2}{2\omega^2L^2} + 1 + 1\right)} = \omega\sqrt{CL}\sqrt{1 + \frac{R^2}{4\omega^2L^2}}$$
$$\cong \omega\sqrt{CL}\left(1 + \frac{R^2}{8\omega^2L^2}\right)$$

giving a phase velocity

$$c' = \frac{\omega}{\beta} = \frac{1}{\sqrt{CL}\left(1 + \frac{R^2}{8\omega^2L^2}\right)} = \frac{c}{1 + \frac{R^2}{8\omega^2L^2}} \quad \text{m/sec} \tag{540}$$

which for some feeder designs may be only 10 to 20 per cent smaller than the velocity $c = 1/\sqrt{CL}$. For linear radiators, we usually deal with phase velocities $c' = 0.95c$ to $0.975c$ along the radiating conductor, which explains why, for instance, a half-wave dipole has a physical length of only 0.95 to 0.975 of the half wavelength in empty space, holding for $c = 3 \times 10^8$ m/sec. The group velocity c'', with which the modulation envelope is being transferred, is found from

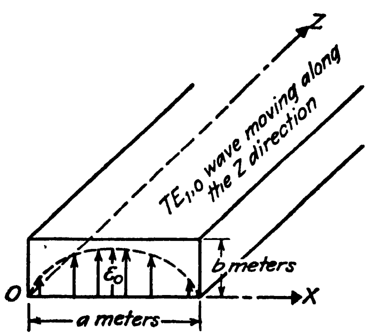
$$\frac{1}{c''} = \frac{d\beta}{d\omega} = \frac{d\left(\omega\sqrt{CL} + \frac{\omega^{-1}R^2\sqrt{CL}}{8L^2}\right)}{d\omega} = \sqrt{CL}$$
$$- \frac{R^2\sqrt{CL}}{8\omega^2L^2} = \frac{(8\omega^2L^2 - R^2)\sqrt{CL}}{8\omega^2L^2} \quad \text{sec/m} \tag{541}$$

The group velocity then becomes

$$c'' = \frac{8\omega^2L^2}{8\omega^2L^2 - R^2}\frac{1}{\sqrt{CL}} = \frac{1}{1 - \Delta}\frac{1}{\sqrt{CL}} = \frac{c}{1 - \Delta} \quad \text{m/sec} \tag{542}$$

for $\Delta = R^2/(8\omega^2L^2)$, in comparison with the phase velocity $c/(1 + \Delta)$. This result is unexpected, indeed, since it states that the modulation power transfer occurs somewhat faster than the velocity c, which does not seem correct. If we had used the expression (538) and carried out the differentiation $d\omega/d\beta$, we should have found a value at best equal to c or smaller than c. The case with phase velocities greater than c due to

a dispersive medium, as in the ionized upper atmosphere, treated elsewhere,[1] is hardly of interest in the UHF band of wave propagation. But in the UHF band of frequencies, we often meet the relationship $c = \sqrt{c'c''}$ meters per second when dealing with wave guides. Suppose we deal with a rectangular wave guide of breadth a meters along the X direction, height b meters along the Y direction, and guiding along the Z direction of a Cartesian coordinate system shown in Fig. 59.



$\mathcal{E}_y = \mathcal{E}_0 \sin \frac{\pi x}{a} \epsilon^{-z\Gamma_z}$. Since $\mathcal{E}_x = 0 = \mathcal{E}_z$, the curl components of Eq. (137) reduce to $H_x = 1/j\omega\mu\ \partial\mathcal{E}_y/\partial z$; $H_y = 0$; $H_z = -1/j\omega\mu\ \partial\mathcal{E}_y/\partial x$; $\partial H_x/\partial y = 0 = \partial H_z/\partial y$; $\partial H_x/\partial z - \partial H_z/\partial x = (\sigma + j\omega\kappa)\mathcal{E}_y$ leading to a cutoff wavelength $\lambda_c = 2a$ m for $\sigma = 0$ mhos/m, since the propagation constant Γ_z along the guide axis is

$$\Gamma_z = \underbrace{\sqrt{\frac{\pi^2}{a^2} - \frac{4\pi^2}{\lambda^2}}}_{\text{for attenuation region}} = j\underbrace{\sqrt{\frac{4\pi^2}{\lambda^2} - \frac{\pi^2}{a^2}}}_{\text{for the pass region}}$$

and $\Gamma_z = 0$ for $\frac{4\pi^2}{\lambda_c^2} = \frac{\pi^2}{a^2}$

For air within the guide the cutoff frequency $f_c = 300/\lambda = 150/a$ Mc/sec.

FIG. 59. Conditions for the dominant TE wave in a rectangular wave guide.

If we consider a mode of transmission for which the electric vector $\mathcal{E}_y$ is the *only* electric field intensity, then $\mathcal{E}_z = \mathcal{E}_x = 0$. Inasmuch as *along* the very good conducting surfaces of the guide, no electric field component can exist, for instance, a field distribution $\mathcal{E}_y$ as indicated is possible. This happens to be a half-period or half-wavelength distribution, *i.e.*, a condition for which $a = \frac{1}{2}\lambda_c$, where λ_c denotes the cutoff wavelength above which wave transmission along the guide is no longer possible. For such a condition, we can have no magnetic vector in *space* phase with $\mathcal{E}_y$; *i.e.*, $H_y = 0$. According to Maxwell's equations (137) on page 148, since also $\mathcal{E}_x = 0 = \mathcal{E}_z$, we find

$$\underbrace{\frac{\partial\mathcal{E}_z}{\partial y}}_{=0} - \frac{\partial\mathcal{E}_y}{\partial z} = -j\omega\mu H_x; \quad \underbrace{\frac{\partial\mathcal{E}_x}{\partial z}}_{=0} - \underbrace{\frac{\partial\mathcal{E}_z}{\partial x}}_{=0} = \underbrace{-j\omega\mu H_y}_{\text{hence, } H_y = 0}; \quad \frac{\partial\mathcal{E}_y}{\partial x} - \underbrace{\frac{\partial\mathcal{E}_x}{\partial y}}_{=0} = -j\omega\mu H_z$$

or

$$H_x = \frac{1}{j\omega\mu}\frac{\partial\mathcal{E}_y}{\partial z}; \quad H_z = -\frac{j}{\omega\mu}\frac{\partial\mathcal{E}_y}{\partial x} \quad \text{amp/m} \tag{543}$$

and for $p = \sigma + j\omega\kappa$, we find

$$\underbrace{\frac{\partial H_z}{\partial y} - \underbrace{\frac{\partial H_y}{\partial z}}_{=0}}_{\text{hence, }=0} = \underbrace{p\mathcal{E}_x}_{=0}; \quad \underbrace{\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = p\mathcal{E}_y}_{\text{only remains}}; \quad \underbrace{\underbrace{\frac{\partial H_y}{\partial x}}_{=0} - \frac{\partial H_x}{\partial y}}_{\text{hence, }=0} = \underbrace{p\mathcal{E}_z}_{=0}$$

[1] *Ibid.*, pp. 16–17, 21–24, 326, 330–332, 373–405.

Therefore,

$$\mathcal{E}_y = \frac{(\partial H_x/\partial z) - (\partial H_z/\partial x)}{\sigma + j\omega\kappa} \qquad \text{volts/m} \tag{544}$$

As is shown in the theory of wave guides, (543) and (544), on elimination of the H components, yield the differential relation

$$\frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} = \mathcal{E}_y \Gamma^2; \qquad \Gamma^2 = -\omega^2\mu\kappa + j\omega\mu\sigma \tag{545}$$

where $\Gamma = j\omega\sqrt{\mu\kappa_e}$ complex radians per meter is the three-dimensional propagation constant of the *unbounded inside* wave-guide space and $\kappa_e = \kappa - j\sigma/\omega$ is again the equivalent dielectric constant of the inside space in farads per meter for $\omega/6.28$ in cycles per second. For an unbounded wave-guide medium, we should also have a characteristic wave impedance $Z_0 = j\omega\mu/\Gamma = j\omega\mu/(j\omega\sqrt{\mu\kappa_e}) = \sqrt{\mu/\kappa_e}$ ohms. For guided one-dimensional wave propagation in the same medium *along the* Z axis, we have then a wave-propagation constant Γ_z and characteristic impedance Z_z that differ from Γ and Z_0. Here also $Z_z = j\omega\mu/\Gamma_z$. The propagation constant Γ_z along the guided direction Z within the guide of general medium (μ, κ, and σ) is $\Gamma_z = \sqrt{\Gamma^2 + K^2}$, and the relation $\partial^2\mathcal{E}_y/\partial x^2 = -K^2\mathcal{E}_y$ is the solution of possible sin (kx) distributions if $K = m\pi/a$ and $m = 1, 2, 3, \cdots$. The case $m = 1$ applies to the $TE_{1,0}$ wave shown in Fig. 59, and $K = \pi/a$. For the $TE_{m,0}$ wave, we have

$$\mathcal{E}_y = \mathcal{E}_0 \sin\frac{m\pi x}{a}\,\epsilon^{-z\Gamma_z} \qquad \text{volts/m} \tag{546}$$

with the transverse magnetic field association

$$H_x = \frac{1}{j\omega\mu}\frac{\partial\mathcal{E}_y}{\partial z} = -\frac{\Gamma_z}{j\omega\mu}\mathcal{E}_0\sin\frac{m\pi x}{a}\,\epsilon^{-z\Gamma_z} = -\frac{\mathcal{E}_y}{Z_z} \qquad \text{amp/m} \tag{547}$$

while the longitudinal magnetic field association

$$H_z = -\frac{1}{j\omega\mu}\frac{\partial\mathcal{E}_y}{\partial x} = -\frac{m\pi\mathcal{E}_0}{j\omega\mu a}\cos\left(\frac{m\pi x}{a}\right)\epsilon^{-z\Gamma_z} \qquad \text{amp/m} \tag{548}$$

For all $TE_{m,0}$ wave excitations, we have

$$\Gamma_z = \sqrt{\Gamma^2 + K^2} = \sqrt{\underbrace{-\omega^2\mu\kappa + j\omega\mu\sigma}_{=\Gamma^2} + \underbrace{\left(\frac{m\pi}{a}\right)^2}_{=K^2}} \qquad \text{complex radians/m} \tag{549}$$

for the propagation constant $\Gamma_z = \alpha_z + j\beta_z$ *within* and *along* the guide (Z direction). Wave guides are used because of their propagation efficiency, which requires that a dissipation-free wave medium ($\sigma = 0$ mhos/m) be used. This eliminates the term with the σ factor; and if we deal with the gravest $TE_{m,0}$ excitation, for which $m = 1$ (shown in Fig. 59), we find

$$\Gamma_z = \sqrt{\left(\frac{\pi}{a}\right)^2 - \omega^2\mu\kappa} = \alpha_z + j\beta_z \qquad \text{generalized radians/m}$$

where α_z is in hyperbolic radians per meter nepers per meter, β_z in circular radians per meter. When $(\pi/a)^2$ is larger than $\omega^2\mu\kappa$, the value of Γ_z is *real*. This means that the guide would act like an *attenuator* ($\Gamma_z = \alpha_z$, since $j\beta_z = 0$). But for $\pi^2/a^2 < \omega^2\mu\kappa$, we have $\Gamma_z = j\beta_z$, and waves can pass without medium and space attenuation α within the guide, and we have the formula

$$\Gamma_z = j\beta_z = j\sqrt{\omega^2\mu\kappa - \frac{\pi^2}{a^2}} \quad \text{circular radians/m} = j\beta \tag{550}$$

Formula (550) is basic for the phase velocity c' and the group velocity c'' of the $TE_{1\,0}$ excitation. We have then the solutions

$$c' = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\kappa - \left(\frac{\pi}{a}\right)^2}} \quad \text{m/sec} \tag{551}$$

$$\frac{d\beta}{d\omega} = \frac{d\left[\omega^2\mu\kappa - \left(\frac{\pi}{a}\right)^2\right]^{1/2}}{d\omega} = \frac{1}{2}\left[\omega^2\mu\kappa - \left(\frac{\pi}{a}\right)^2\right]^{-1/2} 2\omega\mu\kappa = \frac{\omega\mu\kappa}{\sqrt{\omega^2\mu\kappa - (\pi/a)^2}} \quad \text{sec/m}$$

$$\frac{d\beta}{d\omega} = \frac{1}{c''} \quad \text{sec/m}$$

Therefore,

$$c'' = \frac{\sqrt{\omega^2\mu\kappa - (\pi/a)^2}}{\omega\mu\kappa} = \frac{1}{\mu\kappa c'} \quad \text{m/sec} \tag{552}$$

But for air as a dielectric of the guide, we have $\kappa \cong \kappa_0$ and $\mu \cong \mu_0$, and $1/(\mu\kappa) \cong 1/(\mu_0\kappa_0) = c = 3 \times 10^8$ m/sec. Therefore, Eq. (552) yields $c'' = c^2/c'$, confirming the relation $c = \sqrt{c'c''}$ as given in (532). Of course, this outcome holds for the other possible $TE_{m,0}$ waves, since we have only the substitutions $m = 2, 3, 4, 5, \cdots$ instead of $m = 1$, and we find generally, for $m = 1, 2, 3, \cdots$,

$$c' = \frac{\omega}{\sqrt{\omega^2\mu\kappa - (m\pi/a)^2}} \quad \text{m/sec}, \qquad c'' = \frac{\sqrt{\omega^2\mu\kappa - (m\pi/a)^2}}{\omega\mu\kappa} = \frac{1}{\mu\kappa c'} \quad \text{m/sec} \tag{553}$$

These results show that for

$$\omega^2\mu\kappa = \left(\frac{m\pi}{a}\right)^2 \tag{554}$$

for a *pure* dielectric medium ($\sigma = 0$), the propagation constant Γ_z, along the Z direction of the guide, vanishes altogether and no power at all can be transferred along the guide. This is equivalent to saying that the group velocity c'' with which power is being transferred must be zero, which can happen only when the phase velocity c' becomes infinite. This is actually the case, since $c' = \omega/0 = \infty$ and $c'' = 0/(\omega\mu\kappa) = 0$. When the medium within the guide is not a perfect insulator ($\sigma \neq 0$), then relation (549) holds, and we have *always* $\Gamma_z = \alpha + j\beta$, *i.e.*, always an attenuation constant α. The equality expressed by (554) yields the cutoff frequency f_c below

which no $TE_{m,0}$ guide action is possible, and we find, for $m = 1, 2, 3, \cdots$,

$$\omega_c = \frac{m\pi}{a\sqrt{\mu\kappa}}$$

which leads for $\mu = \mu_0$ and $\kappa = \kappa_0$ (holding for air essentially) to $\omega_c = \pi cm/a = 3 \times 10^8\pi m/a$ and

$$f_c = \frac{m\pi \cdot 3 \times 10^8}{2\pi a} 10^{-6} = 150\frac{m}{a} \quad \text{Mc/sec} \tag{555}$$

For a guide width of $a = 0.1$ m and a $TE_{1,0}$ excitation, we have $m = 1$ and $f_c = 150 \times 1/0.1 = 1{,}500$ Mc/sec as the lowest frequency that just could not pass the guide in this mode.

Let us now take up the case of a cylindrical wave guide as in Fig. 60 and consider the simplest mode of excitation, which would be a TM

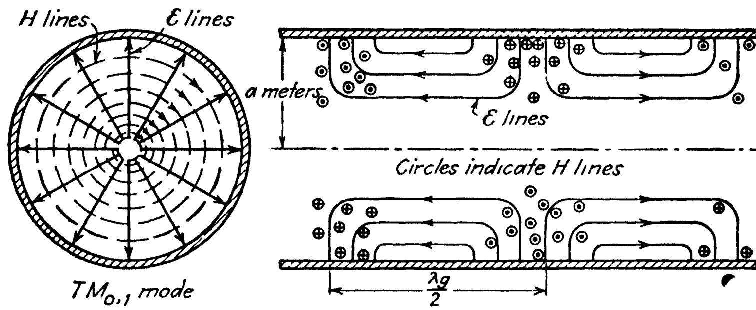


FIG. 60. Dominant longitudinal ℰ wave in a cylindrical wave guide. $\lambda_c = 2.61a$ m; $f_c = \dfrac{3.83 \times 10^{-7}}{a\sqrt{\mu\kappa}}$ Mc/sec for μ henrys/m, κ farads/m, and $\sigma = 0$ mhos/m inside the guide.

wave $TM_{0,1}$. For such a mode, the magnetic H lines are concentric about the Z axis of the guide, and the electric ℰ lines are as indicated. We deal then with a circular magnetic wave, since the H lines form circles. The general form for such waves is expressed by $TM_{0,m}$ of order m, where m is the number of cylinders including the boundary of the guide to which the electric vector is normal. For $m = 1$, we have the circular magnetic wave, shown in Fig. 60 with the lowest cutoff frequency. $TM_{n,m}$ waves, for which n is not equal to zero, are noncircular waves, and n denotes the number of axial planes to which the electric vector is normal, while m has again the above-mentioned meaning.

Here also the criterion is that for a perfectly conducting cylinder surface: no H line *can ever cross* the metal cylinder, and the electric ℰ lines must enter or leave the metal boundary normal to the boundary. In comparison with the simple $TE_{1,0}$ excitation of the rectangular guide of Fig. 59, we note that we now have essentially a longitudinal $\mathcal{E}_z$ field and only a completely transverse H field. If we compare the

possible $\mathcal{E}_z$ field of Fig. 60 with the $\mathcal{E}_y$ field condition of Fig. 59, we have, for $\Gamma = j\omega\sqrt{\mu\kappa_e} = j\omega\sqrt{\mu(\kappa - j\sigma/\omega)}$,

$$\left.\begin{aligned} &\frac{\partial^2\mathcal{E}_y}{\partial x^2} + \frac{\partial^2\mathcal{E}_y}{\partial z^2} - \mathcal{E}_y\Gamma^2 = 0 && \text{for the rectangular guide of width of } a \text{ m and } \textit{transverse } \mathcal{E}_y \text{ waves along the } Z \text{ axis} \\ &\frac{\partial^2\mathcal{E}_z}{\partial r^2} + \frac{1}{r}\frac{\partial\mathcal{E}_z}{\partial r} + \mathcal{E}_z\Gamma^2 = 0 && \text{for the circular guide of radius } a \text{ m and } \textit{transverse H} \text{ waves along the } Z \text{ axis of the guide} \end{aligned}\right\} \quad (556)$$

with respective solutions

$$\left.\begin{aligned} &\mathcal{E}^{(\text{volts/m})} = \mathcal{E}_z = \mathcal{E}_0 J_0(r\Gamma)\epsilon^{-z\Gamma_z} \text{ for} \\ &\Gamma_z = \sqrt{\underbrace{-\omega^2\mu\kappa + j\omega\mu\sigma}_{=\Gamma^2} + \left(\frac{p}{a}\right)^2} && \text{for } p = 2.405,\ 5.52,\ 8.654,\ 11.79,\ \cdots, \text{ and a circular guide} \\ &\mathcal{E}^{(\text{volts/m})} = \mathcal{E}_y = \mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\epsilon^{-z\Gamma_z} \text{ for} \\ &\Gamma_z = \sqrt{-\omega^2\mu\kappa + \underbrace{j\omega\mu\sigma}_{\substack{=0\\ \text{for air}}} + \left(\frac{m\pi}{a}\right)^2} && \text{for } m = 1,\ 2,\ 3,\ \cdots, \text{ and a rectangular guide} \end{aligned}\right\} \quad (557)$$

Since along the cylinder walls, *i.e.*, for $r = a$ meters, the $\mathcal{E}_z$ intensity must vanish, we find $\mathcal{E}_z = 0$ and, therefore, $J_0(a\Gamma) = 0$, giving the consecutive Bessel roots p of Eq. (557), because for $(a\Gamma) = p$, we have $J_0(p) = 0$.

It is again the value under the square root of the propagation constant Γ_z along the guide, resulting in $\Gamma_z = j\beta$, *i.e.*, becoming purely imaginary, that is of concern in this discussion, since we are interested only in conditions for which power is being transferred without attenuation ($\alpha = 0$ in $\Gamma_z = \alpha + j\beta$). Again using air as dielectric within the guide, we have $\sigma = 0$; and for the pass region of the guide,

$$\Gamma_z = j\beta = j\sqrt{\omega^2\mu\kappa - \frac{p^2}{a^2}} \tag{558}$$

since $\omega^2\mu\kappa$ must be larger than $(p/a)^2$. The phase velocity along the guide is then

$$c' = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\kappa - (p/a)^2}} \quad \text{m/sec} \tag{559}$$

Moreover, we find

$$\frac{d\beta}{d\omega} = \frac{d[\omega^2\mu\kappa - (p/a)^2]^{1/2}}{d\omega} = \frac{1}{2}\left[\omega^2\mu\kappa - \left(\frac{p}{a}\right)^2\right]^{-1/2} 2\omega\kappa$$

$$= \frac{\omega\mu\kappa}{\sqrt{\omega^2\mu\kappa - (p/a)^2}} = \frac{1}{c''}$$

yielding the group velocity c'' as

$$c'' = \frac{\sqrt{\omega^2\mu\kappa - (p/a)^2}}{\omega\mu\kappa} = \frac{1}{\mu\kappa c'} \quad \text{m/sec} \tag{560}$$

which again, for dry air as the wave medium, yields practically $1/\sqrt{\mu\kappa} \cong 1/\sqrt{\mu_0\kappa_0} = c = 3 \times 10^8$ m/sec and confirms the relation $c = \sqrt{c'c''}$ for $TM_{n,m}$ modes in circular guides.

According to (557), as brought out above, the Bessel function $J_0(r\Gamma)$ of the zeroth order determines the possible p values and therefore also the possible modes of TM waves. The reason for this is that the J_0 function looks like a damped wave train if plotted against the argument $(r\Gamma)$. For argument $r\Gamma = 0$, *i.e.*, the radius of the guide equal to zero, we have a Bessel value equal to unity as a cosine function of the same argument would have. Since the argument $r\Gamma$ is the product of the three-dimensional propagation constant Γ of the guide medium (without guide envelope) and any radius r of the guide and $a\Gamma$ denotes the actual radius condition for which $J_0(a\Gamma) = 0$ makes $\mathcal{E}_z$ also equal to zero, we satisfy thereby the boundary condition of the guide. Hence, only Bessel values of arguments that render $J_0(r\Gamma) = 0$ are of concern in these formulas. Therefore, wherever the damped wave-train function $J_0(a\Gamma)$ intersects the $r\Gamma$ reference axis, we have a possible TM mode, fixed by $(a\Gamma) = p$. The J_0 Bessel curve intersects the $r\Gamma$ axis the first time for $r\Gamma = 2.4048$, which satisfies the boundary condition of the circular guide (for $\mathcal{E}_z = 0$) when r is equal to the radius a of the guide. This yields then the condition for the lowest transverse H mode, with a phase velocity c' and the group velocity c'' given by

$$c' = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{2.4048}{a}\right)^2}} \text{ m/sec}; \qquad c'' = \frac{c^2}{c'} \text{ m/sec}; \qquad c = 3 \times 10^8 \text{ m/sec} \tag{561}$$

where the radius a is in meters and $\omega/6.28$ in cycles per second. Since $c' = \omega/\beta$, the square-root value $\beta = \sqrt{(\omega/c)^2 - (2.4048/a)^2}$ must vanish for the frequency $f_c = \omega/6.28$, at which and below which no power transfer is possible along the guide. This occurs then for $(2.4048/a)^2 = (2\pi f_c/c)^2$. For $a = 0.1$ m, the cutoff frequency f_c is then found from

$$f_c = \frac{2.4048c}{6.28a} = \frac{2.4048 \times 3 \times 10^8}{6.28 \times 0.1} = 1.148 \times 10^9 \text{ cycles/sec}$$

or $f_c = 1{,}148$ Mc/sec. Since $a = 0.1$ m $= 10$ cm $= 10 \times 2.54$ in., a 25.4 in. radius would have such a cutoff frequency, at which and below which the guide could act only as an attenuator. For a 3-in. wave guide, we have $a = (3/2) \times 2.54 \times 10^{-2}$ m $= 3.8 \times 10^{-2}$ m and $f_c = 1.148 \times 10^8/(3.8 \times 10^{-2}) = 3.012 \times 10^9$ cycles/sec, or 3,012 Mc/sec. For the different $TM_{n,m}$ modes, we have then the following formulas, where μ is in henrys per meter and κ in farads per meter·

$$f_c^{(\text{Mc/sec})} = \frac{47.7p}{a} \quad \text{for air}$$

$$f_c^{(\text{Mc/sec})} = 1.592 \times 10^{-7} \frac{p}{a\sqrt{\mu\kappa}} \quad \text{for medium } \mu, \kappa, \sigma = 0$$

Mode	$TM_{0,1}$	$TM_{0,2}$	$TM_{0,3}$	$TM_{0,4}$
p	2.405	5.52	8.654	11.79

the p's are the consecutive Bessel curve intersections with the zero reference axis (562)

72. Details on the Phase Velocity, Group Velocity, and Cutoff Frequency in Wave Guides. By means of formulas (559), (560), and (562), we can compute the phase velocity c', the group velocity c'', and the cutoff frequency f_c for any perfect dielectric within a circular guide of radius a meters. Suppose we use again a 3-in. copper tube as a guide; then we have $f_c = 1{,}012$ Mc/sec for air and the $TM_{0,1}$ mode of TM waves (taken from computations given already). We must, therefore, choose a frequency higher than f_c if we expect power transfer along the guide. Suppose we choose $f = 4{,}000$ Mc/sec. This frequency is then still below the cutoff frequency of the next higher mode, for which we have $p = 5.52$, according to formula (561). The radius $a = (3/2) \times 2.54 \times 10^{-2} = 3.81 \times 10^{-2}$ m gives for the $TM_{0,2}$ mode a cutoff frequency $f_c = 47.7 \times 5.52/(3.81 \times 10^{-2}) \cong 6{,}900$ Mc/sec. Hence, there is no risk of also exciting the $TM_{0,2}$ mode with a 4,000-Mc/sec frequency. We have then $\omega = 6.28 \times 4{,}000 \times 10^6 = 2.512 \times 10^{10}$, $\omega/c = 2.512 \times 10^{10}/(3 \times 10^8) = 83.8$, $p/a = 2.405/(3.81 \times 10^{-2}) = 63.2$. Since for air $\mu\kappa \cong \mu_0\kappa_0 = 1/c^2$, we find from formula (559) for the phase velocity

$$c' = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{p}{a}\right)^2}} = \frac{2.512 \times 10^{10}}{\sqrt{83.8^2 - 63.2^2}}$$

$$= \frac{2.512 \times 10^{10}}{55.05} = 4.575 \times 10^8 \text{ m/sec}$$

as against 3×10^8 m/sec. The phase velocity is therefore $(4.575 - 3)100/3 = 52.5$ per cent greater than the velocity of light. But the actual power transfer occurs, according to (560), with a velocity of only $c'' = c^2/c' = 9 \times 10^{16}/4.575 \times 10^8 = 1.97 \times 10^8$ m/sec, which satisfies Einstein's statement that the velocity of energy propagation can *never exceed* the speed of light. It is nevertheless the *phase* velocity $c' = 4.575 \times 10^8$ m/sec that is instrumental when we think in terms of field *distributions* along the guide. Thus, the half-wavelength distribution of the $TM_{0,1}$ mode along the cylindrical guide, shown in Fig. 60, is found by means of the phase velocity c' and *not* by means of the group velocity c''.

According to Sec. 57 and Table 10, we find that the propagation constant Γ_z along the guide (Z direction in Figs. 59 and 60) is $\Gamma_z = \sqrt{\Gamma^2 + K^2}$, where $\Gamma = \sqrt{j\omega\mu(\sigma + j\omega\kappa)}$ is the intrinsic three-dimensional propagation constant of the *unbounded* inside wave-guide medium with electromagnetic constants μ henrys per meter, κ farads per meter, and σ mhos per meter, and $\omega/6.28$ is the operating frequency in cycles per second. For a rectangular guide with a cross section ab square meters, where the dimension a is parallel to X and the dimension b is parallel to

Y of a Cartesian XYZ system, according to step 19 of Table 10, we have

$$K = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{563}$$

for possible modes and $m = 1, 2, 3, \cdots ; n = 1, 2, 3, \cdots$ in the case of TM waves, and $m = 0$; $n = 1, 2, 3, \cdots$ as well as $n = 0$, $m = 1, 2, 3, \cdots$ for TE waves, since $TE_{0,0}$ cannot exist. Here, it is again understood that the TE modes are also known as the *longitudinal H* modes and the TM waves as the *longitudinal* $\mathcal{E}$ waves or modes. We have for the possible TE waves ($TE_{m,n}$), for $m = 0$ and $n = 1$, the $TE_{0,1}$ wave, which is also the longitudinal $H_{0,1}$ wave with $K = \pi/b$, and the propagation constant *along* and *within* the guide yields

$$\Gamma_z = \sqrt{K^2 + \Gamma^2} = \sqrt{\left(\frac{\pi}{b}\right)^2 + j\omega\mu(\sigma + j\omega\kappa)}$$

For a perfect dielectric within the guide ($\sigma = 0$, such a dielectric is needed),

$$\left.\begin{aligned}
&\Gamma_z = \alpha + j\beta = \sqrt{\left(\frac{\pi}{b}\right)^2 - \omega^2\mu\kappa} = \alpha \quad \text{and} \\
&j\beta = 0, \quad \text{if } \left(\frac{\pi}{b}\right)^2 > \omega^2\mu\kappa \quad \text{guide acts as } \textit{attenuator}, \text{ since operating in attenuation region} \\
&\Gamma_z = j\beta = j\sqrt{\omega^2\mu\kappa - \left(\frac{\pi}{b}\right)^2} \quad \text{guide is operating in the } \textit{pass} \text{ region for which } \alpha = 0, \text{ since } \left(\frac{\pi}{b}\right)^2 < \omega^2\mu\kappa
\end{aligned}\right\} \tag{564}$$

for the phase constant $\beta = \beta_z$ along the guide direction (Z axis). We have then the phase and group velocities along the guide for the $TE_{0,1}$ rectangular guide mode $c' = \omega/\beta$ and $c'' = d\omega/d\beta = c^2/c'$ for $\mu = \mu_0$ and $\kappa = \kappa_0$, holding essentially for air. Otherwise, we have $c'' = v^2/c'$, where $v = 1/\sqrt{\mu\kappa}$, since, for a perfect dielectric medium, the three-dimensional unbounded propagation velocity is

$$\frac{1}{\sqrt{\mu\kappa}} = \sqrt{c'c''} \qquad \text{m/sec}$$

The outcome for the $TE_{0,1}$ wave is then similar to formulas (551) and (552), except that dimension b instead of dimension a of the $TE_{1,0}$ wave occurs and the $\mathcal{E}_x$ is the only field component for the $TE_{0,1}$ wave. Of

course, for both the $TE_{1,0}$ and $TE_{0,1}$ excitations, we have additional longitudinal H components (H_z fields). The transverse fields for the $T_{1,0}$ excitation are $\mathcal{E}_y$ and H_x; and for the $TE_{0,1}$ excitation, we have $\mathcal{E}_x$ and H_y fields. For the $TE_{0,2}$ mode, we have $K = 2\pi/b$, while for the $T_{2,0}$ wave, we have $K = 2\pi/a$. Hence,

$$\left.\begin{aligned} \beta &= \sqrt{\omega^2\mu\kappa - \left(\frac{2\pi}{a}\right)^2} \quad \text{for } TE_{2,0} \text{ (also called } H_{2,0}\text{) wave} \\ \beta &= \sqrt{\omega^2\mu\kappa - \left(\frac{2\pi}{b}\right)^2} \quad \text{for } TE_{0,2} \text{ (also called } H_{0,2}\text{) wave} \end{aligned}\right\} \tag{565}$$

and

$$\left.\begin{aligned} c' &= \frac{\omega}{\sqrt{\omega^2\mu\kappa - \left(\frac{2\pi}{k}\right)^2}};\ c'' = \frac{d\omega}{d\beta} = \frac{\sqrt{\omega^2\mu\kappa - \left(\frac{2\pi}{k}\right)^2}}{\omega\mu\kappa} = \frac{1}{\mu\kappa c'} = \frac{v^2}{c'} \\ v &= \frac{1}{\sqrt{\mu\kappa}};\quad \begin{aligned} k &= a \quad \text{for } H_{2,0} \text{ wave} \\ k &= b \quad \text{for } H_{0,2} \text{ wave} \end{aligned} \end{aligned}\right\} \tag{566}$$

We have then similar expressions for all other types of rectangular $H_{3,0}$, $H_{4,0}$, $H_{5,0}$, · · · with the $k = a$ value and 3π, 4π, 5π numerators in the K term, and the $k = b$ value for the $H_{0,3}$, $H_{0,4}$, $H_{0,5}$, · · · longitudinal magnetic modes. For the general case of an $H_{m,n}$, *i.e.*, a $TE_{m,n}$, excitation, we have the formulas

$$K = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2};\qquad \beta = \sqrt{\omega^2\mu\kappa - K^2};\qquad c' = \frac{\omega}{\beta};$$

$$c'' = \frac{d\omega}{d\beta} = \frac{\beta}{\omega\mu\kappa} = \frac{\beta}{\mu\kappa c'} = \frac{v^2}{c'} \tag{567}$$

proving that for all possible rectangular modes of the type $TE_{m,n}$, also called $H_{m,n}$ modes, the relation $c'c'' = v^2$ holds. For air within the guide and strictly for empty space, the relation $c'c'' = c^2 = 9 \times 10^{16}$ m/sec² is again satisfied. For an $H_{3,2}$ excitation, we have for a rectangular guide

$$K = K_{3,2} = \sqrt{\left(\frac{3\pi}{a}\right)^2 + \left(\frac{2\pi}{b}\right)^2} \quad \text{and} \quad \beta = \beta_{3,2} = \sqrt{\omega^2\mu\kappa - K^2}$$

for the expressions of (567). We have similar relations for the general case of longitudinal $\mathcal{E}_{n,m}$ waves (also called $TM_{n,m}$) of circular guides and longitudinal $H_{n,m}$ waves (also called $TE_{n,m}$) for a radius of a meters.

We have then the formulas

$$\beta = \sqrt{\omega^2\mu\kappa - K^2}; \qquad K = \frac{p}{a}; \qquad f_c^{(\mathrm{Mc})} = 1.592 \times 10^{-7} vK;$$

$$v = \frac{1}{\sqrt{\mu\kappa}} \quad \text{m/sec} \tag{568}$$

Mode	$H_{0,1}$	$H_{1,1}$	$H_{2,1}$	$H_{3,1}$	$H_{0,2}$	$H_{1,2}$	$H_{2,2}$	$H_{3,2}$
p	3.832	1.1841	3 054	4.20	7.016	5.332	6.705	8.02

Mode	$H_{0,3}$	$H_{1,3}$	$H_{2,3}$	$H_{3,3}$
p	10.173	8.536	9.965	11.35

Longitudinal $H_{n,m}$ waves are transverse $TE_{n,m}$ modes.

Mode	$\mathcal{E}_{0,1}$	$\mathcal{E}_{1,1}$	$\mathcal{E}_{2,1}$	$\mathcal{E}_{3,1}$	$\mathcal{E}_{0,2}$	$\mathcal{E}_{1,2}$	$\mathcal{E}_{2,2}$	$\mathcal{E}_{3,2}$
p	2 405	3.832	5.135	6.38	5.52	7.016	8.417	9.76

Mode	$\mathcal{E}_{0,3}$	$\mathcal{E}_{1,3}$	$\mathcal{E}_{2,3}$	$\mathcal{E}_{3,3}$	$\mathcal{E}_{0,4}$	$\mathcal{E}_{1,4}$	$\mathcal{E}_{2,4}$	$\mathcal{E}_{3,4}$
p	8 654	10 173	11.62	13 02	11 79	13 32	14 8	16 22

Longitudinal $\mathcal{E}_{n,m}$ waves are transverse $TM_{n,m}$ modes.

The phase velocity $c' = \omega/\beta$ and the group velocity $c'' = d\omega/d\beta$ are, therefore, for all cases interlinked with the three-dimensional unbounded medium velocity $v = 1/\sqrt{\mu\kappa}$ meters per second by the relation $c'c'' = v^2$, which satisfies formula (532), since $c'' = 1/(\mu\kappa c')$.

Since the cutoff frequency f_c is given by

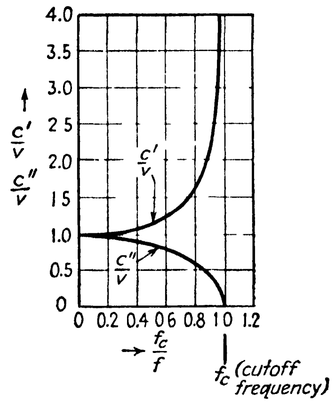
$$f_c = 1.592 \times 10^{-7} v \frac{p}{a} \qquad \text{Mc/sec} \tag{569}$$

and μ is in henrys per meter and κ in farads per meter in $v = 1/\sqrt{\mu\kappa}$ meters per second, we have for f_c in cycles per second and the cutoff wavelength λ_c in meters also the relation $f_c\lambda_c = v = 1/\sqrt{\mu\kappa}$. But for the cutoff frequency, the value of β must just vanish or $K^2 = \omega_c^2\mu\kappa = (\omega/v)^2$ and, therefore, $f_c = vK/6.28$, and

$$v = f_c\lambda_c = \frac{vK}{6.28}\lambda_c = \frac{pv\lambda_c}{6.28a} \qquad \text{or} \qquad \lambda_c = \frac{6.28a}{p} \quad \text{m} \tag{570}$$

where p is read off from the above table and a is the radius of the circular guide in meters.

Since for a circular guide, for the phase constant $\beta = \beta_z$ along the guide and for the cutoff frequency $f_c = \omega_c/6.28$ cycles per second, because



For a pure dielectric of μ henrys/m and κ farads/m within the guide we have the three-dimensional (space) velocity $v = 1/\sqrt{\mu\kappa}$ m/sec which is the wave velocity in absence of the metal envelope making up the guide. The *phase* velocity along the guide direction (Z direction) is

$$c' = \frac{\omega}{\beta_z} = \frac{v}{\sqrt{1 - \nu^2}} \quad \text{m/sec}$$

The group velocity is $c'' = d\omega/d\beta_z = v\sqrt{1 - \nu^2}$; $\nu = \dfrac{f_c}{f}$; f is the operating frequency, $v = 3 \times 10^8$ m/sec for air within the guide.

FIG. 61. Relations for the phase velocity c' and the group velocity c'' along the guide with respect to the operating frequency f.

$\beta = 0$ for such a condition, also $\omega_c/v = p/a$, or $p = a\omega_c/v$, we have

$$\beta = \sqrt{\omega^2\mu\kappa - \left(\frac{p}{a}\right)^2} = \sqrt{\left(\frac{\omega}{v}\right)^2 - \left(\frac{p}{a}\right)^2} = \sqrt{\left(\frac{\omega}{v}\right)^2 - \left(\frac{\omega_c}{v}\right)^2} = \frac{\sqrt{\omega^2 - \omega_c^2}}{v}$$

$$= \frac{\omega\sqrt{1 - (\omega_c/\omega)^2}}{v} = \frac{\omega\sqrt{1 - \nu^2}}{v}; \qquad \nu = \frac{f_c}{f}$$

For the phase velocity c' and group velocity c'', we have

$$c' = \frac{\omega}{\beta} = \frac{v}{\sqrt{1 - \nu^2}}; \qquad c'' = \frac{d\omega}{d\beta} = \frac{v^2}{c'} = v\sqrt{1 - \nu^2}; \qquad \nu^2 = \left(\frac{f_c}{f}\right)^2 \qquad (571)$$

For the rectangular guide, for which $K = m\pi/a$ and $\mathcal{E} = \mathcal{E}_y$, we have $m\pi/a = \omega_c/v$, and

$$\beta = \sqrt{\left(\frac{\omega}{v}\right)^2 - \left(\frac{m\pi}{a}\right)^2} = \sqrt{\left(\frac{\omega}{v}\right)^2 - \left(\frac{\omega_c}{v}\right)^2} = \frac{\sqrt{1 - \nu^2}}{v}$$

for a transverse $\mathcal{E}_y$ field (572)

giving again the same formula for the phase constant along the guide. As a matter of fact, we find the same result for the other rectangular guide modes, since

$$\left.\begin{aligned}
&\beta = \sqrt{\left(\frac{\omega}{v}\right)^2 - K^2}; \quad K = \frac{\omega_c}{v}; && K = \frac{n\pi}{b} \text{ for transverse } \mathcal{E}_x \text{ fields} \\
&\beta = \frac{\omega\sqrt{1-\nu}}{v} && K = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \\
&\nu = \frac{f_c}{f}; \quad c' = \frac{\omega}{\beta} = \frac{v}{\sqrt{1-\nu^2}} && \text{for transverse } \mathcal{E} = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} \text{ fields} \\
&c'' = \frac{v^2}{c'} = v\sqrt{1-\nu^2}; \quad v = \frac{1}{\sqrt{\mu\kappa}}
\end{aligned}\right\} \quad (573)$$

Figure 61 gives plots of this general outcome for the discussed modes of rectangular and circular wave guides.

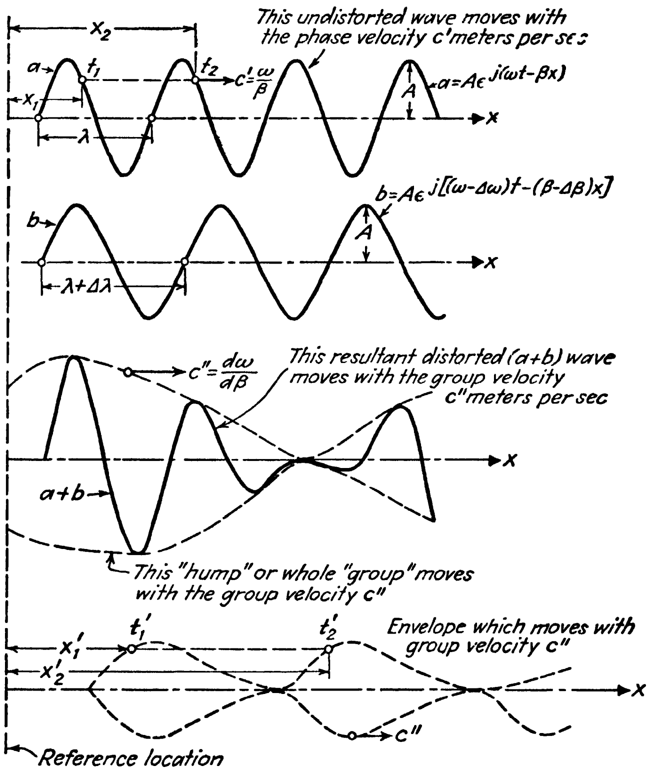


FIG. 62. Details with respect to the group velocity c'' due to two long wave trains of slightly different wavelength (classic Rayleigh explanation).

73. Remarks on Phase- and Group-velocity Formulas. According to Lord Rayleigh, formulas for the group velocity can be derived by means of two *infinitely* long wave trains. In Fig. 62, two waves of the

same amplitude and *slightly different* angular velocities ω and $\omega + \Delta\omega$ are traveling along the X direction. At any instant, we have then the respective intensities

$$a = A\epsilon^{j(\omega t - \beta x)}; \qquad b = A\epsilon^{j[(\omega+\Delta\omega)t-(\beta+\Delta\beta)x}$$

with a resultant effect

$$a + b = A\epsilon^{j(\omega t - \beta x}(1 + \epsilon^{jp}); \qquad p = \Delta\omega t - \Delta\beta x$$

But

$$1 + \epsilon^{jp} = 1 + \cos p + j \sin p = \sqrt{(1 + \cos p)^2 + \sin^2 p}$$
$$= \sqrt{2(1 + \cos p)} = 2\sqrt{\frac{1 + \cos p}{2}} = 2 \cos \frac{p}{2}$$

Hence,

$$\left.\begin{aligned} a &= A\epsilon^{j(\omega t - \beta x)} \\ b &= A\epsilon^{j[(\omega+\Delta\omega)t-(\beta+\Delta\beta)x]} \\ a + b &= \underbrace{2A \cos [\tfrac{1}{2}(\Delta\omega t - \Delta\beta x)]}_{\text{magnitude of resultant wave}}\epsilon^{j(\omega t - \beta x)} \end{aligned}\right\} \tag{574}$$

Let us consider for the a wave of Fig 62, a portion of an infinitely long sinusoidal variation. Hence, at the moment t_1, we are at a distance x_1 and the wave has passed through many wavelengths when reaching plane $x = x_1$. Plane waves that have a fixed phase in this plane are considered. This surface moves with the phase velocity c' along the $+X$ direction, say to a plane $x = x_2$, corresponding to an instant t_2 of arrival. Both positions correspond to

$$a_1 = A\epsilon^{j(\omega t_1 - \beta x_1)} = a_2 = A\epsilon^{j(\omega t_2 - \beta x_2)}$$

since for a fixed phase, $\omega t_1 - \beta x_1 = \omega t_2 - \beta x_2$ and intensity $a_1 = a_2$. Hence,

$$c' = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\omega}{\beta} \qquad \text{m/sec} \tag{575}$$

must be the velocity of propagation of the *phase* or *state*, which does not necessarily imply that power is transferred at the same rate, although this may happen under certain conditions. By "power" is meant any signal or intelligence that carries its own modulation (movement of modulation energy). Inasmuch as the phase velocity c' is a steady-state property of an undistorted pure harmonic wave train, it is possible to have the magnitude of this velocity c' larger and even very much larger than the velocity $c = 3 \times 10^8$ m/sec. A phase-distorted, frequency-distorted, or amplitude-distorted wave train will then, as a rule, propagate the distortion that represents the modulation energy with a velocity

c'', which may be different from the phase velocity c'. The velocity c'', however, can *never* exceed the value of c but can have any value smaller than 3×10^8 m/sec, even as small as zero. The latter would, in the case of certain wave-guide excitations, correspond to a phase velocity that is infinite, since $v \cong c = \sqrt{c'c''}$ and states, in this case, that no power flow occurs along the guide (happens at the cutoff frequency f_c of the guide). This is also true in a *dispersive* medium such as the ionized region of the upper atmosphere. Now the distortion in amplitude modulation is exactly the same type as the $(a + b)$ wave of Fig. 62, and the amplitude or envelope curve within which HF variations occur is a measure for the degree of modulation at any instant as well as at any point in the space through which the AM wave passes. In other words, the amplitude distortion expressed by the envelope of the $(a + b)$ wave (resultant wave) indicates the intelligence to be propagated. We have again the same argument $(\omega t - \beta x)$ as for the unmodulated wave (a wave or b wave) but an amplitude $2A \cos [\frac{1}{2}(\Delta\omega t - \Delta\beta x)]$ instead of the fixed amplitude A of a pure harmonic wave such as a and b. This variable amplitude represents the intelligence propagated, and the $(a + b)$ contour is moved *un*disturbed with a velocity c'' when for times t_1' and t_2' and corresponding positions x_1' and x_2'

$$\Delta\omega t_1' - \Delta\beta x_1' = \Delta\omega t_2' - \Delta\beta x_2'$$

or

$$c'' = \frac{x_2' - x_1'}{t_2' - t_1'} = \frac{\Delta\omega}{\Delta\beta}$$

yielding, for *small* frequency differences of the a and b component waves,

$$c'' = \frac{d\omega}{d\beta} \tag{576}$$

with which the *whole* $(a + b)$ *group* is *moved* along the X axis.

Since $\beta = 2\pi/\lambda$ and $c' = \omega/\beta$, we have, for the group velocity, any of the following relations

$$c'' = \frac{d\omega}{d\beta} = \frac{d(c'\beta)}{d\beta} = c' + \beta\frac{dc'}{d\beta} = c' - \lambda\frac{dc'}{d\lambda} = \frac{df}{d(1/\lambda)} = \frac{1}{d\beta/d\omega} \tag{577}$$

where, for instance, in transmission lines and wave guides, the formula $c'' = 1/(d\beta/d\omega)$ is readily evaluated.

The *assumption* behind the result for c'' is, therefore, that two wave trains *slightly* different in frequency cause a "power hump," which is being propagated with the velocity c''. Thus, we generally have a wavelength $\lambda_1 = \lambda$ and a wavelength $\lambda_2 = \lambda + \Delta\lambda$ for the component waves for the phase velocities c_1 and c_2 and phase constants $\beta_1 = 2\pi/\lambda_1$ and

$\beta_2 = 2\pi/\lambda_2$. The resultant distorted wave then is

$$\begin{aligned} a + b &= A \sin \beta_1(x - c_1t) + A \sin \beta_2(x - c_2t) \\ &= 2A \sin \{\tfrac{1}{2}[\beta_1(x - c_1t) + \beta_2(x - c_2t)]\} \\ &\qquad \times \cos \{\tfrac{1}{2}[\beta_1(x - c_1t) - \beta_2(x - c_2t)]\} \\ &= 2A \sin \pi \left[\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)x - \left(\frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2}\right)t\right] \\ &\qquad \times \cos \pi \left[\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)x - \left(\frac{c_1}{\lambda_1} - \frac{c_2}{\lambda_2}\right)t\right] \end{aligned}$$

Since, for our *assumption*, λ_1 and λ_2 are almost equal, *we have the approximations* $\lambda_1 \cong \lambda \cong \lambda_2$, $c_1 \cong c' \cong c_2$, and

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{2}{\lambda}; \quad \frac{1}{\lambda_1} - \frac{1}{\lambda_2} = d\left(\frac{1}{\lambda}\right); \quad \frac{c_1}{\lambda_1} + \frac{c_2}{\lambda_2} = \frac{2c'}{\lambda}; \quad \frac{c_1}{\lambda_1} - \frac{c_2}{\lambda_2} = d\left(\frac{c'}{\lambda}\right)$$

Hence, for $\beta = 2\pi/\lambda$, we find for the resultant wave

$$a + b = \underbrace{2A\left\{\cos \pi \left[d\left(\frac{1}{\lambda}\right)x - d\left(\frac{c'}{\lambda}\right)t\right]\right\}}_{\text{distorted (AM) amplitude } A'} \sin \beta(x - c't) \tag{578}$$

This expression, like the expression for $a + b$ in Eq. (574), consists of a variable amplitude A' and a phase term $\sin \beta(x - c't)$. The phase term gives a means of finding an expression for the phase velocity c', and the amplitude A' gives a means of finding the value for the group velocity c'', since it is the interference hump that represents the signal energy (intermodulation energy of a on b or b on a) moving out with the hump or group velocity c''. When the hump, *i.e.*, the resultant wave envelope, moves with c'', any point of the envelope must also move with this velocity. A convenient point is the condition for which the variable amplitude A' becomes a maximum. This happens when the argument of the cosine term vanishes, or

$$x \cdot d\left(\frac{1}{\lambda}\right) - t \cdot d\left(\frac{c'}{\lambda}\right) = 0$$

Since x/t is the velocity c'' with which this chosen point moves along the X direction, we find

$$c'' = \frac{x}{t} = \frac{d(c'/\lambda)}{d(1/\lambda)} = \frac{d(2\pi c'/\lambda)}{d(2\pi/\lambda)} = \frac{d\omega}{d\beta} \tag{579}$$

since

$$\frac{2\pi c'}{\lambda} = \frac{2\pi\omega}{\beta\lambda} = \frac{\beta\omega}{\beta} = \omega$$

confirming (577).

It is now of interest to find out what happens when a single group of any envelope shape, instead of periodically recurring groups, is being propagated. We have then a single arbitrarily shaped pulse that, according to the Fourier integral theory, is equivalent to many component harmonic waves, with amplitudes depending on β, *i.e.*, on ω or the frequency. For such a single pulse, the steady-state condition giving the phase velocity c' as velocity of propagation does *not* hold. The comparison wave functions along the X direction then are

$$\left.\begin{aligned} w(x, t) &= A\epsilon^{j(\omega t-\beta x)} && \text{for a harmonic wave with a phase velocity } \omega/\beta \\ w(x, t) &= \int_{-\infty}^{+\infty} A(\beta)\epsilon^{j(\omega t-\beta x)}\,d\beta && \text{for a single pulse} \end{aligned}\right\} \tag{580}$$

We note that for the infinite number of component waves, we now have amplitudes $A(\beta)$, which depend on β, *i.e.*, on the frequency. Components for which the value of $A(\beta)$ is *small* in relation to the center reference amplitude are neglected. According to the definition of the group velocity c'', based on the assumption made for Eq. (578), this velocity has only a meaning, and can be applied to the net effect of several significant component waves $\Sigma A(\beta)\epsilon^{j(\omega t-\beta x)}$ only when these waves occupy a narrow frequency spectrum. This means waves of length $\lambda_0 \pm k\,\Delta\lambda$, which for the largest value of factor k are only a small percentage larger or smaller than the average value λ_0. The increasing value of $k\,\Delta\lambda$, the change in the respective phase velocities c' of the component waves in a medium, where c' depends on the frequency, may become large enough so that the wave function $w(x, t)$ rapidly becomes distorted and the concept of group velocity *no longer has a physical significance*. The $k\,\Delta\lambda$ deviation corresponds to the change $\Delta\beta$. Neglecting insignificant components, the integration of Eq. (580) need be extended only from $\beta_0 - \Delta\beta$ to $\beta_0 + \Delta\beta$, where β_0 stands for the average value of the phase constant $2\pi/\lambda_0$. We have then the wave function

$$w(x, t) = \int_{\beta_0-\Delta\beta}^{\beta_0+\Delta\beta} A(\beta)\epsilon^{j(\omega t-\beta x)}\,d\beta$$

For a sufficiently *small value* of $2\Delta\beta$, which is the total *useful spectrum spread*, it can be shown[1] that the group velocity is $c'' = d\omega/d\beta_0$.

73a. Notes on a Dispersive Medium in Connection with Velocity of Propagation. A medium in which the phase velocity of a wave depends on the frequency of said wave is known as "dispersive." The electromagnetic constants that characterize a medium and affect wave propaga-

[1] STRATTON, *op. cit.*, p. 332.

tion are the dielectric constant κ in farads per meter, magnetic permeability μ in henrys per meter, and the specific conductivity σ in mhos per meter.

In the mks system of units, we have for vacuum, which is zero material medium, $\kappa = \kappa_0 = 8.854 \times 10^{-12}$ farad/m, $\mu = \mu_0 = 1.1257 \times 10^{-6}$ henry/m, and $\sigma = 0$ mhos/m. For any other medium, we have the specific or relative numeric values $\kappa_r = \kappa/\kappa_0$, $\mu_r = \mu/\mu_0$. For water, we have $\kappa_r \cong 80$ at room temperature. The relative constants are numeric, since they are mere ratios. Moreover, in vacuum, we have $1/\sqrt{\mu_0\kappa_0} = 2.998 \times 10^8$ m/sec $= c$ (exact value) and $\sqrt{\mu_0/\kappa_0} = 376.7$ ohms, which is the intrinsic space wave impedance in empty space and practically so in the distant radiation region, since $\mathcal{E}_\theta/H_\varphi = 376.7$ ohms for the significant perpendicular field associations $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter of such a region. Also for wave motion in space, with constants κ, μ, and σ, the three-dimensional unbounded propagation constant Γ is generally complex, *i.e.*, of the form $\alpha + j\beta$, with an attenuation constant α nepers per meter and phase constant β circular radians per meter in any direction for an isotropic medium. Hence,

$$\Gamma = \alpha + j\beta = j\omega\sqrt{\mu\kappa_e}; \qquad \kappa_e = \kappa - j\sigma/\omega$$

where $\omega/6.28$ is cycles per second.

$$\Gamma^2 = -\omega^2\left(\mu\kappa - \frac{j\mu\sigma}{\omega}\right) = \underbrace{-\omega^2\mu\kappa}_{\alpha^2-\beta^2} + \underbrace{j\omega\mu\sigma}_{2\alpha\beta} = (\alpha + j\beta)^2 = \underbrace{\alpha^2 - \beta^2}_{\text{real term}} + \underbrace{j2\alpha\beta}_{\text{imaginary term}}$$

Hence, $\beta = \omega\mu\sigma/(2\alpha)$, and

$$-\omega^2\mu\kappa = \alpha^2 - \beta^2 = \alpha^2 - \frac{\omega^2\mu^2\sigma^2}{4\alpha^2}$$

or

$$\alpha^4 + \mu\kappa\omega^2\alpha^2 - \tfrac{1}{4}\omega^2\mu^2\sigma^2 = 0$$

But $A\alpha^4 + B\alpha^2 + C = 0$ yields $\alpha = \sqrt{(-B \pm \sqrt{B^2 - 4AC})/(2A)}$ and

$$\alpha = \sqrt{\frac{\omega^2\mu\kappa \pm \sqrt{\omega^4\mu^2\kappa^2 + \omega^2\mu^2\sigma^2}}{2}} = \sqrt{0.5\omega^2\mu\kappa \pm 0.5\mu\kappa\sqrt{\omega^4 + \frac{\omega^2\mu^2\sigma^2}{\mu^2\kappa^2}}}$$

$$= \sqrt{0.5\mu\kappa\omega^2 + 0.5\mu\kappa\omega^2\sqrt{1 + \frac{\sigma^2}{\kappa^2\omega^2}}} = \omega\sqrt{0.5\mu\kappa\left[1 + \sqrt{1 + \left(\frac{\sigma}{\omega\kappa}\right)^2}\right]}$$

Therefore, we have the formulas

$$\left.\begin{aligned} &\alpha = \omega\sqrt{a(\sqrt{b} - 1)} \quad \text{nepers/m;} \\ &\beta = \omega\sqrt{a(\sqrt{b} + 1)} \quad \text{circular radians/m;} \qquad a = \frac{1}{2}\mu\kappa = \frac{0.5}{v^2} \\ &b = 1 + \frac{1}{Q^2}; \qquad Q = \frac{\omega\kappa}{\sigma} \end{aligned}\right\} \qquad (581)$$

where v is in meters per second, $\omega/6.28$ in cycles per second, μ in henrys per meter, κ in farads per meter, σ in mhos per meter, since $\beta = \omega\mu\sigma/(2\alpha)$ and $\alpha^2 - \beta^2 = -\omega^2\mu\kappa$,

only (-1) and $(+1)$ distinctions. The quantity Q is the quality of the wave medium. It is understood that $v = 1/\sqrt{\mu\kappa}$ is only an ideal reference velocity, since the actual velocity is ω/β for the general medium and β is found from $\Gamma = \alpha + j\beta$.

Equations (581) are then relations for the general *unobstructed and unbounded medium.* For empty space, we have

$$a = 0.5\mu_0\kappa_0 = 0.5 \times 1.257 \times 10^{-6} \times 8.854 \times 10^{-12}$$
$$= 0.5(1/9) \times 10^{-16} = 5.555 \cdots \times 10^{-18} \text{ sec}^2/\text{m}^2$$

which is half the value of the oscillation constant $\mu_0\kappa_0$ of a vacuum, since $1/\kappa_0$ may be called the "elastivity" and $1/\mu_0$ the "reluctivity" of the vacuum. We have also $b = 1$ for vacuum, since $\sigma = 0$ mhos/m and $\alpha = \alpha_0 = \omega\sqrt{\tfrac{1}{2}\mu_0\kappa_0(1 - 1)} = 0$ nepers/m, proving that for vacuum there can be no medium dissipation. But for the phase constant, we find $\beta = \beta_0 = \omega\sqrt{0.5\mu_0\kappa_0(1 + 1)} = \omega/c = 2\pi f/c = 2\pi/\lambda$. For such an ideal medium (the best medium imaginable), we find that wave propagation occurs with the velocity c of light. Since the phase velocity c' for any medium is ω/β, we find from (581)

$$c' = \frac{1}{\sqrt{a(\sqrt{b} + 1)}} = \frac{1}{\sqrt{\mu_0\kappa_0}\sqrt{0.5\dfrac{\mu\kappa}{\mu_0\kappa_0}\left(\sqrt{1 + \dfrac{\sigma^2}{\kappa^2\omega^2}} + 1\right)}}$$
$$= \frac{3 \times 10^8}{\sqrt{0.5\mu_r\kappa_r\left[\left(\sqrt{1 + \left(\dfrac{\sigma}{\kappa\omega}\right)^2} + 1\right)\right]}} \qquad (582)$$

Hence, when μ, κ, and σ are independent of the frequency, the phase velocity increases with the frequency. For instance, for dry land with $\sigma = 10^{-4}$ mho/m and a relative dielectric constant $\kappa_r = \kappa/\kappa_0 = 10$, we have $\kappa = 10\kappa_0 = 10 \times 8.854 \times 10^{-12} = 8.854 \times 10^{-11}$ farad/m, $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m, since for nonferromagnetic dry ground $\mu_r = \mu/\mu_0 = 1$. Then $a = 0.5\mu\kappa = 0.5 \times 1.257 \times 10^{-6} \times 8.854 \times 10^{-11} = 55.55 \times 10^{-18}$. For 796 cycles/sec we have $\omega = 6.28 \times 796 = 5 \times 10^3$; for $f = 796$ kc/sec, we have $\omega = 5 \times 10^6$; and for 796 Mc/sec, we have $\omega = 5 \times 10^9$. We find then $\sigma^2 = 10^{-8}$ (mho/m)^2 and $\kappa^2 = 78 \times 10^{-22}$ (farad/m)^2. For the chosen frequencies, we find

$$\frac{1}{Q^2} = \frac{\sigma^2}{\omega^2\kappa^2} = \frac{\sigma^2}{\kappa^2}\frac{1}{\omega^2} = \frac{1.28 \times 10^{12}}{\omega^2} = \frac{128 \times 10^{10}}{25 \times 10^6} = 5.12 \times 10^4 \text{ for } f = 796 \text{ cycles/sec}$$

$$\frac{\sigma^2}{\omega^2\kappa^2} = \frac{128 \times 10^{10}}{25 \times 10^{12}} = 5.12 \times 10^{-2} \text{ for } f = 796 \text{ kc/sec}$$

$$\frac{\sigma^2}{\omega^2\kappa^2} = \frac{128 \times 10^{10}}{25 \times 10^{18}} = 5.12 \times 10^{-8} \text{ for } f = 796 \text{ Mc/sec}$$

Hence,

$$\sqrt{b} = \sqrt{1 + \frac{1}{Q^2}} = \sqrt{51201} = 226.274 \text{ at 796 cycles/sec}$$

addition of 1 has *no* effect

$$\sqrt{b} = \sqrt{1 + \frac{1}{Q^2}} = \sqrt{1.0512} = 1.025 \text{ at 796 kc/sec}$$

addition of 1 is *most* significant

$$\sqrt{b} = \sqrt{1 + \frac{1}{Q^2}} = \sqrt{1.0000000512} = 1 \text{ at 796 Mc/sec}$$

addition of 1 is *only important*

$$c' = \frac{1}{\sqrt{a(\sqrt{b} + 1)}} = 0.12 \times 10^6 \text{ m/sec at 796 cycles/sec}$$

$$c' = \frac{1}{\sqrt{a(\sqrt{b} + 1)}} = 1.27 \times 10^8 \text{ m/sec at 796 kc/sec}$$

$$c' = \frac{1}{\sqrt{a(\sqrt{b} + 1)}} = 1.28 \times 10^8 \text{ m/sec at 796 Mc/sec}$$

showing that even at 796 Mc/sec, the phase velocity is only 1.28×10^8 m/sec instead of 3×10^8 m/sec. This is about the highest obtainable phase velocity with increasing frequency, since, according to formula (582), we have $\sqrt{b} + 1 = 2$ when $1/Q^2 = \sigma^2/(\omega\kappa)^2$ is negligible compared with unity so that no frequency effect can be influential on the phase velocity c' at still higher frequencies. It will be noted that at lower frequencies, the phase velocity becomes especially small. This can be understood from Maxwell's equations, which give the current density at any place of a partially conducting medium as

$$\underbrace{\sigma\mathcal{E}}_{\substack{\text{conduction}\\ \text{current density}\\ \text{in amp/m}^2}} + \underbrace{\kappa\frac{\partial\mathcal{E}}{\partial t}}_{\substack{\text{displacement}\\ \text{current density}\\ \text{in amp/m}^2}} = \underbrace{(\sigma + \omega\kappa)\mathcal{E}}_{\substack{\text{for harmonic}\\ \text{waves}}} \tag{583}$$

If $\mathcal{E}$ now denotes the effective field in volts per meter, we have the scalar value of the current density in amperes per square meter as $\mathcal{E}\sqrt{\sigma^2 + \omega^2\kappa^2}$ with a phase angle $\varphi = \tan^{-1}(\omega\kappa/\sigma)$ and $\omega\kappa/\sigma$ gives the ratio of displacement current to conduction current at any point in the medium. Hence, from a certain frequency on, only the displacement current plays a significant part for the square-root value; while for lower frequencies, it is the conduction current that is mostly responsible in current displays. Let us now see what happens when an electromagnetic wave penetrates ocean water. We have then $\kappa_r = \kappa/\kappa_0 \cong 80$; σ is from 3 to 5 mhos/m depending on the salinity and the temperature. Assuming

a value of $\sigma = 5$ mhos/m, we find $\kappa = 80\kappa_0 = 80 \times 8.854 \times 10^{-12} = 71 \times 10^{-11}$ farad/m. The permeability μ of the ocean is essentially the same as for vacuum; *i.e.*, $\mu \cong \mu_0 = 1.257 \times 10^{-6}$ henry/m. We have then $a = 0.5\mu\kappa = 0.5 \times 1.257 \times 10^{-6} \times 71 \times 10^{-11} = 445 \times 10^{-16}$ $(\text{sec/m})^2$. Again for 796 cycles/sec, 796 kc/sec, and 796 Mc/sec, we have $\sigma^2 = 25(\text{mhos/m})^2$, $\kappa^2 = 5041 \times 10^{-22}$ $(\text{farad/m})^2$. Therefore,

$$\frac{1}{Q^2} = \frac{\sigma^2}{\omega^2\kappa} = \frac{25}{25 \times 10^6 \times 5041 \times 10^{-22}} = 1.98 \times 10^{12} \text{ for 796 cycles/sec}$$

$$\frac{1}{Q^2} = \frac{\sigma^2}{\omega^2\kappa^2} = \frac{25}{25 \times 10^{12} \times 5041 \times 10^{-22}} = 1.98 \times 10^{6} \text{ for 796 kc/sec}$$

$$\frac{1}{Q^2} = \frac{\sigma^2}{\omega^2\kappa^2} = \frac{25}{25 \times 10^{18} \times 5041 \times 10^{-22}} = 1.98 \text{ for 796 Mc/sec}$$

$$\left.\begin{aligned} \sqrt{b} &= \sqrt{1.98 \times 10^{12}} = 1.405 \times 10^6 \text{ for 796 cycles/sec} \\ \sqrt{b} &= \sqrt{1.98 \times 10^{6}} = 1.405 \times 10^3 \text{ for 796 kc/sec} \end{aligned}\right\} \text{since the addition of 1 to } (\sigma/\omega\kappa)^2 \text{ has } \textit{no} \text{ effect}$$

$$\sqrt{b} = \sqrt{2.98} = 1.72 \text{ for 796 Mc/sec}$$

We have then the corresponding phase velocities

$$\left.\begin{aligned} c' &= \frac{1}{\sqrt{a(\sqrt{b}+1)}} = \frac{1}{\sqrt{445 \times 10^{-16} \times 1.405 \times 10^6}} \\ &= \frac{476 \times 10^4}{\sqrt{1.405 \times 10^6}} = 4{,}040 \text{ m/sec at 796 cycles/sec} \\ c' &= \frac{1}{\sqrt{a(\sqrt{b}+1)}} = \frac{476 \times 10^4}{\sqrt{14.05 \times 10^2}} \\ &= 12.7 \times 10^4 \text{ m/sec at 796 kc/sec} \\ c' &= \frac{1}{\sqrt{a(\sqrt{b}+1)}} = \frac{476 \times 10^4}{\sqrt{2.72}} \\ &= 289 \times 10^4 \text{ m/sec at 796 Mc/sec} \end{aligned}\right\} \text{determines the phase (or wave) distribution}$$

Therefore, we have again much smaller different phase velocities from $c = 3 \times 10^8$ m/sec holding for empty space. From the comparative values of Table 12, it is noted that both the conductivity and the dielectric constant of the medium affect the phase velocity, the relative degree of the effect depending on whether the conduction or the dispacement current is more significant. Since in formula (582), the term $(\sigma/\omega\kappa)^2$ is responsible for the *dispersive* effect, because for $(\sigma/\omega\kappa)^2 = 0$,

$$c' = c\sqrt{\frac{\mu_0\kappa_0}{\mu\kappa}} = \frac{1}{\sqrt{\mu\kappa}} \tag{584}$$

TABLE 12

Frequency	Propagation in ocean for $\sigma = 5$ mhos/m, $\kappa = 71 \times 10^{-11}$ farad/m, $\mu = 1.257 \times 10^{-6}$ henries/m		Propagation in dry ground for $\sigma = 10^{-4}$ mho/m, $\kappa = 8.854 \times 10^{-11}$ farad/m, $\mu = 1.257 \times 10^{-6}$ henry/m	
	Phase velocity c', m/sec	Percentage of $c = 3 \times 10^8$ m/sec	Phase velocity c', m/sec	Percentage of $c = 3 \times 10^8$ m/sec
796 cycles/sec	$0\ 0000108 \times 10^8$	0 00136%	$0\ 12 \times 10^8$	4%
796 kc/sec	$0\ 00127 \times 10^8$	0.0423%	$1\ 27 \times 10^8$	42 3%
796 Mc/sec	0.0289×10^8	0.96%	1.28×10^8	42 7%

the phase velocity c' becomes *independent* of the frequency as long as the dielectric constant κ and the permeability μ of the medium are independent of the frequency. The term $q = \sigma^2/(\omega\kappa)^2$ for a perfect insulator is zero, since $\sigma = 0$. For such a case, Eq. (584) holds. Any modulated wave, irrespective of the type and shape of modulation, will then be propagated through the medium μ, κ, $\sigma = 0$ without distortion, since even a pulse modulation will in equivalence be a superposition of infinitely long harmonic wave trains, all of which will have the same velocity $1/\sqrt{\mu\kappa}$. But for values of q that are not negligible in comparison with unity, the phase velocity c' becomes a function of the frequency even though we have μ and κ as well as σ independent of the frequency. According to (581), we have both phase and amplitude distortion during the process of wave propagation. The ratio of c c' is, according to (582),

$$\frac{c}{c'} = \sqrt{\frac{\mu\kappa}{2\mu_0\kappa_0}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\kappa^2}} + 1\right)} \qquad \text{numeric} \tag{585}$$

and is sometimes known in electro-optics as the index n of refraction, even though this is strictly true only for $q = 0$ (or at least negligible) in comparison with unity, since

$$\frac{c}{c'} = \sqrt{\frac{\mu\kappa}{\mu_0\kappa_0}} = \sqrt{\mu_r\kappa_r} \quad \text{for all nonconductive mediums}$$

$$\frac{c}{c'} = \sqrt{\kappa_r} \quad \text{for nonconductive mediums with } \mu = \mu_0 \tag{586}$$

where μ_r and κ_r are the relative values μ/μ_0 and κ/κ_0, respectively. According to (585) and Table 12, at 796 Mc/sec for ocean water, we have $c/c' = 3 \times 10^8/(0.0289 \times 10^8) = 103.7$ while for dry ground, it is $3 \times 10^8/(1.28 \times 10^8) = 2.34$ only. Using formula (586), we have $c/c' = \sqrt{80} = 8.91$ as the index n of refraction for ideal distilled water.

At this point, it should be understood that the ratio c/c' in electro-optics deals only with perfect dielectrics and that for a complex medium (σ not equal to zero or σ/ω not negligible in comparison with the value of κ in $\kappa_e = \kappa - j\sigma/\omega$ with $\omega/6.28$ in cycles per second), the ratio of the propagation constants of the complex medium and of vacuum determines the index of refraction. This is dealt with on pages 57–59 and 798–863.

The dielectric constant κ, at very high and ultrahigh frequencies is not necessarily constant, though holding for quite a range. This seems especially true with sea water, for instance. This constant is also a function of the temperature. At room temperature, ordinary water has a value $\kappa_r \cong 80$ but a value of only 4 in the case of ice. It is also found that in all mediums which are not ionized, $c' < c$ and $c/c' > 1$; hence, the wavelength λ must be shorter than in empty space. In the ionized region of the upper atmosphere, $c' > c$ and very long wavelengths in comparison with the wavelength in vacuum may occur.

We noted that the magnitude of the term $\sigma^2/(\omega\kappa)^2$ with respect to unity gives an idea as to what extent the dielectric displacement current density at any space point in a partially conducting medium plays a part in comparison with the conduction current density. The formulas for the attenuation constant α and the phase constant β as given in Eq. (581) are then analyzed with respect to $q = \sigma^2/(\omega\kappa)^2$ either very much smaller than unity or very much larger than unity with the following details.

Since $\alpha = \omega\sqrt{a(\sqrt{b} - 1)}$ and $\beta = \omega\sqrt{a(\sqrt{b} + 1)}$, we have $\sqrt{a(\sqrt{b} \mp 1)}$ as the characteristic expression in each case, with $\sqrt{b}$ as the governing term. Since $b = 1 + q$, we find for $q \ll 1$, from the binomial theorem,

$$\sqrt{b} = \sqrt{1+q} = (1+q)^{1/2} = 1 + \frac{1}{2}q + \frac{\frac{1}{2}(\frac{1}{2} - 1)q^2}{1 \times 2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)q^3}{1 \times 2 \times 3}$$
$$+ \cdots = 1 + \frac{q}{2} - \frac{q^2}{8} + \frac{q^3}{16} - \cdots \cong 1 + \frac{q}{2}$$

and

$$\alpha \cong \omega\sqrt{a\left(1 + \frac{q}{2} - 1\right)} = \omega\sqrt{\frac{aq}{2}} = \omega\sqrt{\frac{\mu\kappa}{2}\frac{q}{2}} = \frac{\omega}{2}\sqrt{\mu\kappa\frac{\sigma^2}{\omega^2\kappa^2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}$$

$$\beta \cong \omega\sqrt{a\left(1 + \frac{q}{2} + 1\right)} = \omega\sqrt{a\left(2 + \frac{q}{2}\right)} = \omega\sqrt{\frac{\mu\kappa}{2}\left(2 + \frac{q}{2}\right)} = \omega\sqrt{\mu\kappa}\sqrt{1 + \frac{q}{4}}$$
$$\cong \omega\sqrt{\mu\kappa}\left(1 + \frac{q}{8} - \cdots\right)$$

negligible

$$= \omega\sqrt{\mu\kappa}\left(1 + \frac{\sigma^2}{8\omega^2\kappa^2} - \frac{\sigma^4}{16 \times 8\omega^4\kappa^4} + \frac{\sigma^8}{64 \times 16\omega^8\kappa^8} - \cdots\right)$$

For $q \gg 1$, we find

$$\sqrt{b} = \sqrt{1+q} \cong \sqrt{q} = \frac{\sigma}{\omega\kappa}; \qquad \sqrt{b}+1 \cong \frac{\sigma}{\omega\kappa} \cong \sqrt{b}-1$$

$$\alpha = \beta \cong \omega\sqrt{a\frac{\sigma}{\omega\kappa}} = \omega\sqrt{\frac{\mu\kappa}{2}\frac{\sigma}{\omega\kappa}} = \sqrt{\frac{1}{2}\omega\mu\sigma}$$

The formulas for the extreme ranges $q \ll 1$ and $q \gg 1$ are then

$$\left.\begin{aligned}
&\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}}; \qquad \beta = \omega\sqrt{\mu\kappa}\left(1 + 0.125\frac{\sigma^2}{\omega^2\kappa^2}\right) \\
&\qquad\qquad \text{for } \frac{\sigma^2}{\omega^2\kappa^2} \ll 1 \qquad \text{mostly } \textit{displacement} \text{ current} \\
&\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = 1.772 \times 10^3\sqrt{\mu\sigma f} \\
&\qquad\qquad \text{for } \frac{\sigma^2}{\omega^2\kappa^2} \gg 1 \qquad \text{mostly } \textit{conduction} \text{ current}
\end{aligned}\right\} \quad (587)$$

where σ is in mhos per meter, μ in henrys per meter, κ in farads per meter, $\omega/6.28$ in cycles per second, and f in megacycles per second. It is seen that the particular medium is responsible for either the extreme ranges or the frequency or both. For instance, let us take again the case of soil that is wet, and we have an average conductivity of about 5×10^{-3} mho across the surfaces of a 1-m cube, which means $\sigma = 5 \times 10^{-3}$ mho/m; the dielectric constant is then about $25 \times \kappa_0 = 25 \times 8.854 \times 10^{-12} = 2.2135 \times 10^{-10}$ farad/m, while μ for nonferromagnetic soil is the value holding for vacuum, or $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m. For a frequency of 10 Mc/sec, we have then

$$q = \frac{\sigma^2}{(\omega\kappa)^2} = \frac{25 \times 10^{-6}}{(6.28 \times 10 \times 10^6 \times 2.2135 \times 10^{-10})^2} = 0.13$$

while for the same medium but for 100 and 1,000 Mc/sec, respectively, we find $q = 0.13/10^2 = 0.0013$ and $0.13/10^4 = 0.000013$, so that the 100- and 1,000-Mc wave propagations in wet soil could be readily computed for the $q \ll 1$ condition. For dry soil, we have $\kappa \cong 10\kappa_0 = 8.854 \times 10^{-11}$ farad/m, $\sigma \cong 10^{-5}$ mho/m, and μ as before for nonferrous ground. Hence, at 10 Mc/sec,

$$q = \frac{10^{-10}}{(6.28 \times 10^7 \times 8.854 \times 10^{-11})^2} = 3.27 \times 10^{-6}$$

Hence, even for the 10-Mc wave, $q \ll 1$. For the 100- and 1,000-Mc waves, we have the respective values $q = 3.27 \times 10^{-8}$ and 3.27×10^{-10}.

For the dry soil, we have then

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\kappa}} = 5 \times 10^{-6}\sqrt{\frac{1.257 \times 10^{-6}}{8.854 \times 10^{-11}}} = 5 \times 10^{-6}\sqrt{1.42 \times 10^{4}}$$
$$= 5.96 \times 10^{-4} \text{ nepers/m}$$

for all frequencies above 10 Mc and as a matter of fact for quite a range below 10 Mc. But we obtain a phase constant β and a phase velocity c'

$$\beta = \omega\sqrt{\mu\kappa}\,(1 + 0.125q)$$
$$= 6.28 \times 10^{7}\underbrace{\sqrt{1.257 \times 10^{-6} \times 8.854 \times 10^{-11}}}_{1.058\times 10^{-8}}(1 + 0.125 \times 3.27 \times 10^{-6})$$
$$= 6.28 \times 10^{7} \times 1.058 \times 10^{-8} = 0.662 \text{ radians/m}$$
$$\beta = \omega\sqrt{\mu\kappa}\,(1 + 0.125q)$$
$$= 6.28 \times 10^{8} \times 1.058 \times 10^{-8}(1 + 0.125 \times 3.27 \times 10^{-8})$$
$$= 6.28 \times 10^{8} \times 1.058 \times 10^{-8} = 6.62 \text{ radians/m}$$
$$\beta = \omega\sqrt{\mu\kappa}\,(1 + 0.125q)$$
$$= 6.28 \times 10^{9} \times 1.058 \times 10^{-8}(1 + 0.125 \times 3.27 \times 10^{-10})$$
$$= 6.28 \times 10^{9} \times 1.058 \times 10^{-8} = 66.2 \text{ radians/m}$$
$$c' = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\kappa}\,(1 + 0.125q)} \cong \frac{1}{\sqrt{\mu\kappa}} = \frac{1}{1.058 \times 10^{-8}}$$
$$= 0.945 \times 10^{8} \text{ m/sec}$$

Hence, in the 10- to 1,000-Mc/sec range and, as a matter of fact, for all frequencies above 10 Mc/sec and quite a range below 10 Mc/sec, we have a phase velocity of 0.945 m/sec. Since according to this numerical outcome, for all frequencies above 10 Mc/sec, $\beta \cong \omega\sqrt{\mu\kappa}$, we have also $d\beta/d\omega = \sqrt{\mu\kappa}$ and the group velocity

$$c'' = \frac{d\omega}{d\beta} = \frac{1}{\sqrt{\mu\kappa}} = c' = 0.945 \times 10^{8} \text{ m/sec}$$

Therefore, dry soil in the frequency range above 10 Mc/sec is an example where phase and group velocity are numerically equal, but well below the velocity $c = 3 \times 10^{8}$ m/sec. If the value of q had not been so small in comparison with unity, but still $q \ll 1$, we should have the solution

$$\frac{d\beta}{d\omega} = \frac{d\left[\omega\sqrt{\mu\kappa}\left(1 + 0.125\frac{\sigma^2}{\omega^2\kappa^2}\right)\right]}{d\omega} = \frac{d(\omega\sqrt{\mu\kappa})}{d\omega} + 0.125\frac{d\left(\omega^{-1}\sigma^2\frac{\sqrt{\mu\kappa}}{\kappa^2}\right)}{d\omega}$$
$$= \sqrt{\mu\kappa} - 0.125\frac{\sigma^2}{\omega^2\kappa^2}\sqrt{\mu\kappa}$$

Hence,

$$c'' = \frac{d\omega}{d\beta} = \frac{1}{\sqrt{\mu\kappa}\left(1 - 0.125\,\dfrac{\sigma^2}{\omega^2\kappa^2}\right)} = \frac{1}{\sqrt{\mu\kappa}\,(1 - 0.125q)} \quad \text{m/sec}$$

$$c' = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\kappa}\,(1 + 0.125q)} \quad \text{m/sec} \tag{588}$$

Suppose we apply these formulas to dry soil and to a condition for which $0.125q = 0.01$. Then

$$q = 0.08 = \frac{\sigma^2}{(\omega\kappa)^2} = \frac{10^{-10}}{78.32 \times 10^{-22}}\,\frac{1}{\omega^2} = \frac{1.28 \times 10^{10}}{\omega^2}$$

Such a condition then occurs for a frequency of

$$f = \sqrt{\frac{1.28 \times 10^{10}}{6.28^2 \times 0.08}} = \sqrt{40.5 \times 10^8} = 6.37 \times 10^4 \text{ cycles/sec}$$

or a frequency as low as 63.7 kc/sec for dry ground. We then find the phase and group velocities as

$$c' = \frac{1}{\sqrt{\mu\kappa}\,(1 + 0.125q)} = \frac{0.945 \times 10^8}{1 + 0.01} = 0.936 \times 10^8 \text{ m/sec}$$

$$c'' = \frac{0.945 \times 10^8}{1 - 0.125q} = \frac{0.945 \times 10^8}{0.99} = 0.953 \times 10^8 \text{ m/sec}$$

In this case, the geometric mean of c' and c'', namely, $\sqrt{c'c''}$, is no longer equal to 0.945×10^8 m/sec. We also note that the group velocity is now greater than the phase velocity.

For cases where q belongs neither to the $q \ll 1$ nor to the $q \gg 1$ region, we cannot use the abridged formulas (587) but must apply the phase constant β as given by (581). No difficulty then arises when evaluating the phase velocity c', since $c' = \omega/\beta$ and the ω factor of the β relation merely cancels out. But for the group velocity c'', we have the operation $c'' = d\omega/d\beta$ with the ω factor before the main square root of the β relation and ω^2 also included in the b value of the subsquare root, since $b = 1 + \sigma^2/(\omega\kappa)^2$. In some applications, it seems then more convenient to compute several β values for the desired frequency range and plot β against ω; the ascend (negative tangent) for a certain ω value on the β curve then represents the group velocity.

Moreover, the formulas of Eq (587) give a means of determining to what extent we have amplitude and phase distortion, respectively. For instance, for $q \ll 1$, no amplitude distortion can take place, since the electromagnetic field attenuation α does not contain a frequency term.

But the phase constant does contain such a term, and phase distortion occurs, because the component waves that make up a modulated wave are propagated with *different* phase velocities. It is the group velocity c'' with which power, *i.e.*, the signal or modulation intelligence, is being propagated that concerns us in the case of sustained carrier supply. According to the derivation of the group-velocity formulas (576) and (579), the value of c'' is found by the ratio of the difference radian frequency $\Delta\omega$ to the corresponding difference $\Delta\beta$ of the phase constant β, which sometimes is also known as the "wave number." Hence,

$$\left.\begin{aligned} \Delta\beta &= \frac{\Delta\omega}{c'} \quad \text{for a } \textit{nondispersive} \text{ medium;} \\ \Delta\beta &\neq \frac{\Delta\omega}{c'} \quad \text{for a } \textit{dispersive} \text{ medium} \end{aligned}\right\} \tag{589}$$

For a nondispersive medium, we have then

$$c' = \frac{\Delta\omega}{\Delta\beta} = c'' = \frac{d\omega}{d\beta} \tag{590}$$

We note that the group velocity c'' cannot be equal to the phase velocities c', which by themselves are unequal, on account of the frequency dependency. This is obvious if we realize that for amplitude modulation, we deal with three frequencies, namely, the unmodulated carrier frequency F and the upper and lower side frequencies $F + f$ and $F - f$, respectively, where f is the frequency of the modulating signal or intelligence. For both frequency and phase modulation, we have, in addition to this symmetrical pair of side frequencies, also significant pairs $F \pm 2f$; $F \pm 3f$, $\cdots$, even in the case of a single harmonic intelligence modulation.

As already brought out, we have for $q \ll 1$

$$c' = \frac{\omega}{\omega\sqrt{\mu\kappa}\,(1 + 0.125q)} = \frac{\omega^2\kappa^2}{\sqrt{\mu\kappa}\,(\omega^2\kappa^2 + 0.125\sigma^2)} \quad \text{m/sec} \tag{591}$$

$$\frac{1}{c''} = \frac{d\beta}{d\omega} = \frac{d(\omega\sqrt{\mu\kappa})}{d\omega} + \frac{d\left(\dfrac{0.125\sigma^2\sqrt{\mu\kappa}}{\omega\kappa^2}\right)}{d\omega}$$

$$= \sqrt{\mu\kappa} - \frac{0.125\sigma^2\sqrt{\mu\kappa}}{\omega^2\kappa^2} \quad \text{sec/m}$$

$$c'' = \frac{\omega^2\kappa^2}{\omega^2\sqrt{\mu\kappa^5} - 0.125\sigma^2\sqrt{\mu\kappa}} \quad \text{m/sec} \tag{592}$$

Formulas (591) and (592) are identical with the expressions given in (588). They usually apply in the upper HF range. For the case $q \gg 1$, we have both amplitude and phase distortion, since $\alpha \cong \beta$ and the

attenuation term as well as the phase term contains the frequency $\omega/6.28$. We then have a phase velocity

$$c' = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{0.5\omega\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad \text{m/sec} \tag{593}$$

We have

$$\frac{1}{c''} = \frac{d\beta}{d\omega} = \frac{d(\omega^{1/2}\sqrt{0.5\mu\sigma})}{d\omega} = \frac{\sqrt{0.5\mu\sigma}}{2\sqrt{\omega}} = \sqrt{\frac{0.125\mu\sigma}{\omega}} \qquad \text{sec/m}$$

with a group velocity

$$c'' = \sqrt{\frac{8\omega}{\mu\sigma}} \qquad \text{m/sec} \tag{594}$$

Formulas (593) and (594) find applications in the higher frequency range only when dealing with mediums of *high* specific conductivity σ. Comparing the two formulas for $q \gg 1$, we find that the group velocity

$$c'' = 2c' \qquad \text{for } \frac{\sigma^2}{\omega^2\kappa^2} \gg 1 \tag{595}$$

If we examine wave propagation in ocean, for which $\kappa \cong 80 \times 8.854 \times 10^{-12} = 7.0832 \times 10^{-10}$ farad/m at 20°C below 100 Mc/sec, $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m. Assuming an average salinity so that $\sigma = 5$ mhos/m, we find the values of Table 13. For a frequency of 10 Mc/sec,

TABLE 13

Frequency f, cycles/sec	$q = \dfrac{1.26 \times 10^{18}}{f^2}$	Remarks	
		Special	General
10^6 10^7 10^8	1.26×10^6 1.26×10^4 1.26×10^2	$q \gg 1$ eqs. (594) and (595)	$q = \dfrac{\sigma^2}{39.4\kappa^2 f^2}$ $= \dfrac{25}{39.4 \times 50.3 \times 10^{-20} f^2}$ for ocean water
10^9	1.26	1.26 as against 1	
10^{10} 3×10^{10}	0.0126 0.0014	$q \ll 1$ and Eqs. (591) and (592)	

the formulas for $q \gg 1$ apply and

$$c' = \sqrt{\frac{2\omega}{\mu\sigma}} = \sqrt{\frac{2 \times 6.28 \times 10^7}{1.257 \times 10^{-6} \times 5}} = 4.5 \times 10^6 \text{ m/sec}$$

According to (595), the group velocity $c'' = 2c' = 9 \times 10^6$ m/sec in comparison with $c = 3 \times 10^8$ m/sec. We have, therefore, again a larger value for the group velocity than for the phase velocity and a drastic difference between both of them. From Table 13 we note that from about 10,000 Mc/sec on, formulas for $q \ll 1$ apply. This assumes that κ still has, in the frequency range up to 10,000 Mc/sec and above it, a value that makes $q \ll 1$. For dry-ground propagation, assuming this

TABLE 14

Frequency f, cycles/sec	$q = \frac{3.24 \times 10^{10}}{f^2}$	Remarks: Special	Remarks: General
10^6	3.24×10^{-2}		$q = \frac{\sigma^2}{\omega^2\kappa^2} = \frac{10^{-8}}{39.4 \times 7.83 \times 10^{-21} f^2}$ for dry land
10^7	3.24×10^{-4}	$q \ll 1$ and Eqs. (591) and (592) apply	
10^8	3.24×10^{-6}		
10^9	3.24×10^{-8}		
10^{10}	3.24×10^{-10}		
3×10^{10}	1.08×10^{-10}		

time $\sigma = 10^{-4}$ mho/m and again $\kappa = 10\kappa_0 = 8.854 \times 10^{-11}$ farad/m, we obtain the data of Table 14. We note that the entire VHF and UHF ranges give the case of $q \ll 1$. For 796 Mc/sec, we have $\omega = 5 \times 10^9$, $\omega^2 = 25 \times 10^{18}$, $\kappa^2 = 7.83 \times 10^{-21}$, $\sigma^2 = 10^{-8}$, $\mu = 1.257 \times 10^{-6}$ henry/m, $\sqrt{\mu\kappa} = \sqrt{1.257 \times 10^{-6} \times 8.854 \times 10^{-11}} = 1.058 \times 10^{-8}$, $q = \sigma^2/(\omega^2\kappa^2) = 10^{-8}/(25 \times 10^{18} \times 7.83 \times 10^{-21}) = 5.1 \times 10^{-8}$, and $0.125 \times 5.1 \times 10^{-8} = 6.37 \times 10^{-9}$. Therefore, in formulas (588) for the group and phase velocities, we have the $(1 + 0.125q)$ essentially equal to unity and $c' = c'' = 1/\sqrt{\mu\kappa} = 0.945 \times 10^8$ m/sec

CHAPTER V

TRANSMISSION LINES AND RADIATION

At VHF's and UHF's, transmission lines can be used for delivering HF power to a radiator or delivering the received power from an antenna to the input network of a receiver. Transmission lines can also be used for impedance matching and in place of capacitors and inductors, respectively. For matched lines, the current distribution along the line is fixed, except for the small self-attenuation effect due to the ohmic resistance of the line and any dielectric losses. A matched single feeder does radiate electromagnetic field energy. Even a matched Lecher wire system has a definite radiation resistance. A coaxial line, which is not closed, likewise shows radiation effects at such ends. Such radiation effects cannot always be ignored at UHF's.

74. Characteristic Impedance and Other Factors of Feeders. In the characteristic impedance formulas given in Fig. 63, it is assumed that the dielectric between and around good conductors (specific conductivity σ in mhos per meter very high) has essentially zero conductivity. In all cases shown, we deal with go and return conductors with a principal electromagnetic wave (TEM wave) guided by the respective feeders. Such a wave propagation is then transverse electromagnetic (TEM) with respect to the direction of energy transfer along the feeder. For air as the dielectric in which respective conductors are embedded, the velocity of propagation is $1/\sqrt{\mu\kappa} \cong 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec, if $\sigma = 0$ for the dielectric and $\sigma = \infty$ for respective conductors. For a Lecher wire system, we have, for *each meter* length of the double line,

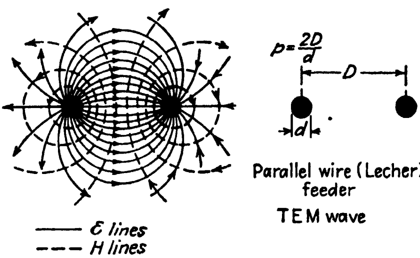
$$C = \frac{\pi\kappa}{\log_\epsilon (p)} \text{ farads;} \qquad L = \frac{\mu \log_\epsilon (p)}{\pi} \text{ henrys} \tag{596}$$

Even though these relations are not derived by means of Maxwell's *field equations*, we find by means of customary transmission-line theory the phase velocity

$$c' = \frac{1}{\sqrt{CL}} = \frac{1}{\sqrt{\mu\kappa}} \qquad \text{m/sec} \tag{597}$$

which "happens" to be the correct expression for the TEM wave, since κ in farads per meter and μ in henrys per meter are the effective medium constants of the TEM wave for no medium conductivity. Since for air

also $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m and $\mu \cong \mu_0 = 1.257 \times 10^{-6}$ henry/m, we have $c' = c = 3 \times 10^8$ m/sec for such an ideal case with no losses. By applying the other transmission-line relation for $\sigma = \infty$ in the con-



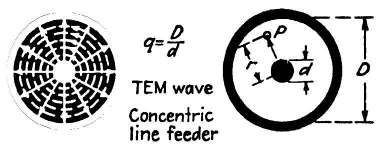
$Z_0^{(\text{ohms})} = 0.734\sqrt{\frac{\mu}{\kappa}}\log_{10}(p)$ for d small compared with D

For air $Z_0^{(\text{ohms})} = 276\log_{10}(p)$

For a perfect nonmagnetic dielectric of relative numeric dielectric constant

$$\kappa_r = \frac{\kappa}{\kappa_0} = \frac{\kappa\ \text{farads/m}}{8.854}\,10^{12}\ \text{farads/m}$$

$$Z_0 = \frac{276}{\sqrt{\kappa_r}}\log_{10}(p)\ \text{ohms}$$

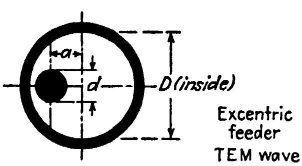


$Z_0 = 0.366\sqrt{\frac{\mu}{\kappa}}\log_{10}(q)$ ohms for $d < D$

For air $Z_0 = 138\log_{10}(q)$ ohms

For a perfect nonmagnetic dielectric ($\sigma = 0$; $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m)

$$Z_0 = \frac{138}{\sqrt{\kappa_r}}\log_{10}(q)\ \text{ohms}$$

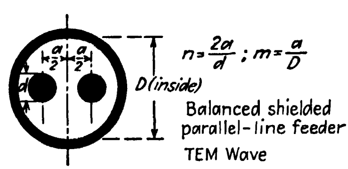


$Z_0 = 0.366\sqrt{\frac{\mu}{\kappa}}\log_{10}\frac{0.25D^2 - a^2}{0.25Dd}$ ohms for $d < D$.

For air $Z_0 = 138\log_{10}\frac{0.25D^2 - a^2}{0.25Dd}$ ohms

For a perfect nonmagnetic dielectric

$$Z_0 = \frac{138}{\sqrt{\kappa_r}}\log_{10}\frac{0.25D^2 - a^2}{0.25Dd}\ \text{ohms}$$



$Z_0 = 0.734\sqrt{\frac{\mu}{\kappa}}\log_{10}\left(n\frac{1 - m^2}{1 + m^2}\right)$ ohms

For air $Z_0 = 276\log_{10}\left(n\frac{1 - m^2}{1 + m^2}\right)$ ohms

For a perfect nonmagnetic dielectric

$$Z_0 = \frac{276}{\sqrt{\kappa_r}}\log_{10}\left(n\frac{1 - m^2}{1 + m^2}\right)\ \text{ohms}$$

For nonmagnetic good conductors such as copper $\sqrt{\frac{\mu}{\kappa}} = 376.7$ ohms

FIG. 63. Characteristic impedance of unshielded and shielded double-line feeders in ohms.

ductors and $\sigma = 0$ in the dielectric around and between the conductors, we find

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{\sqrt{\mu/\kappa}}{\pi}\log_\epsilon(p) = \frac{2.302}{\pi}\sqrt{\frac{\mu}{\kappa}}\log_{10}(p)$$

$$= 0.734\sqrt{\frac{\mu}{\kappa}}\log_{10}(p) \qquad \text{ohms} \qquad (598)$$

where $\sqrt{\mu/\kappa}$ would be the characteristic *space* impedance of a plane wave at its wave front with field values of $\mathcal{E}$ volts per meter and H amperes per meter, and $\mathcal{E}/H = \sqrt{\mu/\kappa}$ with a value essentially equal to 376.7 ohms for an air dielectric. Hence, a *Lecher wire* Z_0 value *depends not only* upon the factor $\sqrt{\mu/\kappa}$ *but also* upon the relative dimensions of the wire spacing and diameter of the wires, since $p = 2D/d$ in Fig. 63. Formula (598) applies then for the above-mentioned ideal conditions as regards the conductivity of the wires and the dielectric (must be perfect). For annealed copper, we have, for instance, $\sigma \cong 5.8 \times 10^7$ mhos/m at 20°C, which is very large (is the conductivity across opposite faces of a cubic meter of copper). The second formula with $276 \log_{10} (p)$ ohms applies to perfectly conducting Lecher wires in air, while the third expression $Z_0 = [276 \log_{10} (p)]/\sqrt{\kappa_r}$ refers to such a Lecher feeder in a perfect non-magnetic dielectric, for which $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m and $\kappa_r = \kappa/\kappa_0 = 10^{12}\kappa/8.851$. If we had *ideal* distilled water, κ_r would have a value of about 80 and the characteristic impedance would be only about one-ninth of the value that the Lecher wires would have when submerged in air. This ignores the *unavoidable* conductivity of the order given in Table 1. For fused quartz, we have less than 2×10^{-17} mho/m at normal room temperature and a relative dielectric constant $\kappa_r \cong 5$, for which we can expect a characteristic impedance of about $0.446Z_0$, where Z_0 is the value for air as a dielectric about the conductors.

In a similar way, all the other formulas in Fig. 63 are found. Figure 64 gives the modified expressions if bead spacers are used to separate any inside conductors within a metal cylinder. For ideal distilled water with fused quartz beads with $b = 3$ mm and $c = 150$ mm dimensions, we have $\kappa_r \cong 80$, $\kappa_r' \cong 5$, and

$$\sqrt{\kappa_r + \frac{\kappa_r' - \kappa_r}{c}\,b} = \sqrt{80 - \frac{5 - 80}{150}\,3} = \sqrt{81.5} \cong 9$$

showing that for a design of this kind, the characteristic impedance Z_0' is also about one-ninth of the value without beads. But if we use the same beads with the same dimensions in an air dielectric instead of distilled water, we must apply the second formula of Fig. 64 and find

$$\sqrt{1 + \frac{\kappa_r' - 1}{c}\,b} = \sqrt{1 + \frac{5 - 1}{150}\,3} = \sqrt{1.08} = 1.039$$

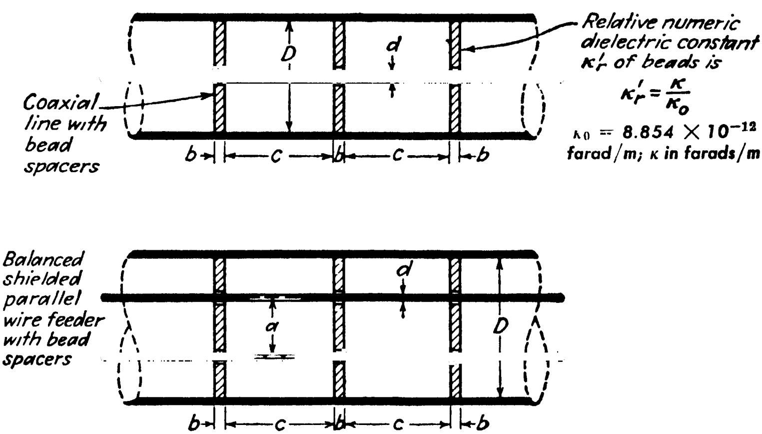
and the characteristic impedance with beads is still about 100/1.039 = 96 per cent of the value without the beads.

Shielded feeders, like a concentric line, may have a shielding envelope that has other forms. If it is, for instance, such that the cross section

is a square of length l with a circular conductor of diameter d, located at the mid-point of the square, the characteristic impedance is

$$\left.\begin{aligned} Z_0^{(\text{ohms})} &= 0.366\sqrt{\frac{\mu}{\kappa}}\log_{10}\left(1.078\,\frac{l}{d}\right) && \text{for a medium } \mu,\ \kappa \\ Z_0^{(\text{ohms})} &= 138\log_{10}\left(1.078\,\frac{l}{d}\right) && \text{for air} \\ Z_0^{(\text{ohms})} &= \frac{138}{\sqrt{\kappa_r}}\log_{10}\left(1.078\,\frac{l}{d}\right) && \text{for a nonmagnetic perfect dielectric of relative dielectric constant } \kappa_r \end{aligned}\right\} \quad (599)$$

Nothing is gained by such a shape. It is also more difficult to build, since it takes more material and space, and the significant term under



$$Z' = \frac{Z_0}{\sqrt{\kappa_r + \frac{\kappa_r' - \kappa_r}{c}\,b}}$$ ohms; κ_r is the relative numeric dielectric constant of the medium about the beads and Z_0 represents the respective values given in Fig. 63. For air as the dielectric about the beads, we have $\kappa_r = 1$ and $Z_0' = \frac{Z_0}{\sqrt{1 + \frac{\kappa_r' - 1}{c}\,b}}$ ohms

FIG. 64. Effect of standoff beads in shielded feeder lines.

the log is essentially multiplied by unity. We have, therefore, approximately the same formula as for the customary concentric line, for which, according to Fig. 63, the ratio D/d in place of l/d is the controlling term. If r denotes any radius within the concentric line with a length dimension along the X axis, with cylindrical coordinates r, φ, x, then the electric field $\mathcal{E}_r$ is, according to Fig. 63, along such a radius, while the associated magnetic vector H_φ is normal to $\mathcal{E}_r$ and tangent to a circle. Since these

field associations constitute the principal TEM wave of a concentric line, the ratio $\mathcal{E}_r/H_\varphi$ must be equal to $\sqrt{\mu/\kappa}$ ohms for $\sigma = 0$ in the space between the central conductor and the coaxial envelope. The propagation then depends on the constant $\Gamma = j\omega\sqrt{\mu\kappa}$, which for air is $\Gamma = j\omega\sqrt{1.257 \times 10^{-6} \times 8.854 \times 10^{-12}} = j\omega/c$, where $c = 3 \times 10^8$ m/sec. This assumes that the conductivity of the conductors is infinite. Since the electric field decreases inversely as the radial distance r and $\mathcal{E}_r/H_\varphi$ is fixed, H_φ is also proportional to $1/r$, and we have

$$\mathcal{E}_r = \sqrt{\frac{\mu}{\kappa}}\frac{H_0}{r}\epsilon^a;\quad H_\varphi = \frac{H_0}{r}\epsilon^a;\quad a = j\omega t - x\Gamma \quad \text{principal wave of a concentric feeder} \tag{600}$$

From this result, we note that at the surface of the *inner* conductor $r = \frac{1}{2}d$, the electromagnetic field is *strongest* and decreases toward the inside surface of the outside conductor ($r = \frac{1}{2}D$). This shows that for finite conductivities of the conductors, such as 5.8×10^7 mhos/m in the case of annealed copper, the *heat losses* are *mostly in* the *inside* conductor, and all the more the smaller d is chosen in comparison to D. If d were infinitesimally small, according to (600) the respective field associations would be infinitely large. The above formula does *not*, however, account for the fact that for a radial $\mathcal{E}_r$ field, there *must* always be an axial conductor present, so that $\mathcal{E}$ lines may terminate on it. Hence, the principal TEM wave cannot exist when $d = 0$ (for a hollow guide).

75. Review of Well-known Transmission-line Relations.[1] These relations refer to uniform lines for which the characteristic line constants per unit length (per meter) are fixed. Unless otherwise mentioned, we have ω in radians per second, $\omega/6.28$ as the frequency f in cycles per second, λ_0 meters the corresponding free-space wavelength, where $f\lambda_0 = 3 \times 10^8$ m/sec. If f is in megacycles per second, then $f\lambda_0 = 300$. The *phase* velocity c' along the line determines the voltage and current distributions, while the group velocity c'' is the velocity in meters per second with which energy is being transferred along the line. Neither c' nor c'' is generally equal to $c = 3 \times 10^8$ m/sec.

We have the definitions l the length of the line in meters, R the resistance along unit length (of go and return for a double line) in ohms per meter, L the self-inductance per unit length in henrys per meter, G the leakage conductance across unit length in mhos per meter, C the capacitance across unit length in farads per meter, α the attenuation constant in nepers per meter (pure hyperbolic radians per meter), β the phase constant in circular radians per meter, $\Gamma = \alpha + j\beta$ the propagation constant in generalized radians per meter, R_0 the resistance component of the characteristic line impedance Z_0 in ohms, X_0 the reactance component of Z_0 in ohms, $Z_0 = R_0 - jX_0$

[1] Complete derivations are in A. Hund, "Phenomena in High-frequency Systems," pp. 406–453, McGraw-Hill Book Company, Inc., New York, 1936.

the characteristic line impedance in ohms, E_1 volts and I_1 amperes the voltage and current at the input end (impressed values), E_2 and I_2 the voltage and current at the far end of the line, $Z_2 = E_2/I_2 = R_2 \pm jX_2$ the load impedance in ohms at the far end, $Z_1 = E_1/I_1 = R_1 \pm jX_1$ the impedance in ohms, looking into the line at the input end.

$Z = R + j\omega L$ impedance along unit length of the line in ohms per meter
$Y = G + j\omega C$ admittance across unit length of the line in mhos per meter
$\Gamma = \sqrt{YZ} = \alpha + j\beta$ propagation constant in generalized radians per meter
$Z_0 = \sqrt{\frac{Z}{Y}}$ characteristic impedance in ohms

The expression $l\Gamma = \alpha l + j\beta l$ is the generalized electrical length of the line of l meters physical length in generalized radians [nepers plus j(circular radians)]

We note that Z and Y are generally complex with the absolute amount $Z = \sqrt{R^2 + (\omega L)^2}$ ohms and $\theta_z = \tan^{-1}(\omega L/R)$, $Y = \sqrt{G^2 + (\omega C)^2}$, and $\theta_y = \tan^{-1}(\omega C/G)$ degrees. The surge (characteristic) impedance Z_0 is also generally complex and given by

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt[4]{\frac{R^2 + (\omega L)^2}{G^2 + (\omega C)^2}} \Big/ 0.5 \tan^{-1}\left[\left(\frac{\omega L}{R}\right) - \tan^{-1}\left(\frac{\omega C}{G}\right)\right] \text{ deg} \quad (601)$$

Generally, we also have

$$\alpha = \sqrt{0.5(\sqrt{a^2 + b^2} + d)} \text{ nepers/m;}$$
$$\beta = \sqrt{0.5(\sqrt{a^2 + b^2} - d)} \text{ circular radians/m} \quad (602)$$

for

$$a = GR + \omega^2 CL; \qquad b = \omega(GL - RC); \qquad d = GR - \omega^2 CL$$

If $GL = RC$, we find $\alpha = \sqrt{GR}$ and $\beta = \omega\sqrt{CL}$. Then we also find $Z_0 = \sqrt{R/G} = \sqrt{L/C}$ ohms. For this condition, Z and Y also have the same phase angle $\theta_z = \theta_y$. Putting

$$A = 1 + \left(\frac{R}{\omega L}\right)^2; \qquad B = 1 + \left(\frac{G}{\omega C}\right)^2; \qquad D = 1 - \frac{GR}{\omega^2 CL}; \qquad F = 1 + \frac{GR}{\omega^2 CL}$$

we also generally have (exact formulas when all line constants are taken into account)

$$\alpha = \omega\sqrt{CL}\sqrt{0.5(\sqrt{AB} - D)} \text{ nepers/m;}$$
$$\beta = \omega\sqrt{CL}\sqrt{0.5(\sqrt{AB} + D)} \text{ circular radians/m} \quad (603)$$

$$R_0 = \frac{\sqrt{\frac{L}{C}}}{\sqrt{B}}\sqrt{0.5(\sqrt{AB} + F)} \text{ ohms;} \qquad X_0 = \frac{\sqrt{\frac{L}{C}}}{\sqrt{B}}\sqrt{0.5(\sqrt{AB} - F)} \text{ reactive ohms} \quad (604)$$

At VHF's and UHF's, the quantity G can often be neglected ($G = 0$), and $\omega L \gg R$, *i.e.*, $R/(\omega L) \ll 1$, and

$$\alpha = \sqrt{0.5[\sqrt{\omega^2C^2(R^2 + \omega^2L^2)} - \omega^2CL]} = \sqrt{0.5\omega C\left[\omega L\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} - \omega L\right]}$$

$$= \sqrt{0.5\omega C\left[\omega L\left(1 + \frac{R^2}{2\omega^2L^2}\right) - \omega L\right]} \cong 0.5R\sqrt{\frac{C}{L}} = \frac{\beta R}{2\omega L} \qquad (605)$$

if $\sqrt{1 + (R^2/\omega^2L^2)}$ is expanded binomially and terms above the second degree are neglected. Moreover,

$$\left.\begin{aligned} \beta = \omega\sqrt{CL} = \frac{2\pi}{\lambda}; \qquad R_0 = \sqrt{\frac{L}{C}}; \qquad X_0 = \frac{R}{2\omega L}\sqrt{\frac{L}{C}} = \frac{R}{2\omega L}R_0; \\ \Gamma = \alpha + j\beta = \left(\frac{R}{2\omega L} + j\right)\beta \\ Z_0 = R_0 - jX_0 = \left(1 - j\frac{R}{2\omega L}\right)\sqrt{\frac{L}{C}} \end{aligned}\right\} \qquad (606)$$

In many cases, the term $R/(2\omega L)$ in Γ and Z_0 is neglected, so that $\Gamma \cong j\beta = j2\pi/\lambda = j\omega/c'$ and $Z_0 \cong \sqrt{L/C}$, making $\alpha \cong R/(2Z_0)$. The imaginary term $jR/(2\omega L)$ in the relation $Z_0 = [1 - j(R/2\omega L)]\sqrt{L/C}$ *must be retained*, however, when the input impedance Z_1 of a line, with the far end short-circuited, is computed.

The general relations for the voltage and the current at any distance x *from the receiving end* are

$$\left.\begin{aligned} E_x^{(\text{volts})} &= E_2 \cosh(x\Gamma) + Z_0I_2 \sinh(x\Gamma) \\ &= E_1 \cosh(x\Gamma) - Z_0I_1 \sinh(x\Gamma) \\ I_x^{(\text{amp})} &= I_2 \cosh(x\Gamma) + \frac{E_2}{Z_0}\sinh(x\Gamma) \\ &= I_1 \cosh(x\Gamma) - \frac{E_1}{Z_0}\sinh(x\Gamma) \end{aligned}\right\} \qquad (607)$$

for $\Gamma = \sqrt{YZ}$ and $Z_0 = \sqrt{Z/Y}$. For a length l meters of a line, we find, for $\Omega = l\Gamma$,

$$\left.\begin{aligned} E_1 = E_2 \cosh\Omega + Z_0I_2 \sinh\Omega; \quad I_1 = I_2\cosh\Omega + \frac{E_2}{Z_0}\sinh\Omega \\ E_2 = E_1 \cosh\Omega - Z_0I_1 \sinh\Omega; \quad I_2 = I_1\cosh\Omega - \frac{E_1}{Z_0}\sinh\Omega \end{aligned}\right\} \qquad (608)$$

The sending-end impedance $Z_1 = E_1/I_1$ or

$$Z_1 = Z_0\frac{Z_2\cosh\Omega + Z_0\sinh\Omega}{Z_0\cosh\Omega + Z_2\sinh\Omega} \qquad (609)$$

Hence, for the line short-circuited at the far end, $Z_2 = 0$ and $Z_1 = Z_0 \tanh \Omega$. For the far end open, we have $I_2 = 0$ and $E_1 = E_2 \cosh \Omega$ and $I_1 = (E_2/Z_0) \sinh \Omega$. Hence, $Z_1 = E_1/I_1 = Z_0 \coth \Omega$. For $Z_2 = Z_0$, we have also $Z_1 = Z_0$. The transfer impedance or *driving* impedance is $Z_d = E_1/I_2$, since the impressed voltage is the cause and the output current I_2 the useful result. Therefore,

$$Z_d = Z_2 \cosh \Omega + Z_0 \sinh \Omega \tag{610}$$

We also have

$$E_2 = \frac{Z_0}{Z_2 \cosh \Omega + Z_0 \sinh \Omega} E_1; \quad I_2 = \frac{Z_0}{Z_2 \sinh \Omega + Z_0 \cosh \Omega} I_1 \tag{611}$$

In HF lines, we can often neglect the attenuation constant α and find $\Omega = l\Gamma \cong j\beta l$, rendering the driving impedance

$$Z_d = \frac{E_1}{I_2} \cong Z_2 \cos \beta l + jZ_0 \sin \beta l \tag{612}$$

which, for $l = \lambda/4$, yields $E_1/I_2 = jZ_0$, showing that the output current I_2 passing through the load impedance $Z_2 = E_2/I_2$ at the far end of the quarter-wavelength line has a *constant* relation to the generator voltage E_1.

From (609), we note that for an open-ended line, $Z_2 = \infty$, making only the terms with factor Z_2 significant. The impedance looking into the line is then

$$Z_1 = Z_{op} = Z_0 \coth \Omega$$

Since $\Omega = l\sqrt{YZ} = p + jq = \alpha l + j\beta l$,

$$Z_{op} = Z_0 \frac{\sqrt{\sinh^2 p + \cos^2 q}\ \underline{/\varphi_1}}{\sqrt{\sinh^2 p + \sin^2 q}\ \underline{/\varphi_2}} \quad \begin{Bmatrix} \tan \varphi_1 = \tan q \tanh p \\ \tan \varphi_2 = \tan q \coth p \end{Bmatrix} \tag{613}$$

When the far end is short-circuited, $Z_2 = 0$ and (609) reduces to

$$Z_1 = Z_{sc} = Z_0 \tanh \Omega = Z_0 \frac{\sqrt{\sinh^2 p + \sin^2 q}\ \underline{/\varphi_2}}{\sqrt{\sinh^2 p + \cos^2 q}\ \underline{/\varphi_1}} \tag{614}$$

Both (613) and (614) show that Z_{op} and Z_{sc} oscillate in value as the length l of the line is changed, since the circular trigonometric functions oscillate between +1 and −1. The function $\sinh^2 p = \sinh^2 \alpha l$ denotes a gradual increase only, which to begin with will not greatly interfere with trigonometric fluctuations. The value of $q = \beta l = 2\pi l/\lambda$ depends on λ as well as the physical length l of the line. It determines the potential and current distributions along the line. Both distributions are known as the "phase" distribution. We note from (613) that when $l/\lambda = 0$, we have for Z_{op} a maximum value and for Z_{sc} a minimum value; for $l/\lambda = \frac{1}{4}$, we have for Z_{op} a minimum value and for Z_{sc} a maximum, etc.

We must distinguish among velocity of propagation in empty space ($c = 3 \times 10^8$ m/sec $= f\lambda_0$), where the wavelength is in meters and f in cycles per second; the phase velocity $c' = f = \omega/\beta$, where λ meters is the wavelength developed along the line; and the group velocity $c'' = d\omega/d\beta$ with which energy is being transferred along the line. This is dealt with in detail in Sec. 41.

The final solutions of (607) are the outcome of expressions

$$E_x = A\epsilon^{x\Gamma} + B\epsilon^{-x\Gamma} \qquad \text{and} \qquad I_x = \frac{A\epsilon^{x\Gamma} - B\epsilon^{-x\Gamma}}{Z_0} \tag{615}$$

where $x\Gamma$ is the generalized electrical length of the line, reckoned from the *far load end* of the line. Since $\Gamma = \alpha + j\beta$ and the real term α accounts for the attenuation, respective exponents are $\alpha + j\beta$ and $-\alpha - j\beta$. Hence, we deal generally with two waves: one traveling toward the far load end and the other returning toward the input end. For *negative* exponents, the *amplitude* $A\epsilon^{-\alpha x}$ *decreases*, while it increases for the wave with amplitude $A\epsilon^{+\alpha x}$. Incident and reflected waves add sometimes in phase and sometimes in antiphase, explaining why we generally note maxima and minima effects along a line. The values of constants A and B are

$$A = \tfrac{1}{2}(E_2 + Z_0 I_2); \qquad B = E_2 - A = \tfrac{1}{2}(E_2 - Z_0 I_2) \tag{616}$$

If we take the distance x_1 from the *input* end, then $x_1 = l - x$, and we have for $\Omega = l\Gamma$

$$\left.\begin{aligned} E_{x_1} &= A_1\epsilon^{-x_1\Gamma} + B_1\epsilon^{x_1\Gamma}; \qquad I_{x_1} = \frac{A_1\epsilon^{-x_1\Gamma} - B_1\epsilon^{x_1\Gamma}}{Z_0} \\ A_1 &= \frac{(Z_0 + Z_2)\epsilon^{\Omega}}{(Z_0 + Z_2)\epsilon^{\Omega} - (Z_0 - Z_2)\epsilon^{-\Omega}} E_1; \\ B_1 &= \frac{(Z_2 - Z_0)\epsilon^{-\Omega}}{(Z_0 + Z_2)\epsilon^{\Omega} - (Z_0 - Z_2)\epsilon^{-\Omega}} E_1 \end{aligned}\right\} \tag{617}$$

where Ω and Z_0 can also be computed from

$$\Omega = \tanh^{-1}\sqrt{\frac{Z_{sc}}{Z_{op}}}; \qquad Z_0 = \sqrt{Z_{sc}Z_{op}} \tag{618}$$

since the input impedance $Z_1 = E_1/I_1$ looking into the line is as in (609) and can be written in the form

$$Z_1 = Z_0 \frac{(Z_0 + Z_2)\epsilon^{\Omega} - (Z_0 - Z_2)\epsilon^{-\Omega}}{(Z_0 + Z_2)\epsilon^{\Omega} + (Z_0 - Z_2)\epsilon^{-\Omega}} \tag{619}$$

Formulas (618) give a method for computing the propagation constant Γ and the characteristic impedance Z_0 from respective input impedances

Z_{sc} and Z_{op} when the far end of the line is either short-circuited or open. The effective values of voltage and current at any distance x_1 meters from the input end can be computed from

$$\left.\begin{aligned} E_{x_1} &= \sqrt{A_1^2\epsilon^{-2\alpha x_1} + B_1^2\epsilon^{2\alpha x_1} + 2A_1B_1\cos\psi} && \text{volts} \\ I_{x_1} &= \frac{\sqrt{A_1^2\epsilon^{-2\alpha x_1} + B_1^2\epsilon^{2\alpha x_1} - 2A_1B_1\cos\psi}}{Z_0} && \text{amp} \\ \psi &= \psi_{A_1} - \psi_{B_1} - 2\beta x_1 \qquad \text{deg} && \end{aligned}\right\} \tag{620}$$

where ψ_{A_1} is the phase angle of A_1 and ψ_{B_1} is the angle of B_1. These angles are found from $A_1 = a + jb$ and $B_1 = d + je$ by means of

$$\psi_{A_1} = \tan^{-1}\frac{b}{a} \qquad \text{and} \qquad \psi_{B_1} = \tan^{-1}\frac{e}{d}$$

For negligible attenuation α, we deal with 100 per cent reflection, and $A_1 = B_1$ and $E_{x_1} = \sqrt{2A_1^2 - 2A_1^2\cos\psi} = 1.414A_1\sqrt{1 - \cos\psi}$ with voltage maxima equal to $2A_1$ and minima equal to zero. Maximum current effects occur at places for which $E_{x_1} = 0$, and zero current effects occur at places for which $E_{x_1} = 2A_1$. Equality of A_1 and B_1 happens when

$$\epsilon^{2\Omega} = \pm\frac{Z_2 - Z_0}{Z_2 + Z_0}$$

Ideal standing waves occur when Z_2 is either zero or infinite and any multiple of $\frac{1}{4}\lambda$ is the physical length l of the line. It is important that an open line, a short-circuited line, or a line terminated into a pure reactance *cannot* absorb power if ideal standing waves occur. If a line is terminated into its characteristic impedance ($Z_2 = Z_0 = R_0$), all the arriving energy is absorbed by the termination $Z_2 = \sqrt{L/C}$ ohms and no standing waves can occur, since there is no reflected wave and, therefore, also no interference between outgoing and returning wave trains. But for $Z_2 \neq Z_0$, we have partial power absorption in the load Z_2 and partial wave reflection at Z_2. Therefore, standing waves are superimposed on the energy flow toward Z_2. We then no longer have true nodal points. The ratio of minimum voltage to maximum voltage is known as the "standing-wave" ratio. If we have, for instance, $E_{min} = 2$ volts and $E_{max} = 8$ volts, the standing-wave ratio of $\frac{2}{8} = 0.25$ means that the load impedance Z_2 is either four times larger or smaller than the characteristic line impedance Z_0. Whether it is larger or smaller than Z_0 can be found from the condition that for $Z_2 = R_2 > Z_0$ the voltage across the load R_2 must be near a location of a voltage maximum while for $R_2 < Z_0$ the voltage across R_2 must be near a location of a voltage

minimum. For a short-circuited line, the voltage at this end is theoretically zero and the first voltage maximum occurs $\lambda/4$ from the short-circuited end. For an open-ended line, the voltage maximum must occur at the end of the line and the first voltage minimum (zero) a quarter-wavelength distance from this end. For a condenser loading at the far end, we have $Z_2 = 1/(\omega C)$; and when equal to Z_0 ohms, the first voltage zero occurs a distance $\lambda/8$ from the condenser load. For a pure inductive load $Z_2 = \omega L = Z_0$ ohms, the first voltage zero occurs a distance $3\lambda/8$ from the inductance. For an impedance load $Z_2 = R_2 \pm jX$, the ratio E_{min}/E_{max} gives only an approximate idea of the mismatch.

Moreover, it is also possible to express the attenuation constant α in terms of the characteristic impedance Z_0. We find for an HF line, for which $G = 0$ mhos/m, $R \ll \omega L$ ohms per meter, that $c' = 1/\sqrt{CL}$ meters per second and $Z_0 = \sqrt{L/C}$ ohms. Hence,

$$\Gamma = \underbrace{\frac{R}{2Z_0}}_{\alpha} + j\underbrace{\frac{2\pi}{\lambda}}_{\beta} \tag{621}$$

Since $\alpha \ll 1$, we find in addition

$$Z_{op} = Z_0 \coth(\alpha l + j\beta l); \quad \frac{Z_{op}}{Z_0} = \frac{1 + j\alpha l \tan(\beta l)}{\alpha l + j \tan(\beta l)}$$

$$\frac{Z_{sc}}{Z_0} = \tanh(\alpha l + j\beta l) = \frac{\alpha l + j \tan(\beta l)}{1 + j\alpha l \tan(\beta l)}$$

compared with the input impedance $Z_1 = E_1/I_1$

$$Z_1 = Z_0 \frac{\sinh(l\Gamma) + (Z_2/Z_0)\cosh(l\Gamma)}{\cosh(l\Gamma) + (Z_2/Z_0)\sinh(l\Gamma)}$$

Neglecting the resistance R ohms per meter of the HF line, we find

$$Z_{op} = -jZ_0 \cot(\beta l); \qquad Z_{sc} = +jZ_0 \tan(\beta l);$$

$$Z_1 = \frac{Z_2 Z_0 + jZ_0^2 \tan(\beta l)}{Z_0 + jZ_2 \tan(\beta l)} \tag{622}$$

and lines *shorter* than $\lambda/4$ act as *condensers when open* at the *far end* and like *inductances* when the *far end* is *short-circuited* Opposite and equal reactances occur when we deal with line lengths of l and $\lambda/2 - l$.

If a line of characteristic impedance Z_0 is correctly matched into a load $Z_2 = R_2$, only an incident forward voltage wave E_i will exist, since $R_2 = Z_0 = \sqrt{L/C}$ ohms consumes all of the incident power, arriving at termination R_2. For a mismatch, however, we shall also have a reflected voltage wave E_r, which travels backward toward the input end of the line. The ratio $\rho = E_r/E_i$ is known as the "reflection coefficient" at the load; and if k is the ratio of E_{min}/E_{max} of the resultant voltage distribu-

tion along the line, we have

$$k = \frac{E_{\min}}{E_{\max}} = \frac{I_{\min}}{I_{\max}} = \frac{1 - \rho}{1 + \rho} = \frac{1 - E_r/E_i}{1 + E_r/E_i} = \frac{E_i - E_r}{E_i + E_r};$$

$$\rho = \frac{(E_{\max}/E_{\min}) - 1}{(E_{\max}/E_{\min}) + 1} \qquad (623)$$

Often, the load is an impedance Z_2 for the HF line when R is negligible compared with ωL and G is negligible compared with ωC. We have then

$$E_x = E_2 \cos(\beta x) + jZ_0 I_2 \sin(\beta x) = E_2 \cos(\beta x) + jZ_0 \frac{E_2}{Z_2} \sin(\beta x) \qquad (624)$$

where x is reckoned with respect to the *load end* and Z_2 is the termination. We have then the reflection factor

$$\rho = \rho\epsilon^{j\varphi} = \frac{Z_2 - Z_0}{Z_2 + Z_0} /\varphi; \qquad \rho \text{ is magnitude} \qquad (625)$$

and

$$E_x = \frac{E_2}{Z_2} \frac{Z_2 + Z_0}{2} [(1 + \rho) \cos(\beta x) + j(1 - \rho) \sin(\beta x)]$$

$$= \frac{E_2}{Z_2} \frac{Z_2 + Z_0}{2} \{[\cos(\beta x) + j \sin(\beta x)] + \rho[\cos(\beta x) - j \sin(\beta x)]\}$$

$$= \frac{E_2}{Z_2} \frac{Z_2 + Z_0}{2} \epsilon^{j\beta x}[1 + \rho\epsilon^{j(\varphi - 2\beta x)}] \qquad (626)$$

since $\rho\epsilon^{j\varphi}$ is the true value (including the phase φ) of ρ. We find then the absolute value of the voltage at any distance x from the load end as

$$E_x = \frac{(Z_2 + Z_0)E_2}{2Z_2} \sqrt{1 + \rho^2 + 2\rho \cos(\varphi - 2\beta x)} \qquad \text{volts} \qquad (627)$$

and the absolute amount value of the current

$$I_x = \frac{(Z_2 + Z_0)I_2}{2Z_0} \sqrt{1 + \rho^2 - 2\rho \cos(\varphi - 2\beta x)} \qquad \text{amp} \qquad (628)$$

In both relations, E, I, and Z may now be taken as the effective absolute magnitudes, where $Z_2 = \sqrt{R_2^2 + X_2^2}$. The square-root factor is exactly the same as found by Fresnel reflection of plane electromagnetic waves (pages 798–824). Calling x_0 the distance of the voltage maximum (which is also a current minimum location) to the *load* impedance Z_2, we find

$$E_{\max} = \frac{(Z_2 + Z_0)E_2}{2Z_2} \sqrt{1 + \rho^2 + 2\rho \cos \underbrace{(\varphi - 2\beta x_0)}_{= 0}}$$

or

$$\varphi = 2\beta x_0 = \frac{4\pi x_0}{\lambda} \qquad \text{radians} \qquad (629)$$

For a *matched* HF *line*, we have only an outgoing wave train and $E_x = E_2\epsilon^{x\Gamma}$, $I_x = I_2\epsilon^{x\Gamma}$, where x is counted from the load and where E_2 and I_2 act. Since the attenuation is $\alpha = 0.5R\sqrt{C/L} = 0.5R/Z_0$ nepers per meter, we have the formulas

$$\left.\begin{aligned}
\alpha &= \frac{1.5 \times 10^{-7}\sqrt{f}}{r\log_{10}\left(\frac{s}{r}\right)} \text{ nepers/m} = \frac{1.3 \times 10^{-6}\sqrt{f}}{r\log_{10}\left(\frac{s}{r}\right)} \\
&\qquad\qquad \text{db/m for parallel-wire line} \\
\alpha &= \frac{3 \times 10^{-7}\sqrt{f}}{\log_{10}\left(\frac{D}{d}\right)}\left(\frac{1}{d} + \frac{1}{D}\right) \text{ nepers/m} \\
&= \frac{2.6 \times 10^{-6}\sqrt{f}}{\log_{10}\left(\frac{D}{d}\right)}\left(\frac{1}{d} + \frac{1}{D}\right) \text{ db/m for concentric line}
\end{aligned}\right\} \quad (630)$$

where the frequency f is in megacycles per second, r is the radius and s the spacing of the parallel-wire system between centers in meters, d denotes the *outside* diameter of the *inner* conductor, and D the *inside* diameter of the *outside* conductor of a concentric line expressed in meters. Since for the HF line

$$Z_0 = \sqrt{\frac{R + j\omega L}{j\omega C}} = \sqrt{\frac{L}{C} + \frac{R}{j\omega C}} \cong \sqrt{\frac{L}{C}} - j\frac{R}{2\omega\sqrt{CL}} \cong \sqrt{\frac{L}{C}}$$

the characteristic impedance Z_0, to a fair degree of approximation, acts as an ohmic resistance, and we can use the physical-dimension relations

$$\left.\begin{aligned}
Z_0 &= 276\log_{10}\left(\frac{s}{r}\right) \quad \text{for parallel-wire line} \\
Z_0 &= 138\log_{10}\left(\frac{D}{d}\right) \quad \text{for concentric line}
\end{aligned}\right\} \text{ ohms} \quad (631)$$

When the parallel wires are close together, the *proximity* effect must also be taken into account. According to pages 408–412, for the far end short-circuited and open, respectively, we have an input line impedance

$$\left.\begin{aligned}
Z_1 &= \frac{E_1}{I_1} = Z_{sc} = Z_0\tanh(\alpha + j\beta)l \\
&\cong Z_0\frac{\alpha l + j(1 - \alpha^2l^2)\sin(\beta l)\cos(\beta l)}{\alpha^2l^2 + (1 - \alpha^2l^2)\cos^2(\beta l)} \\
Z_1 &= \frac{E_1}{I_1} = Z_{op} = Z_0\coth(\alpha + j\beta)l \\
&\cong Z_0\frac{\alpha l - j(1 - \alpha^2l^2)\sin(\beta l)\cos(\beta l)}{\alpha^2l^2 + (1 - \alpha^2l^2)\sin^2(\beta l)}
\end{aligned}\right\} \text{ ohms} \quad (632)$$

if α is small enough to make $\sinh \alpha l = \alpha l$ and $\cosh \alpha l = 1.0$. Hence, for l shorter than $\lambda/4$, the Z_{sc} value acts like a coil and the Z_{op} value acts like a condenser; while between $l = \lambda/4$ and $\lambda/2$, the Z_{sc} value acts like a condenser and the Z_{op} value like a coil. For $l = \lambda/4$, either line (far end short-circuited or open, respectively) behaves like a pure resistance of value

$$\left.\begin{aligned} Z_1 &= Z_{sc} = \frac{2Z_0^2}{Rl} = \frac{8Z_0^2}{R\lambda} \quad \text{a maximum value} \\ Z_1 &= Z_{op} = \frac{Rl}{2} \quad \text{a minimum value} \end{aligned}\right\} \text{ohms} \qquad (633)$$

The *short-circuited* $\lambda/4$ line then acts like a parallel resonance circuit with a value $2Z_0^2/(Rl)$ ohms, and the *open* $\lambda/4$ line acts like a series resonance circuit with a minimum value of $0.5Rl$ ohms. The merit of a line is its Q value, *i.e.*, the ratio X_1/R_1. Doing this for (632), we find for l considerably smaller than $\lambda/4$

$$Q_{sc} = Q_{op} = \frac{(1 - \alpha^2 l^2) \sin \beta l \cos \beta l}{\alpha l} \qquad (634)$$

The Q values[1] may, therefore, be considerably more favorable than for networks using customary coils and condensers as reactances.

For a transmission line of length l meters and with negligible dissipation, we have, for the input impedance Z_1 and admittance Y_1 when looking toward a complex load $Z_2 = R_2 + jX_2$ ohms,

$$\left.\begin{aligned} Z_1 &= R_1 + jX_1 = Z_0 \frac{Z_2 + jZ_0 \tan(\beta l)}{Z_0 + jZ_2 \tan(\beta l)} \quad \text{ohms} \\ Y_1 &= \frac{1}{Z_1} = \frac{R - jX_1}{R_1^2 + X_1^2} = G_1 - jB_1 \\ &= Y_0 \frac{Y_2 + jY_0 \tan(\beta l)}{Y_0 + jY_2 \tan(\beta l)} \quad \text{mhos} \end{aligned}\right\} \qquad (635)$$

for a characteristic line impedance $Z_0 = 1/Y_0$ ohms, a phase constant $\beta = 2\pi/\lambda$ radians per meter, and a wavelength λ meters along the line. The load impedance Z_2 in terms of the load admittance Y_2 is given by $Z_2 = 1/Y_2$. Therefore,

[1] Generally, the Q value of a system is (ω_0 times the energy in a circuit) divided by the power loss or

$$Q = \frac{\pi \text{ times energy stored}}{\text{energy lost in half a cycle}}$$

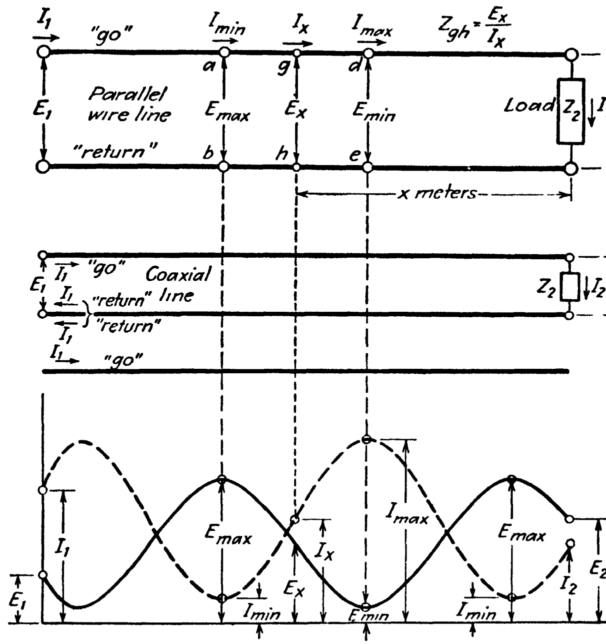
$$\left.\begin{aligned}
R_1 &= Z_0 Z_2 \frac{1 + \tan^2(\beta l)}{Z_0^2 + Z_2^2 \tan^2(\beta l)} \quad \text{ohms}; \\
X_1 &= \frac{Z_0^2 - Z_2^2}{Z_0^2 + Z_2^2 \tan^2(\beta l)} \tan(\beta l) \quad \text{ohms} \\
G_1 &= \frac{R_1}{R_1^2 + X_1^2} \quad \text{mhos}; \qquad B_1 = \frac{X_1}{R_1^2 + X_1^2} \quad \text{mhos}
\end{aligned}\right\} \qquad (636)$$

The Z_1 and Y_1 expressions of Eqs. (635) are often useful, since they provide a means, by a proper choice of the length l of the line, of obtaining, in effect, loaded lines for which Z_1 is either $R_1 + jX_1$ or $R_1 - jX_1$, corresponding to admittances $G_1 - jB_1$ and $G_1 + jB_1$, respectively. Hence, such lines with a complex impedance loading $Z_2 = R_2 \pm jX_2$ at the far end can be used for adjusting to a condition for which the loaded line behaves like a coil of resistance R_1 and reactance X_1 ohms or like a condenser of negative reactance X_1 ohms and series resistance R_1 ohms. Hence, there must also exist lengths for which X_1 vanishes altogether and the loaded line behaves like an ohmic resistance R_1. The case of the open and short-circuited lines is well known in the art of radio and refers to $Z_2 = \infty$ and $Z_2 = 0$ ohms. According to (635), for the short-circuited case $Z_{sc} = Z_1 = jZ_0 \tan \beta l$ and the open-ended case $Z_{op} = Z_1 = -jZ_0 \cot \beta l$, we have the corresponding input admittances $Y_{sc} = 1/Z_{sc} = -jY_0 \cot \beta l$ and $Y_{op} = jY_0 \tan \beta l$. We can, therefore, use such lines to inject suitable parallel admittances across a line so as to neutralize an undesirable susceptance. This is the subject in Secs. 79 to 84 and made use of in matching stubs when the length of an open-ended stub or a short-circuited stub injects the compensating susceptance and the location of the stub to the first voltage minimum of the line next to the load gives the proper termination of the stub to the line (satisfies the surge impedance). For an open-ended stub less than $\lambda/4$ long, the stub acts like a capacitance but like an inductance when short-circuited at the far end.

Moreover, for negligible line losses, we find *at* a voltage maximum of the line, for the standing-wave ratio $r = E_{max}/E_{min}$, for the input impedance the expression $Z = rZ_0$ and *at* a voltage minimum the expression $Z = Z_0/r$ ohms. The corresponding admittances are $Y = 1/Z = 1/(rZ_0) = Y_0/r$ at E_{max} positions and $Y = 1/Z = r/Z_0 = rY_0$ mhos at E_{min} positions along the line. Hence, we must have a location along the line where $Y = Y_0$, *i.e.*, the impedance $Z = Z_0$ ohms.

76. Fundamental Actions for Traveling and Standing Waves on Transmission Lines. This discussion applies particularly to Lecher wires (parallel wires) and coaxial lines (concentric lines). From the review on line formulas in Sec. 75, we note that an incident wavelike voltage

disturbance E_i travels along the line[1] toward the far end and is generally reflected at this end and returned as a voltage disturbance E_r toward the source of input voltage E_1 and input current I_1. The same happens for the corresponding line current, and we have an incident wavelike current disturbance I_i and a reflected disturbance I_r coming from the far end,



$Z_{ab} = \frac{E_{max}}{I_{min}} = rZ_0$ ohms; $Z_{de} = \frac{E_{min}}{I_{max}} = \frac{Z_0}{r}$ ohms; $Z_0 = \sqrt{\frac{L}{C}}$ ohms; L and C are inductance and capacitance per unit distance.

Standing-wave ratio $r = \frac{E_{max}}{E_{min}} = \frac{I_{max}}{I_{min}}$

Reflection coefficient at the load end $\rho = \frac{r-1}{r+1}$, $\rho = \frac{Z_2 - Z_0}{Z_2 + Z_0}$ numeric

FIG. 65. Standing-wave ratio and reflection coefficient.

at which we have a terminal voltage E_2 and a current flow I_2 (when loaded with an impedance Z_2). Such reflections do not necessarily require a load Z_2 at the far end, since for an open-ended line, we have $I_2 = 0$ amp at the far end, even though an input voltage E_1 starts an incident current disturbance I_i toward the far end. This necessitates then that at the open end, the reflected current disturbance must start out with a value $I_r = -I_i$ in order to cause a net current of $I_2 = 0$ amp,

[1] With respect to the promising traveling-wave tube, reference is made to papers by J. R. Pierce, L. M. Field, and others, *Proc. IRE*, **38**, 366, April, 1950.

since at every point of the line, the addition of reflected and incident current effects at that point accounts for the current measured there. Hence, at the far end, $I_2 = 0 = I_\iota + I_r$ requires $I_r = -I_\iota$. But for the resultant terminal voltage across the open end, we have $E_2 = E_r + E_\iota = 2E_\iota = 2E_r$ volts or $E_r = +E_\iota$. For an alternating voltage E_1 impressed across the input end of a perfect line, the incident wavelike voltage disturbance E_ι is a vector that rotates, so to speak, sinusoidally about the conductor axis toward the far end in a counterclockwise sense, while the reflected disturbance is a vector E_r that winds itself clockwise backward toward the input side of the line. We have then only to add *vectorially* the corresponding E_ι and E_r values in order to obtain the actual line voltage E_x at any distance x from the far end. The same holds for the actual current I_x at such a line point, since it is equal to the geometric addition of I_ι and I_r. Hence, for points where incident and reflected disturbances are in phase, we must obtain maximum effects, whereas antiphase values result in minimum effects. For line points where incident and reflected disturbances have no such phase relations, the resultant E_x and I_x values are somewhere between E_{max} and E_{min} or the I_{min} and I_{max} conditions. Across line points (on go and return conductors) corresponding to a distance x from the far end, for which we have neither maximum E_x nor minimum E_x effects, we look into an impedance and into a reactance for the case of an ideal line. For points for which $E_x = E_{max}$ and $E_x = E_{min}$, we look into an ohmic resistance. Figure 65 shows the voltage (E_x) and the current (I_x) distributions along the line. They are relatively displaced by 90 space degrees corresponding to a distance of one-quarter of a wavelength along the line.

The characteristic impedance Z_0 of a line is the impedance offered to a wavelike disturbance moving along the line without an obstruction. Hence, Z_0 must be equal to E_ι/I_ι ohms as well as to E_r/I_r ohms. Since the incident wave disturbance moves just as fast in one direction as the returning wave reflected at the far end moves in the opposite direction, the resultant E_x and I_x distributions stand still, or are stationary. For E_x as the voltage across the line and I_x the corresponding current flow at a distance x from the far end, we find $Z_x = E_x/I_x$ as the impedance looking into the go and the return between such go and return points. When the line is terminated into a pure resistance R_2 ohms equal to the characteristic impedance $Z_0 = \sqrt{L/C}$ ohms of a well-designed HF line, then all the arriving wave energy is being absorbed by the load R_2 and no wave reflection can take place at such a termination. For such a condition, $Z_x = E_x/I_x = E_\iota/I_\iota = E_1/I_1 = E_2/I_2 = Z_0$ ohms, since $E_r = 0 = I_r$ and only E_ι and I_ι traveling outgoing waves exist. Such a condition is known as a "flat" distribution, and the line is "matched."

For a short-circuited line, there can be no terminal voltage at the far end and $E_2 = 0$ volts. Hence, E_r must be equal to $-E_i$ at such a termination, and $E_2 = E_i + E_r = E_i - E_i = 0$. For the corresponding current through the zero-ohm termination, on account of $I_r = I_i$, we find $I_2 = I_i + I_r = 2I_i$ amperes.

We have seen that for $Z_2 = 0$ ohms, I_i and I_r are equal and in phase. We also found that for $Z_2 = R_2 = \sqrt{L/C}$ ohms, only an outward-traveling wave opposed by a wave impedance $E_1/I_1 = E_x/I_x = E_i/I_i = E_2/I_2 = Z_0 = \sqrt{L/C}$ ohms exists and $I_2 = I_1$ amperes for an ideally flat line. For a pure resistance mismatch $R_2 \neq Z_0$ ohms, the power transfer to R_2 ohms is always smaller than would exist for a perfect match ($R_2 = Z_0$), regardless of whether $R_2 < Z_0$ or $R_2 > Z_0$ resistive ohms. For a pure resistance loading R_2 smaller than the characteristic line impedance Z_0, the reflected E_r value across R_2 must be in phase opposition to the E_i value across R_2 in order to account for the smaller actual value $E_2 = E_i - E_r = I_2R_2$ volts. For a loading R_2 larger than Z_0 ohms, more resultant voltage E_2 must occur across R_2 and $E_2 = E_i + E_r = I_2R_2$ volts, since E_r is in phase with E_i. For $R_2 > Z_0$ and $R_2 < Z_0$, the reflected voltage E_r across R_2 is smaller than the incident value E_i across R_2.

The reflection coefficient ρ for a pure resistance load $Z_2 = R_2$ is $\rho = E_r/E_i$, since ρE_i is the portion that is being reflected back by the impact or incident voltage E_i and ρ is, at best, equal to unity (perfect reflection). For an impedance loading $Z_2 = R_2 + jX_2$, the magnitude of the reflection coefficient ρ is still equal to the ratio of reflected-to-incident voltage at the load impedance Z_2, but we must also take a phase angle into account, since the reflected voltage E_r at the load is rotated with respect to the incident voltage E_i at Z_2 and due to Z_2. For a marked reactive component X_2 of Z_2, the standing-wave ratio $r = E_{max}/E_{min} = I_{max}/I_{min}$, can give only an approximate indication of the mismatch of Z_2 with respect to Z_0 of the line.

From the line formulas in Sec 75, the reflection coefficient ρ at the load of Z_2 ohms is, according to (625),

$$\rho = \frac{Z_2 - Z_0}{Z_2 + Z_0} \tag{637}$$

where for a complex value for Z_2, the factor ρ must also be complex and, therefore, has the form $A \pm jB$ with a magnitude $\sqrt{A^2 + B^2}$ and phase $\tan^{-1} (\pm B/A)$ time degrees This relation is actually the outcome of (615) in Sec. 75, since the line voltage E_x and the corresponding line current I_x at any distance x from the load are

$$E_{x,t} = (E_i\epsilon^{x\Gamma} + E_r\epsilon^{-x\Gamma})\epsilon^{j\omega t}; \qquad I_{x,t} = \left(\frac{E_i}{Z_0}\epsilon^{x\Gamma} - \frac{E_r}{Z_0}\epsilon^{-x\Gamma}\right)\epsilon^{j\omega t}$$

where $E_i/Z_0 = I_i$ and $E_r/Z_0 = I_r$ are incident and reflected constituents at any line point. Hence, the impedance, at any distance x of the go and return from the load looking into the line, is found by forming the ratio of the space functions. We have then

$$Z_x = \frac{E_{x,t}}{I_{x,t}} = Z_0\frac{E_i\epsilon^{x\Gamma} + E_r\epsilon^{-x\Gamma}}{E_i\epsilon^{x\Gamma} - E_r\epsilon^{-x\Gamma}} \quad \text{ohms} \tag{638}$$

Suppose we connect a complex load of Z_2 ohms across the far end of the line, for which $x = 0$. We have then $Z_x = Z_2$ and

$$Z_2 = Z_0\frac{E_i + E_r}{E_i - E_r} = Z_0\frac{1 + \rho}{1 - \rho} \tag{639}$$

since the reflection coefficient ρ expresses what portion of the incident wavelike disturbance E_i the reflected disturbance E_r is, or $E_r = \rho I_i$. Solving for ρ then yields (637). From (623) of Sec. 72 and the standing-wave ratio $r = E_{max}/E_{min} = I_{max}/I_{min}$, we have $r = (1 + \rho)/(1 - \rho)$, or $\rho = (r - 1)/(r + 1)$. We have the important formulas for the reflection coefficient

$$\rho = \frac{(E_{max}/E_{min}) - 1}{(E_{max}/E_{min}) + 1} = \frac{Z_2 - Z_0}{Z_2 + Z_0} \tag{640}$$

It is the expression to the right that gives both the magnitude and the phase of ρ if Z_2 is known, while the expression to the left seems useful for straight resistance loading and cases where the reactive component of the loading Z_2 is not too pronounced. The standing-wave ratio[1] r, for the ratio of respective maximum to minimum values, is also useful when we wish to find the magnitude and phase of an unknown load. It can also be employed for matching an impedance load against a line or if one or two matching stubs are used (Secs. 80 and 81).

77. Standing-wave Ratio and Its Application to UHF Measurements. Such a ratio defines either the ratio of maximum to minimum values of line voltage and current $r = E_{max}/E_{min} = I_{max}/I_{min}$ or its reciprocal $k = 1/r$. In connection with some work, it is more convenient to use the reciprocal and to call $k = E_{min}/E_{max} = I_{min}/I_{max}$ the standing-wave ratio. Formulas using such a ratio are given in Sec. 76; and by expressing it by E_{min}/E_{max}, we have

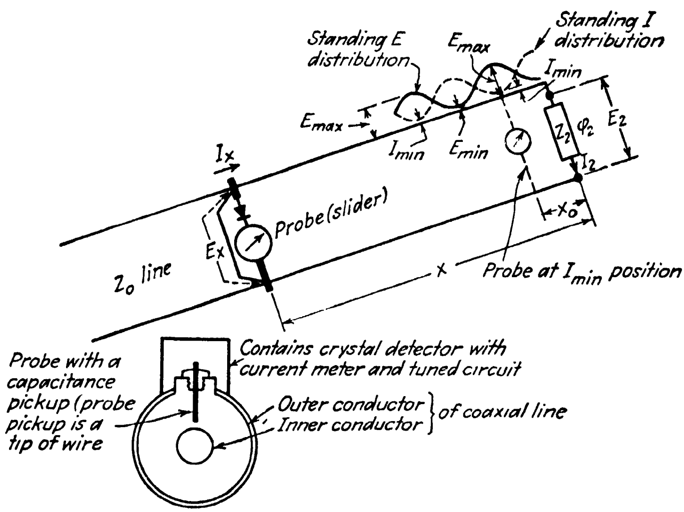
$$Z_2 = Z_0\sqrt{\frac{A}{B}} \quad \text{ohms}; \qquad \varphi_2 = \sin^{-1}\left(\frac{C}{D}\right) \quad \text{time degrees} \tag{641}$$

[1] In many texts, the standing-wave ratio is also taken as the reciprocal value, namely, minimum value/maximum value.

for the load impedance Z_2 and its phase at the far end of the line and the characteristic impedance Z_0 of the line. We have then

$$\left.\begin{aligned} A &= 1 + k^2 + (1 - k^2)\cos\theta; \quad B = 1 + k^2 - (1 - k^2)\cos\theta; \\ &\qquad\qquad\qquad\qquad\qquad C = (1 - k^2)\sin\theta \\ D &= \sqrt{(1 + k^2)^2 - (1 - k^2)^2\cos^2\theta} \\ \theta &= \frac{720x_0}{\lambda} \text{ space degrees}; \quad k = \frac{E_{\min}}{E_{\max}} = \frac{I_{\min}}{I_{\max}} \end{aligned}\right\} \quad (642)$$

where x_0 denotes the distance from the load end to the first voltage maximum in the same measure as the wavelength along the line. The first voltage maximum corresponds also to the first current minimum.



$\beta = \dfrac{2\pi}{\lambda}$ radians per unit distance

$\beta = \dfrac{360}{\lambda}$ deg per unit distance

Load $Z_2 = R_2 \pm jX_2$; $\varphi_2 = \tan^{-1}\left(\dfrac{\pm X_2}{R_2}\right)$;

$R_2 = Z_2\cos\varphi_2$ ohms; $X_2 = Z_2\sin\varphi_2$ ohms;
$I_{max} = (1 + \rho)I$; $I_{min} = (1 - \rho)I$;

$k = \dfrac{E_{min}}{E_{max}} = \dfrac{I_{min}}{I_{max}}$

FIG. 66. Determination of the magnitude and phase of an unknown impedance. A shunted probe system, as shown for the parallel-wire line, behaves like a current probe and indicates current maxima and minima; the voltage probe, shown for the coaxial line, indicates voltage maxima and minima.

The measurement for the determination of the value of an unknown termination Z_2 ohms and φ_2 time degrees is then based on the arrangement given in Fig. 66. In the case of a parallel-wire system (Lecher wires),

a slider bridge is moved along the line in place of the proximity pickup, producing into the coaxial line through a slot opening. The slider bridge usually contains a sensitive direct-current meter in series with a crystal rectifier shunted by a No. 14 wire. For the probe of a concentric line, the crystal rectifier with the current indicator is generally within the shielded box, and a short tip of a wire couples capacitively to the central conductor of the concentric line. Great care has to be taken that the distance of the tip of the central conductor remains exactly the same as the pickup is moved along the slot opening for noting the voltage distribution. The R.F. voltmeter of the pickup can also use a diode instead of a crystal rectifier, with a current meter and tuned input.

Formulas (641) and (642) are easy to apply, as is shown below. The formulas are based on the well-known transmission-line relations

$$I_x = \underbrace{\frac{1}{2} I_2 \left(1 + \frac{Z_2}{Z_0}\right) \overset{\text{space degrees}}{\underline{/+\beta x}}}_{\text{incident } I_i \text{ disturbance at any place } x} + \underbrace{\frac{1}{2} I_2 \left(1 - \frac{Z_2}{Z_0}\right) \overset{\text{space degrees}}{\underline{/-\beta x}}}_{\text{reflected } I_r \text{ disturbance at any place } x \text{ from the load end}}$$

$$= \frac{I_2(Z_0 + Z_2)}{2Z_0} \epsilon^{j\beta x} + \frac{I_2(Z_0 - Z_2)}{2Z_0} \epsilon^{-j\beta x} \tag{643}$$

for $\beta = 360/\lambda$ space degrees per meter length of the line (unit length), and $Z_2 = R_2 + jX_2$, where X_2 may be positive or negative, depending upon whether the loading at the far end has inductive or capacitive reactance effect, besides resistance R_2 dissipation. We then generally have at the load end also a phase $\varphi_2 = \tan^{-1} (\pm X_2/R_2)$ time degrees. At the far end of the line, where the load Z_2, φ_2 is located, we have $x = 0$, $\epsilon^{j\beta x} = 1$, and $I_x = I_2$ or

$$I_x = I_2 = \frac{1}{2} I_2 \left(1 + \frac{Z_2}{Z_0}\right) + \frac{1}{2} I_2 \left(1 - \frac{Z_2}{Z_0}\right) = \frac{1}{2} I_2 + \frac{1}{2} I_2$$

or

$$I_x = \underbrace{\frac{I_2(Z_0 + Z_2)}{2Z_0}}_{I_i \text{ at } x = 0} + \underbrace{\frac{I_2(Z_0 - Z_2)}{2Z_0}}_{I_r \text{ at } x = 0} = I_2 \tag{644}$$

where $Z_0 + Z_2$ generally has a phase angle $\theta' = \tan^{-1}[\pm X_2/(Z_0 + R_2)]$ and $Z_0 - Z^2$ has a phase angle $\theta'' = \tan^{-1}[+ X_2/(Z_0 - R_2)]$ degrees if Z_0 is taken pure resistive. Inasmuch as the first part of the I_2 relation represents the incident current value I_i at the load Z_2 of the line and the second part the reflected value I_r, we have, for the reflection coefficient at the load Z_2,

$$\rho = \frac{I_r}{I_i} = \frac{Z_0 - Z_2}{Z_0 + Z_2} = a + jb \tag{645}$$

where b may be positive or negative, since generally $Z_2 = R_2 \pm jX_2$; *i.e.*, X_2 may be positive or negative. Hence, we also generally have a phase of the reflection coeffi-

cient that is $\varphi = \tan^{-1} (\pm b/a)$ degrees. For the distance x_0 from the load end to the first current minimum I_{min}, we have $2\beta x_0 + \theta' + \theta'' = 180$ deg, since $\beta = 360/\lambda$. Twice the electrical distance βx_0 *must* be used, since incident and reflected portions *each have* an absolute phase shift of βx_0 degrees, explaining why the value $720 = 2 \times 360$ occurs in formulas (642). A maximum value I_{max} occurs at places along the line whenever the incident I_i wave is in phase with the reflected I_r wave disturbance, *i.e.*, when $I_i + I_r = I_{max}$. An antiphase condition prevails for minimum current positions along the line, *i.e.*, whenever $I_i - I_r = I_{min}$. We find then from (643) and (644), for the antiphase and inphase conditions,

$$\left.\begin{aligned} I_{min} &= I_i - I_r = \frac{I_2}{2Z_0}[Z_0 + Z_2 - (Z_0 - Z_2)] \\ &= \frac{(Z_0 + Z_2)I_2}{2Z_0}\left(1 - \frac{Z_0 - Z_2}{Z_0 + Z_2}\right) = (1 - \rho)\frac{(Z_0 + Z_2)I_2}{2Z_0} \\ I_{max} &= I_i + I_r = (1 + \rho)\frac{(Z_0 + Z_2)I_2}{2Z_0} \end{aligned}\right\} \quad (646)$$

The standing-wave ratio k then yields

$$k = \frac{I_{min}}{I_{max}} = \frac{1 - \rho}{1 + \rho} \quad (647)$$

which is the factor used in (642). If the result is solved for ρ, we have

$$\rho = \frac{1 - k}{1 + k} \quad (648)$$

When the far end is short-circuited, then $Z_2 = 0$, I_{min} of (646) yields zero value, and $I_{max} = I_2$.

78. Determination of the Magnitude and the Phase of an Unknown Impedance. According to the preceding section, formulas (641) and (642) can be used when the distance x_0 to the first voltage maximum (the same as the first current minimum) from the load Z_2, φ_2 is found experimentally by means of a traveling probe indicator. The same probe is also used to find the standing-wave ratio $k = E_{min}/E_{max} = I_{min}/I_{max}$. Suppose the exciting frequency is 100 Mc/sec and a voltage E_1 is impressed at the input side of the line with the unknown impedance Z_2 at the far end of the line. Assuming that the voltage and current distributions along the feeder are determined essentially by a phase velocity equal to that in empty space, we have $\lambda = 300/f = 300/100 = 3$ m. Suppose that the first voltage maximum from the load end occurs 50 cm from Z_2; then $x_0 = 0.5$ m, and according to (642)

$$\theta = \frac{720x_0}{\lambda} = \frac{720 \times 0.5}{3} = 120°$$

If, in addition, the sliding probe indicates a maximum value of 7 and a minimum value of 3.5, we then have, for the standing-wave ratio, $k = 3.5/7 = 0.5$. For (642), we find

$$A = 1 + k^2 + (1 - k^2)\cos\theta = 1 + 0.25 + 0.75\cos 120^\circ = 1.25 - 0.75\underbrace{\sin 30^\circ}_{0.5} = 0.875$$

$$B = 1 + k^2 - (1 - k^2)\cos\theta = 1.25 + 0.75 \times 0.5 = 1.625$$

$$C = (1 - k^2)\sin\theta = 0.75\sin 120^2 = 0.75\underbrace{\cos 30^\circ}_{0.866} = 0.65$$

$$D = \sqrt{(1 + k^2)^2 - (1 - k^2)^2\cos^2\theta} = \sqrt{1.25^2 - 0.75^2\underbrace{\cos^2 120^\circ}_{(-0.5)^2}} = \sqrt{1.562 - 0.562 \times 0.25} = \sqrt{1.4215} = 1.19$$

$$\sqrt{\frac{A}{B}} = \sqrt{\frac{0.875}{1.625}} = 0.735$$

The magnitude of the load impedance is, according to (641),

$$Z_2 = Z_0\sqrt{\frac{A}{B}} = 75 \times 0.735 = 55 \text{ ohms}$$

if the line has a 75-ohm surge impedance. Besides, Z_2 has a phase angle

$$\varphi_2 = \sin^{-1}\frac{C}{D} = \sin^{-1}\frac{0.65}{1.19} = \sin^{-1} 0.545 = 33^\circ$$

We have then a resistance $R_2 = Z_2 \cos\varphi_2 = 55 \times 0.839 = 46$ ohms and an inductive reactance $X_2 = Z_2 \sin\varphi_2 = 55 \times 0.545 = 30$ ohms. This would correspond to a self-inductance $L_2 = X_2/\omega = 30/(6.28 \times 100 \times 10^6)$ henrys $= (30/628)\mu$h $= 0.048\ \mu$h, which is a very small inductance and yet causes 30 ohms reactive drop at 100 Mc/sec.

79. Practical Hints on Load Impedance Determinations. When a parallel-wire system is used, it is an easy matter to proceed as just described, since the slider bridge used for indicating either voltage maxima and minima or the corresponding current extremes can be moved all along the line from the load end. This is not practically feasible along a standard slotted coaxial line, since the slot does not extend all the way to the far end.

It is important that the slotted coaxial line section A of Fig. 67 has the same characteristic impedance Z_0 as the associated concentric line B to which the unknown impedance Z_2 is connected at the far end. It is also important that the capacitance coupling (distance of the end of probe tip to the outer surface of the inner conductor of the slotted coaxial

line) is always the same. The VHF or UHF source need not be matched but, if matched, delivers most power to the measuring system.

The procedure of determination is then as follows: (1) Remove the load Z_2, connect across terminals a, b, a good short circuit, and slide the voltage probe along the slot until a decided maximum effect is observed on the indicating meter. (2) Increase the power input of the source until the pointer of the probe meter gives a full deflection. (3) Slide the probe to a position P_0 for which a voltage minimum occurs. The exact value of such a position is obtained by observing positions of equal deflections on each side of P_0 and placing P_0 halfway between such slight equal deflections. The point P_0 on the scale of the slot is taken as the reference point. Therefore, it is convenient to choose a voltage minimum

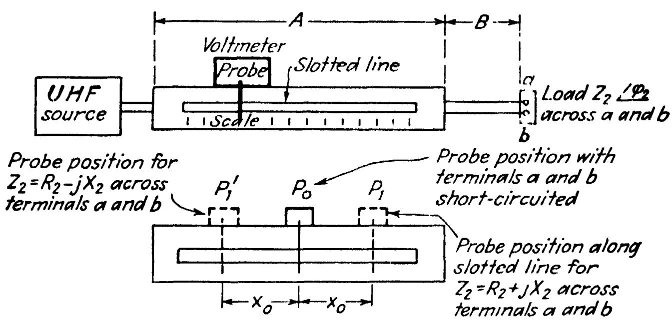


FIG. 67. Slotted line and sliding probe positions for different load conditions at the far end (across terminals a and b).

position P_0 near the center of the slot length so that large left- and right-hand displacements of the probe are possible. (4) Replace the short circuit, and connect across the end terminals a, b the actual load Z_2, φ_2. (5) Move the probe again until a minimum voltage effect is noted, and take down the E_{min} reading. If the location of this E_{min} is to the left of P_0, then we have a point such as P_1', showing that a capacitive reactance $-X_2$ of the load has moved the voltage minimum toward the source. If the probe is to be moved to the right, such as a point P_1, the load reactance X_2 is positive (inductive). This can be understood from the following:

For a loading $Z_2 = R_2 = Z_0 = \sqrt{L/C}$ pure resistive ohms, the line is matched and a pure traveling wave toward the load end exists with no wave return, since all the incident wave energy is being absorbed by R_2 ohms, which forms the load at the far end. For $Z_2 = R_2$ but either larger or smaller than Z_0 ohms, partial wave reflection will occur at the pure resistance termination R_2 ohms. Such a wave reflection is in phase or in antiphase, respectively, with the incident voltage wave. For either case, the reflected voltage wave at the load R_2 is smaller than the incident voltage

value at R_2, since some of the arriving wave energy is being absorbed by R_2. For an impedance loading $Z_2 = R_2 + jX_2$ or $R_2 - jX_2$ ohms, the reflected voltage vector at R_2 is rotated in one direction due to $+X_2$ and in the other direction due to $-X_2$ reactance effect. Hence, all minima and maxima are shifted either closer to the load or farther away from it.

The next steps in the procedure are then as follows: (6) The distance $x_0 = P_0P_1'$ or P_0P_1, respectively, thus gives a means for computing the angle θ of formulas (612), since $360x_0/\lambda$ degrees is the phase rotation due to the reactance X_2 of impedance loading Z_2. (7) The probe is now moved to a position where a voltage maximum occurs, and E_{max} read off. This, together with the minimum reading E_{min} a distance x_0 away from the reference point P_0, gives the factor $k = E_{min}/E_{max}$ of (612). It is therefore seen that we need either a *linear* probe indicator or an indicator whose *calibration is known* if some other deflection law is followed. We

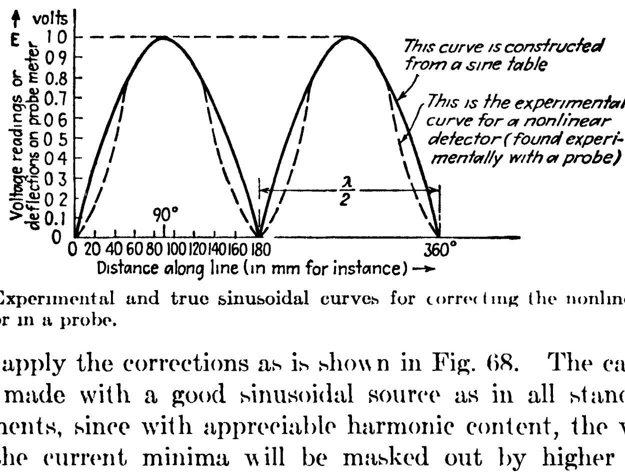


FIG. 68. Experimental and true sinusoidal curves for correcting the nonlinearity effect of a detector in a probe.

can then apply the corrections as is shown in Fig. 68. The calibrations must be made with a good sinusoidal source as in all standing-wave measurements, since with appreciable harmonic content, the voltage as well as the current minima will be masked out by higher harmonic effects. The calibration is carried out for the far end open or short-circuited so that $E_{min} = 0$ volts results. An experimental curve then looks like the dotted curve in Fig. 68, since nonlinearity shows up especially near the minimum readings of the meter. The maximum indication is so chosen that it corresponds to full meter deflection of the probe voltmeter. The full-line curve, which holds for a perfect linear probe detector, is constructed over the same reference base (time axis) as the calibration curve by means of sine values obtained from customary trigonometric tables. Hence, if the probe voltmeter for an actual Z_2 loading reads $E_{min} = 0.23$ and $E_{max} = 0.78$, the value of 0.23 on the experimental curve yields about $E_{min} = 0.35$ on the full-line curve and the experimental value of 0.78 gives $E_{max} = 0.83$ on the correct full-line curve. The actual standing-wave ratio is then $k = 0.35/0.83 = 0.42$.

The power loss due to the mismatch is given by the ratio

$$\frac{\text{Power loss in mismatched line}}{\text{Power loss in matched line}} = \frac{1 + k^2}{2k} \tag{649}$$

and is equal to $(1 + 0.42^2)/(2 \times 0.42) = 1.176/0.84 = 1.4$.

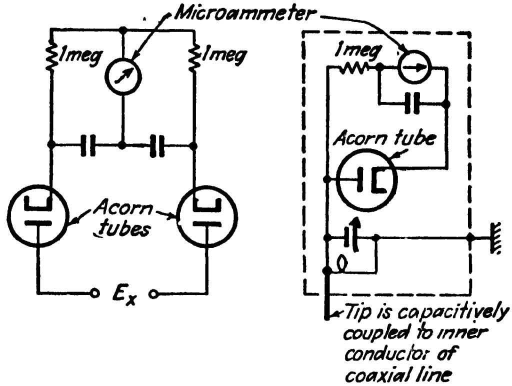


FIG. 69. Detector arrangements for probes.

Figure 69 shows a two-ended and a single-ended voltage indicator for standing-wave measurements. The single-ended network applies to slotted coaxial lines.

80. The Procedure of Single-stub Impedance Matching. This method is based on locating the positions of successive voltage maxima and minima with a probe and utilizing in Fig. 70 either the location of the first voltage minimum next to the load Z_2 or, as the reference location, the first voltage maximum next to Z_2. It is then possible to place either a short-circuited stub or an open stub of suitable length l a definite distance d from said reference location, which will bring about impedance matching of the load to the feeder line.

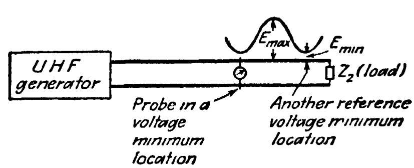
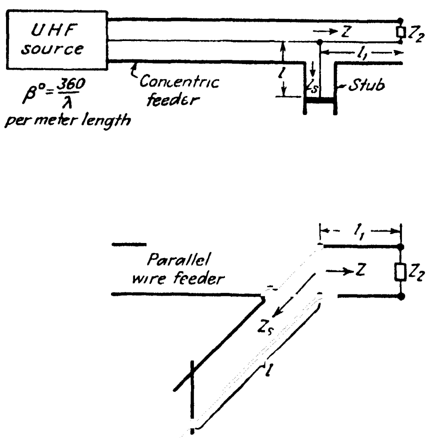


FIG. 70. Procedure for locating minimum and maximum locations along a feeder. $r = E_{max}/E_{min}$.

Even though any of the minimum and maximum locations, respectively, could be used as a reference, it is preferable to use the location closest to the load Z_2. The reason for this is that the purpose of the stub is to cancel the reactive effect of the load. The ideal case would then be to connect a suitable stub length across the load terminals. For practical reasons, this is not always possible and especially when the load is a dipole. It is also possible to place the stub on either side of the reference location, but again the preference is to place it toward the load side of

this location. Places of voltage maximum and voltage minimum are used as references because such locations represent conditions for which the ratio of line voltage to line current is purely resistive, with resistance values rZ_0 and Z_0/r ohms for the standing-wave ratio $r = E_{max}/E_{min}$ and the characteristic impedance Z_0 ohms of the feeder to be matched.

The stub consists conveniently of the same line (same Z_0 value) as the feeder. We know that the input impedance of an open-ended line looks essentially like $Z_0 \cot \beta l$ ohms and for the far end short-circuited like $Z_0 \tan \beta l$ ohms. Hence, for an electrical length equal to 45 electrical degrees or $l = \lambda/8$ (since $\beta l = (360/\lambda)(\lambda/8) = 45$ deg), both the open-ended and the short-circuited lines act like a pure resistance of Z_0 ohms.



Z_2 complex ohms is the load to be matched

$Z_s = jZ_0 \tan (\beta l)$
Z_0 is the characteristic impedance of the stub (conveniently of same value as the line)

FIG. 71. Single stubbing for matching a load impedance Z_2 to a feeder.

For a physical length l shorter than one-quarter of 1 wavelength, the open-ended stub acts like a capacitance and the short-circuited stub like an inductance. Between $l = \lambda/4$ and $\lambda/2$ and larger than $\lambda/4$, the open-ended stub behaves like an inductance and the short-circuited stub like a capacitance. Hence, it is necessary to use stubs only about $\lambda/4$ long, for which the length l is decreased or increased by telescoping, for the open-ended stub. For the short-circuited stub, the short-circuiting bar is moved toward the input side in order to produce the desired reactance effect.

Figure 71 shows the case of a closed stub of length l for a concentric line and a Lecher wire system. The impedance Z looking toward the load is generally complex and of the form $Z = R \pm jX$. The impedance Z_s looking into the closed stub is essentially a reactance $jZ_0 \tan \beta l$, the polarity and magnitude being affected by the length l of the stub. The magnitude and phase of the impedance Z depend on the length l_1 and

the magnitude and phase of the load impedance Z_2. The resistance R and the reactance X act in series with equivalent values R' and X' acting in parallel. Hence, for a certain distance l_1, the equivalent resistance R' becomes equal to the characteristic impedance $Z_0 = \sqrt{L/C}$ of the feeder; and for a certain stub length l, the parallel reactance X' due to the l_1 and Z_2 effect can be made equal and opposite to $jZ_0 \tan \beta l$, since for parallel resonance at the stub joint, we have the condition $Z_s = jZ_0 \tan \beta l = -jX'$. Hence, when the length l_1 and the impedance Z_2 are such that X' comes out to be a capacitive reactance, then the closed stub length l must be shorter than $\lambda/4$ in order to behave like an inductance. If X' comes out as a positive reactance acting like a parallel inductance to $R' = Z_0$ ohms, then an open-ended stub of a suitable length less than $\lambda/4$ long will cancel out the X' effect at the stub joint. Suppose X' is $-j60$ ohms; then the closed stub of Fig. 71 must be such that $jZ_0 \tan \beta l = +j60$ ohms; and for a 100-ohm characteristic line impedance, $100 \tan \beta l = 60$ or $\tan \beta l = 0.6$, making the electric length $\beta l = 31$ electrical degrees for the closed stub. Now 90 electrical degrees corresponds to a length $l = \frac{1}{4}\lambda$, and 31 deg yields $0.25 \times 31\lambda/90 = 0.086\lambda$ long. Suppose the upper arrangement of Fig. 71 is of concern and for a 600-Mc excitation. Then $\lambda = 300/600 \cong 0.5$ m or 50 cm long. The physical length l of the concentric closed stub is then $l = 0.086\,\lambda = 0.086 \times 50 \cong 4.3$ cm in length since λ along the stub is somewhat shorter.

For such applications, it should be remembered that the phase constant for a uniform line as well as for a uniform stub can be expressed in electrical degrees $\beta = 360/\lambda$ per unit length, *i.e.*, per meter, or in circular radians $\beta = 6.28/\lambda$ per meter, since 360 deg denotes the phase distribution along a feeder for the duration of one complete cycle, *i.e.*, for 1 wavelength, and $2\pi = 6.28$ is the radian equivalent for it.

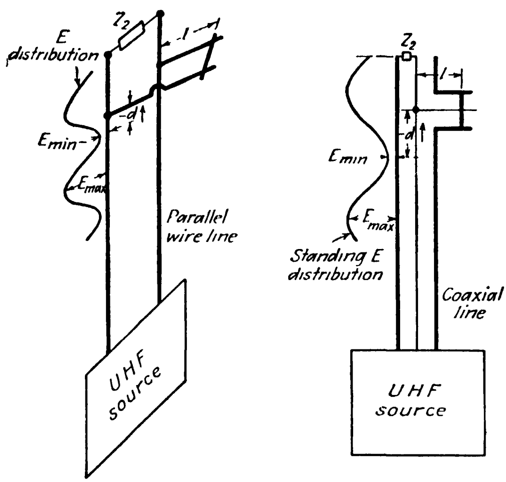
When a closed Lecher wire feeder as in the lower case of Fig. 71 is used for very high frequencies, such as centimeter waves, the length of the short-circuiting bar can no longer be neglected, since it represents a certain portion of the wave distribution. Normally, if s denotes the spacing between the centers of the go and return wires, the length s acts like an inductance, for which a formula can be derived. Hence, the stub length l must be chosen shorter. Often, subtracting $0.5s$ from the computed value found from $Z_0 \tan \beta l$ gives about the correct stub length.

In actual practice, we proceed semiexperimentally, since from the standing-wave ratio $r = E_{max}/E_{min}$ and the reference position, which may be the E_{min} or the E_{max} location, we find the location to the stub joint and then use a stub about $\frac{1}{4}\lambda$ long and shorten it until a suitable impedance match is obtained. We then have results as shown in the representation of Figs. 72 and 73. For the open-stub matching, we have

results as in Fig. 74, where, for the sake of simplicity, the coaxial line only is shown. From the inscribed formulas, it is seen that when $(r - 1)/\sqrt{r}$ is equal to unity, then the open- and the closed-stub lengths l happen to be of the same length. This occurs for $E_{max}/E_{min} = 2.62$, since

$$\frac{2.62 - 1}{\sqrt{2.62}} = \frac{1.62}{1.62} = 1$$

The application of these formulas is simple. Suppose we find a standing-wave ratio of $r = E_{max}/E_{min} = 2$ experimentally; then Fig. 72 shows that for a closed-



Z_2 is load, and $-d$ (negative) is result.

$$\cot(\beta l) = \pm\frac{r-1}{\sqrt{r}}, \qquad \tan(\beta d) = \mp\frac{1}{\sqrt{r}}$$

βl and βd are computed

d is measured from E_{min} *toward* Z_2

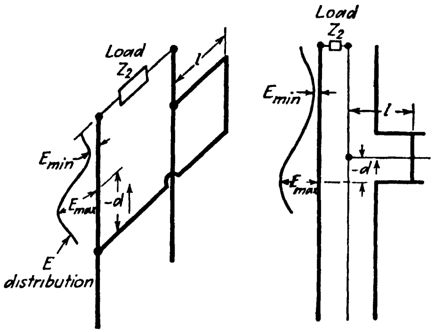
$r = \dfrac{E_{max}}{E_{min}}$ found experimentally

FIG. 72. Formulas and procedure for closed-stub matching. $+d$ is reckoned from E_{min} location away from the impedance load Z_2.

stub matching with respect to the minimum voltage location, $r - 1 = 1$, $\sqrt{r} = 1.41$, $\cot \beta l = (r - 1)/\sqrt{r} = 1/1.41 = 0.707$, $\beta l = 54.7$ electrical degrees, $\tan \beta d = 1/\sqrt{r} = 1/1.41 = 0.707$, or $\beta d = 35.3$ electrical degrees. The closed-stub length is then $l = 54.7\lambda/360 = 0.152\lambda$ and a distance $d = 35.3\lambda/360 = 0.098\lambda$ from the voltage minimum next to load Z_2. For a 300-Mc excitation, we have a wavelength of 300/300 = 1 m, showing that the closed stub is 0.152 m, or 15.2 cm, long and 0.098 m, or 9.8 cm, from the E_{min} location. It is to be noted that, strictly, we have to use the wavelength along the stub and along the feeder, which can be readily found by experiment.

If an open stub is used for $r = 2$, and again with respect to the voltage minimum setting, the formulas to the left of Fig. 74 show that the open stub must be moved through exactly the same distance of 0.098 wavelength but *away* from the load, since $\tan \beta d = \pm 1/\sqrt{r}$ is the formula for it and no longer $\tan \beta d = \mp 1/\sqrt{r}$. The stub length l is now computed from a tangent instead of from a contangent function, and we have then, for $r = 2$, the evaluations $\tan \beta l = 1/1.41$ and also $\beta l = 35.3$ electrical degrees, *i.e.*, 0.098 wavelength long. Table 15 gives for closed and open-ended stubs $\frac{1}{4}\lambda$ and shorter in length the computations that cover all single-stub cases, and Fig. 75 is the plot of these results. Because of the different signs of the tangent function

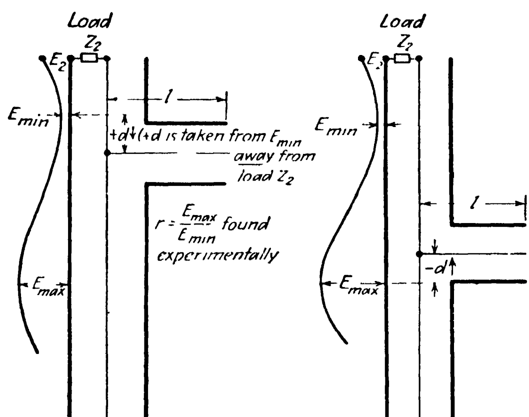
for the displacement of the stub, the same displacement toward the load for a closed stub holds, but toward the source for an open stub from the voltage minimum. The length l of the stub is, however, different.



$$\cot(\beta l) = \pm \frac{r-1}{\sqrt{r}}$$
$$\tan(\beta d) = \pm \sqrt{r}$$
βl and βd are computed

$$r = \frac{E_{max}}{E_{min}}$$ found experimentally

FIG. 73. Formulas for closed-stub impedance matching and a voltage maximum as a reference location.



$$\tan(\beta l) = \pm \frac{r-1}{\sqrt{r}} \qquad \tan(\beta l) = \pm \frac{r-1}{\sqrt{r}}$$

$$\tan(\beta d) = \mp \frac{1}{\sqrt{r}} \qquad \tan(\beta d) = \pm \sqrt{r}$$

βl and βd are computed

FIG. 74. Formulas for stub impedance matching for voltage maximum and voltage minimum reference locations and open-stub lengths. E_2 are generally no voltage maxima but merely any voltage drop across a particular impedance loading of Z_2 complex ohms.

81. Notes on Double-stub Impedance Matching. Figure 76 shows such a method applied to a coaxial line. It should be understood that the distance d between two closed stubs must be other than $\frac{1}{2}\lambda$ or any integer multiple thereof, since such a length behaves like a 1:1 impedance-

TABLE 15

$r = \frac{E_{max}}{E_{min}}$ measured	$r - 1$	$\sqrt{r}$	$\frac{1}{\sqrt{r}}$	For either stub: βd deg $\beta d = \tan^{-1}\frac{1}{\sqrt{r}}$	For either stub: d in wave-lengths $d = \frac{\beta d°\lambda}{360°}$	$\frac{r-1}{\sqrt{r}}$	Open-stub length: $\tan^{-1}\left(\frac{r-1}{\sqrt{r}}\right) = \beta l$ deg	Open-stub length: l in wave-lengths $l = \frac{\beta l°\lambda}{360°}$	Closed-stub length: $\cot^{-1}\left(\frac{r-1}{\sqrt{r}}\right) = \beta l$ deg	Closed-stub length: l in wave-lengths $l = \frac{\beta l°\lambda}{360°}$
1	0	1	1	45	0 125λ	0 0	0	0λ	90	0 25λ
1 5	0 5	1 225	0 815	39 2	0 109λ	0 1075	22 15	0 0615λ	87 85	0 244λ
2	1	1 41	0 707	35 3	0 098λ	0 707	35 3	0 098λ	54 7	0 152λ
3	2	1 73	0 58	30 1	0 083λ	1 16	49 2	0 138λ	40 8	0 113λ
4	3	2	0 5	26 5	0 073λ	1 5	56 4	0 157λ	33 6	0 0935λ
5	4	2 24	0 447	24 1	0 067λ	1 988	63 3	0 176λ	26 7	0 074λ
6	5	2 45	0 408	22 2	0 062λ	2 04	63 9	0 178λ	26 1	0 0725λ
7	6	2 65	0 378	20 7	0 057λ	2 268	66 2	0 181λ	23 8	0 066λ
8	7	2 83	0 354	19 5	0 054λ	2 47	68	0 189λ	22	0 061λ
9	8	3 0	0 333	18 4	0 051λ	2 664	69 4	0 193λ	20 6	0 057λ
10	9	3 16	0 315	17 5	0 0485λ	2 835	70 6	0 196λ	19 4	0 054λ
12	11	3 46	0 289	16 1	0 0446λ	3 179	72 55	0 201λ	17 45	0 0485λ
15	14	3 87	0 259	14 5	0 0403λ	3 62	74 6	0 206λ	15 4	0 0428λ
20	19	4 47	0 223	12 6	0 035λ	4 21	76 7	0 213λ	13 3	0 037λ

Remarks For the closed stub, the distance d is scaled off from a voltage minimum location toward the load end. For the open stub the same distance for same r value is scaled off toward the source end.

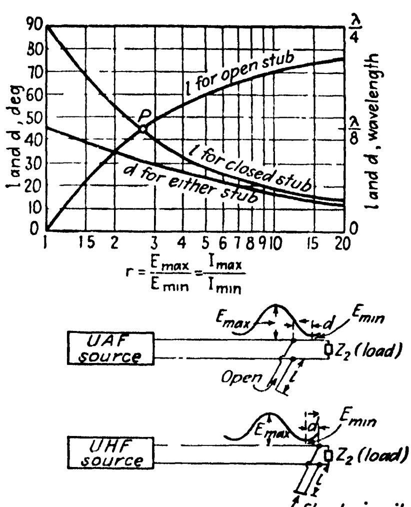


FIG. 75. Curves for the stub length l and the location d of the stub for stubs which are either open-ended or short-circuited. For intersection P corresponding to $r = 2.62$, both stubs have the same length, and for $r > 1$, closed stub becomes shorter. Distances d are read off with respect to E_{min} reference position.

matching transformer with no impedance transformation. Such a transformation is needed for the sake of canceling the susceptance due to a complex load. Moreover, a double stub with any fixed distance d between respective stubs cannot match all conditions of loading but has the decided advantage that the stub positions are fixed, and only variations of the respective closed-stub lengths l_1 and l_2 are needed to bring about an impedance match. This advantage is especially important in the case of coaxial feeders, since the stub positions are not so readily changed. Inasmuch as a fixed spacing d will generally not make possible

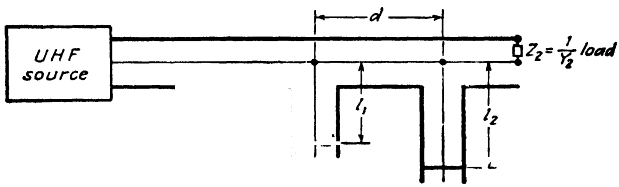


FIG. 76. Double-stub impedance matching.

an impedance match to all Z_2 loads, a spacing d has to be chosen to cover the range that the loading Z_2 may require.

As for a single-stub impedance matching, the procedure is based on the admittance relation

$$Y_1 = Y_0 \frac{Y_2 + jY_0 \tan (\beta l)}{Y_0 + jY_2 \tan (\beta l)} = G_1 - jB_1 \tag{650}$$

where Y_1 mhos denotes the admittance of a line of length l for negligible attenuation and a load admittance of Y_2 mhos, which is generally complex. The load impedance $Z_2 = 1/Y_2$ is at the far end of the line. The quantity Y_0 is the characteristic admittance $1/Z_0$ of the line. For a line with negligible attenuation, $Y_0 \cong G_0$ mhos, since the susceptance B_0 of the characteristic admittance is also negligible. Since $Y_1 = 1/Z_1$,

$$Z_1 = Z_0 \frac{Z_2 + jZ_0 \tan \beta l}{Z_0 + jZ_2 \tan \beta l} = R_1 + jX_1 \tag{651}$$

is the corresponding input impedance relation for any impedance loading $Z_2 = 1/Y_2 = R_2 + jX_2$ ohms. The quantity G_1 denotes the input conductance in mhos, and B_1 the input susceptance in mhos. These are well-known transmission-line relations, and form (650) is useful when stubs of lengths l_1 and l_2 are shunted across a feeder length d as in Fig. 76, since the purpose of the respective stub lengths is to neutralize susceptance effects and of the feeder length d is to bring about appropriate impedance transformation. The ratio Y_1/Y_0 of formula (650) is then numeric and equal to the numeric $(Y_2 + jY_0 \tan \beta l)/(Y_0 + jY_2 \tan \beta l) = (G_1 - jB_1)/Y_0$. For the corresponding numeric ratio Y_2/Y_0 of the load admittance Y_2, we have

$$\frac{Y_2}{Y_0} = \frac{Z_0}{Z_2} = \frac{Z_0}{R_2 + jX_2} = \frac{Z_0(R_2 - jX_2)}{R_2^2 + X_2^2} = \frac{Z_0}{R_2^2 + X_2^2} R_2 - j \frac{Z_0}{R_2^2 + X_2^2} X_2 = a - jb \tag{652}$$

Therefore, we have for the numeric load components

$$a = \frac{Z_0}{R_2^2 + X_2^2} R_2; \qquad b = \frac{Z_0}{R_2^2 + X_2^2} X_2 \tag{653}$$

where a stands for the numeric load conductance and b for the numeric load susceptance per characteristic feeder conductance $Y_0 = 1/Z_0$ mhos. In these relations, the reactance X_2 of the load may be positive (inductive) or negative (capacitive), in the general case.

The electrical stub lengths βl_1 and βl_2, in terms of any electrical spacing βd between the stubs, can then be expressed in terms of (652) by

$$\cot(\beta l_1) = \pm\sqrt{\frac{p}{a}} - \cot(\beta d); \qquad \cot(\beta l_2) = \pm\sqrt{ap} - b - \cot(\beta d);$$
$$p = 1 - a + \cot^2(\beta d) \tag{654}$$

Inasmuch as we deal with square-root terms $\sqrt{p/a}$ and $\sqrt{ap}$ for respective stub lengths βl_1 and βl_2, only positive values under the square root can give useful, *i.e.*, real, values. This necessitates that p be a positive quantity, or $p \geqq 0$, yielding $\cot^2(\beta d) \geqq a - 1$. The largest numeric conductance that can be matched is then given by the limit

$$\cot^2 \beta d = a_{max} - 1 \qquad \text{or} \qquad a_{max} = 1 + \cot^2 \beta d$$

Hence, for a spacing d between stubs equal to $\frac{1}{4}\lambda$, we should have $\beta d = 360\lambda/(4\lambda) = 90$ deg, $\cot \beta d = 0$, and $a_{max} = 1 + \cot^2 \beta d = 1$, which is the largest value to be used in the expression for a in (652). The numeric susceptance b of (651) and (652) needs no consideration, since it can be compensated by means of a suitable length l_2.

From this discussion, it is seen that for a certain spacing d between stubs, the numerical values a and b defining the load conditions can be found from the respective lengths l_1 and l_2, which bring about an impedance match. The numerics a and b are then computed from (654).

Even though the spacing d can be chosen arbitrarily, except equal to $\frac{1}{2}\lambda$ for which no impedance conversion can take place, it is convenient to use a suitable spacing such as $d = \frac{1}{4}\lambda$, for instance. We have then $\beta d = 90$ deg, $\cot \beta d = 0 = \cot^2 \beta d$, and (654) simplifies, owing to $p = 1 - a$, to

$$\cot \beta l_1 = \pm\sqrt{\frac{1-a}{a}}; \qquad \cot \beta l_2 = \pm\sqrt{a(1-a)} - b \tag{655}$$

At a voltage minimum, the impedance looking toward the load Z_2 is Z_0/r ohms; while at a voltage maximum location, it is rZ_0 ohms for $r = E_{max}/E_{min}$. Hence, at a voltage maximum of a feeder, the feeder acts as though loaded at the far end by a numeric conductance $1/r$, *i.e.*, by E_{min}/E_{max}, a ratio that is less than unity. The closed stub next to the load is then located at a voltage maximum, and the spacing d between the stubs may then be any distance d for bringing about an impedance

conversion. This also satisfies (655). By applying (655) for a load that is $Z_2 = 74 - j42$ ohms and a 74-ohm line and 74-ohm stubs, we find for Z_0, according to (652), $R_2 = 74$ ohm, $X_2 = -42$ ohm, and

$$a = \frac{74}{74^2 + 42^2} 74 = 1.02 \times 10^{-2} \times 74 = 0.758; \qquad b = 1.02 \times 10^{-2}(-42) = -0.43$$

Using these numerics in (655) yields

$$\beta l_1 = \cot^{-1} \sqrt{\frac{1-a}{a}} = \cot^{-1} \sqrt{\frac{0.212}{0.758}} = \cot^{-1} 0.565 = 60.5^\circ$$

$$\beta l_2 = \cot^{-1}[\sqrt{a(1-a)} - b] = \cot^{-1} [\underbrace{\sqrt{0.242 \times 0.758}}_{0.428} + 0.43]$$

$$= \cot^{-1} 0.858 = 49.4^\circ$$

For a 100-Mc current, we have a wavelength $\lambda \cong {}^{300}/_{100} = 3$ m, $\beta = 360$ deg/λ = 120 deg/m. The respective stub lengths are then $l_1 \cong 60.5$ deg/β = 60.5/120 = 0.505 m, or 50.5 cm, and $l_2 \cong 49.4/120 = 0.41$ m, or 41 cm. Inasmuch as formulas (655) hold for a spacing $\beta d = 90$ deg, the spacing d is $\frac{1}{4}\lambda$ long or $d = \frac{3}{4} = 0.75$ m, or 75 cm. For a half-wave dipole as the load at the end of the feeder, the effective load would be the conjugate value, *i.e.*, $Z_2 = 74 + j42$, rendering $a = 0.758$ as above, but $b = +0.43$. Hence βl_1 is again equal to 60.5 deg but $\beta l_2 = \cot^{-1}$ (0.428 − 0.43) $\cong \cot^{-1}$ (0), calling for a stub length closest to the load end of $\beta l_2 = 90$ electrical degrees or one-quarter of a wavelength long.

82. Remarks on Successive Stubbing toward an Improved Impedance Match. This relates, in effect, also to a multiple-stub procedure that can be reduced to a double-stub arrangement as described in the preceding section. There is, however, a difference, in that this method uses the single-stub procedure and in repetition. Therefore, the procedure consists in locating E_{max} and E_{min} positions, giving also the standing-wave ratio $r = E_{max}/E_{min}$ and finding the distance d and length l in electrical degrees or in terms of the wavelength from the curves of Fig. 75. For a closed stub, d then denotes the distance from the voltage minimum next to the load and in direction toward the load. The length l is the length of the closed stub. After making such d and l adjustments, the standing-wave ratio is again found experimentally by means of the voltage probe and may be still appreciably more than unity. If so, we obtain a new ratio $r' = E'_{max}/E'_{min}$, which is better than the original r value. The curves of Fig. 75 yield for r' the new distance d' from the new voltage minimum E'_{min} and a new length l' for a second stub. With this additional stub in place, the impedance match should prove satisfactory.

83. Impedance Match with a Quarter-wavelength Transformer. An impedance transformer is based on the relation $Z_0 = \sqrt{Z_1 Z_2}$, where Z_1

and Z_2 are the input and output terminations (impedances) and Z_0 is the characteristic impedance of the impedance transformer. Such a transformer may be a lumped network with a series reactance jX and a shunt reactance $-jX$ or vice versa. For UHF excitations, a quarter-wave line of characteristic impedance Z_0 is a more convenient expedient. It is then necessary only to find the standing-wave ratio $r = E_{max}/E_{min}$. At the E_{max} location, we have a pure resistance rZ_0; and at the E_{min} location, we have a pure resistance Z_0/r ohms, respectively. A matching of the loaded line is possible by inserting a $\frac{1}{4}\lambda$ line section of characteristic impedance $Z_0\sqrt{r}$ ohms between the E_{max} location and the load end of the loaded line. When the $\frac{1}{4}\lambda$ section is inserted between the E_{min} location and the remainder of the line, the inserted section requires a characteristic impedance of $Z_0/\sqrt{r}$ ohms. In each case, Z_0 ohms denotes the characteristic impedance of the feeder line to be matched against its load Z_2 at the far end. A similar procedure is used in connection with formulas (655) in Sec. 81, since the double stubs are also spaced 90 electrical degrees apart. The more general solution is given by formulas (654) of Sec. 81, which, for a spacing of $d = \lambda/8$, give an electrical length of 45 deg, rendering $\cot \beta d = \cot^2 \beta d = 1$ and simplifying (654) to

$$\cot (\beta l_1) = \pm \sqrt{\frac{p}{a}} - 1; \qquad \cot (\beta l_2) = \pm \sqrt{ap} - b - 1;$$
$$p = 2 - a \qquad (656)$$

showing that the numerical conductance a, looking toward the load Z_2 (Fig. 76) must be equal to or smaller than 2; otherwise the square-root terms cannot yield real values. With respect to Fig. 76 and formulas (655), we also note another difference from the method described in Sec. 81, since, as in Fig. 76, the quarter wavelength $d = \frac{1}{4}\lambda$, which holds according to (655) for a section with the same Z_0 value that the feeder has. In the procedure described in this section, the $\lambda/4$ section that is inserted has a characteristic impedance either $Z_0\sqrt{r}$ or $Z_0/\sqrt{r}$, depending upon where the section is inserted.

84. Notes on the Theory Basic to Stub Formulas and Curves. When connecting a complex load Z_2 at the far end of a line, we obtain distributions for the voltage E and the current I, as indicated in Fig. 77. At any distance x from the load end, the impedance is $Z_x = E_x/I_x$ ohms and generally complex, except at locations for which E_{max}/I_{min} and E_{min}/I_{max} are the ratios. The reflection coefficient at the load Z_2 has a magnitude ρ and a phase φ, which may be a positive or a negative time angle. We have then

$$\rho e^{j\varphi} = \frac{Z_2 - Z_0}{Z_2 + Z_0} = A + jB; \qquad \varphi = \tan^{-1}\left(\frac{A}{B}\right) \qquad (657)$$

where B may be taken positive or negative. We have also the standing-wave ratio

$$r = \frac{E_{max}}{E_{min}} = \frac{I_{max}}{I_{min}} = \frac{1+\rho}{1-\rho} \tag{658}$$

yielding the magnitude of the reflection coefficient as

$$\rho = \frac{r-1}{r+1} \tag{659}$$

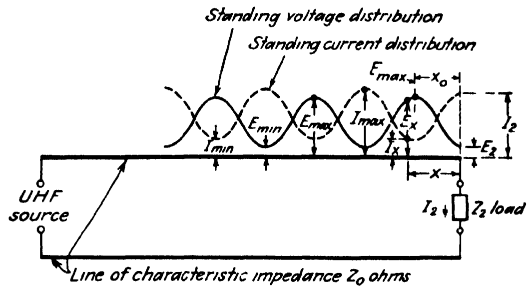


FIG. 77. Standing potential and current distributions along a loaded double line. $I_{max} = (1+\rho)I$, $I_{min} = (1-\rho)I$, $\rho\epsilon^{j\varphi}$ is reflection factor of magnitude ρ and phase φ. (φ may be positive or negative.)

The phase φ can also be computed from the distance x_0 from the load to the first E_{max} or I_{min} location by means of

$$\varphi = 2\beta x_0 = 720\frac{x_0}{\lambda}\ \text{deg} \tag{660}$$

where λ is the wavelength determining the phase distribution along the line.

It is now of interest to match impedances. If we had a quarter-wavelength section of the characteristic impedance Z_0 as in Fig 78, the characteristic impedance of the impedance-matching line is the geometric mean of the two terminations; i e, $Z_0 = \sqrt{Z_1 Z_2}$. Suppose Z_2 represents a coil of ohmic resistance R_2 and reactance $+jX_2$ ohms. Then Z_1 must be such that

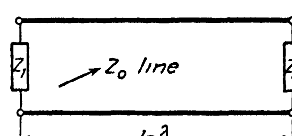


FIG. 78 Quarter-wave line section between two impedances Z_1 and Z $Z_2 = R + jX_2$, $Z_0 = \sqrt{Z_1 Z_2}$ X_2 may be negative or positive.

$$Z_1 = \frac{Z_0^2}{Z_2} = \frac{Z_0^2}{R_2 + jX_2} = \frac{Z_0^2(R_2 - jX_2)}{R_2^2 + X_2^2} = R_1 - jX_1;$$

$$R_1 = \frac{R_2}{R_2^2 + X_2^2} Z_0^2; \qquad X_1 = \frac{X_2}{R_2^2 + X_2^2} Z_0^2 \tag{661}$$

and generally for $Z_1 = R_1 \pm jX_1$, we require a loading $Z_2 = R_2 \mp jX_2$, since an *inductive* source of internal impedance $Z_1 = R_1 + jX_1$ requires a *capacitive* impedance $Z_2 = R_2 - jX_2$ and vice versa for a quarter-wave matching line. From (661), we note that if $Z_0 = \sqrt{L/C} = \sqrt{R_2^2 + X_2^2}$, *i.e.*, the characteristic line impedance Z_0 is equal to the absolute value of the load, the quarter-wave section will transform an impedance $Z_1 = R_1 \pm jX_1$ into an impedance $Z_2 = R_1 \mp jX_1$, which is conjugate. For the general case expressed by (661), a load impedance $Z_2 = R_2 \pm jX_2$ is likewise transformed, but the conjugate $R_2 \mp jX_2$ must be multiplied by the ratio $Z_0^2/(R_2^2 + X_2^2)$ to yield $Z_1 = R_1 \mp jX_1$.

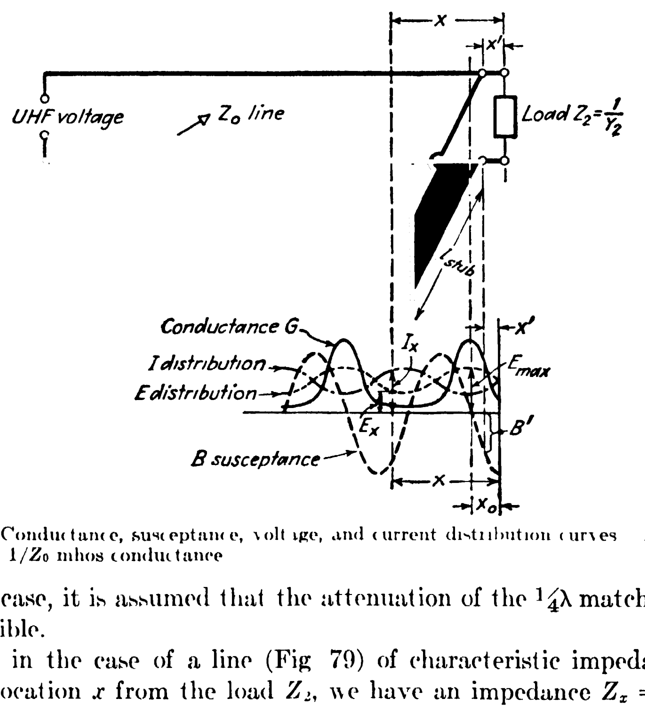


FIG. 79. Conductance, susceptance, voltage, and current distribution curves. At X' we have $G' = 1/Z_0$ mhos conductance.

In each case, it is assumed that the attenuation of the $\frac{1}{4}\lambda$ matching line is negligible.

Now in the case of a line (Fig. 79) of characteristic impedance Z_0 at any location x from the load Z_2, we have an impedance $Z_x = E_x/I_x = R_x + jX_x$, where X_x may again assume positive and negative values. The corresponding admittance Y_x at such a location is

$$Y_x = \frac{1}{Z_x} = \frac{1}{R_x + jX_x} = \frac{R_x}{R_x^2 + X_x^2} - j\frac{X_x}{R_x^2 + X_x^2} = G_x - jB_x \text{ mhos} \quad (662)$$

The admittance Y_x is then composed of the conductance G_x and the susceptance B_x. The susceptance may be negative or positive as the reactance X_x is positive or negative, while, of course, the conductance G_x oscillates only between positive maximum and positive minimum values

as we move along the line. At points for which the conductance G_x is equal to the reciprocal of the characteristic line impedance Z_0 ohms are places where a matching stub should be located. A distance x' from the load Z_2 is such a location, since $G' = 1/Z_0$ mhos. Inasmuch as the stub is in shunt with the admittance $Y_{x'}$ looking toward the load and this admittance is $G' - jB'$, where B' may be positive or negative, we must choose the stub length l such that the susceptance $1/X_l$ of the stub just compensates the susceptance B' due to the feeder length x' and the effect of the load Z_2. The quantity X_l represents the reactance offered by the stub, with negligible attenuation. Hence, for an admittance $G' - jB'$, we must have $(j/X_l) - jB' = 0$ or $1/X_l$ an equal positive susceptance. For $G' + jB'$, the stub must act like an equal negative susceptance.

According to transmission-line relations (Sec. 75), the current I_x and the voltage E_x at any location x from a load Z_2 are, for a reflection coefficient $\rho\epsilon^{j\varphi}$ and $\theta = \varphi - 2\beta x$,

$$I_x = \frac{I_2}{Z_0}\frac{Z_2 + Z_0}{2}\epsilon^{j\beta x}(1 - \rho\epsilon^{j\theta}); \qquad E_x = \frac{E_2}{Z_2}\frac{Z_2 + Z_0}{2}\epsilon^{j\beta x}(1 + \rho\epsilon^{j\theta}) \tag{663}$$

causing the admittance Y_x looking toward the load Z_2

$$Y_x = \frac{I_x}{E_x} = \frac{1}{Z_0}\frac{1 - \rho\epsilon^{j\theta}}{1 + \rho\epsilon^{j\theta}} \tag{664}$$

But

$$\frac{1 - \rho\epsilon^{j\theta}}{1 + \rho\epsilon^{j\theta}} = \frac{1 - \rho(\cos\theta + j\sin\theta)}{1 + \rho(\cos\theta + j\sin\theta)} = \frac{\overbrace{1 - \rho\cos\theta}^{u} - j\rho\sin\theta}{\underbrace{1 + \rho\cos\theta}_{v} + j\rho\sin\theta}$$

$$= \frac{(u - j\rho\sin\theta)(v - j\rho\sin\theta)}{v^2 + \rho^2\sin^2\theta} = \frac{uv - \rho^2\sin^2\theta - j\rho(u + v)\sin\theta}{v^2 + \rho^2\sin^2\theta}$$

Hence

$$Y_x = \frac{1}{Z_0}\left[\frac{uv - \rho^2\sin^2\theta}{v^2 + \rho^2\sin^2\theta} - j\frac{\rho(u + v)\sin\theta}{v^2 + \rho^2\sin^2\theta}\right] = G_x - jB_x$$

corresponding to the expression (662). The conductance component looking toward the load at a distance x from it is then the real part

$$G_x = \frac{1}{Z_0}\frac{uv - \rho^2\sin^2\theta}{v^2 + \rho^2\sin^2\theta} = \frac{1}{Z_0}\frac{1 - \rho^2\cos^2\theta - \rho^2\sin^2\theta}{1 + 2\rho\cos\theta + \rho^2\cos^2\theta + \rho^2\sin^2\theta}$$

$$= \frac{1}{Z_0}\frac{1 - \rho^2}{1 + \rho^2 + 2\rho\cos\theta} \tag{665}$$

The susceptance component yields

$$B_x = -\frac{1}{Z_0}\frac{\rho(u + v)\sin\theta}{v^2 + \rho^2\sin^2\theta} = -\frac{1}{Z_0}\frac{2\rho\sin\theta}{1 + \rho^2 + 2\rho\cos\theta} \quad \text{mhos} \tag{666}$$

Now if x is a distance x_0 from the load for which $G_x = G' = 1/Z_0$ mhos, we have the correct location for a matching stub. This happens when in (665)

$$1 - \rho^2 = 1 + \rho^2 + 2\rho \cos \theta \tag{667}$$

or $\cos \theta = -\rho$ and occurs when

$$\cos [\theta + (2p + 1)180°] = -\rho \tag{668}$$

if p is an integer. Now $\theta = \varphi - 2\beta x$; and from (663), we find the value of the current

$$I_x = \frac{I_2}{Z_0}\frac{Z_2 + Z_0}{2}\sqrt{1 + \rho^2 - 2\rho \cos \theta} \tag{669}$$

Since $1 - \rho\epsilon^{j\theta} = 1 - \rho \cos \theta - j\rho \sin \theta$, we find $\sqrt{(1 - \rho \cos \theta)^2 + \rho^2 \sin^2 \theta} = \sqrt{1 + \rho^2 - 2\rho \cos \theta}$. In Fig. 79, the distance to a voltage maximum is $x = x_0$ and is the same distance from the load to a current minimum. But $I_x = I_{min}$ when the square-root value of (669) becomes a minimum. This happens for $\cos \theta = 1$ or $\theta = 0$. Hence, $\theta = \varphi - 2\beta x_0 = 0$, or

$$\varphi = 2\beta x_0 = 720 \frac{x_0}{\lambda} \qquad \text{deg} \tag{670}$$

since $\beta = 360/\lambda$ space degrees per unit length is the phase constant of a uniform line. We can now use this result in (668) and find

$$-\rho = \cos [\varphi - 2\beta x + (2p + 1)180°] = \cos \left[720° \frac{x_0}{\lambda} - 720° \frac{x}{\lambda} + (2p + 1)180°\right]$$

and

$$\frac{x - x_0}{\lambda} 720° - (2p + 1)180° = \pm\cos^{-1} \rho \tag{671}$$

Introducing the standing-wave ratio $r = E_{max}/E_{min}$, which is found experimentally and is also equal to I_{max}/I_{min}, we have $\rho = (r - 1)/(r + 1)$ and

$$\frac{d}{\lambda} = \frac{x - x_0}{\lambda} = 0.25(2p + 1) \pm \frac{\left[\cos^{-1}\left(\frac{r - 1}{r + 1}\right)\right]^\circ}{720°} \tag{672}$$

where $d = x - x_0$ fixes the location of the stub, since it is the distance from a voltage maximum (or a current minimum). The ratio $(x - x_0)/\lambda$ is this distance per wavelength along the line. For a short-circuited stub, $x - x_0$ is then measured from the E_{max} location toward the source; and for an open-ended stub, it is measured toward the load. For practical reasons, the stub is located close to the load Z_2 to be matched; *i.e.*, d is measured from the voltage maximum next to the load. An application of (672) is then for the case $p = 0$ as in Table 16. These results check the outcome of Table 15, which are computations with expressions derived at the end of this section. For the case $p = 1$, in (672) we have the value $\frac{1}{4}(2p + 1) = 0.75$ and 0.75 is to be used instead of 0.25 in the d/λ column of Table 16. For $p = 2$, we find the value $\frac{1}{4}(2 \times 2 + 1) = 1.25$, etc. This means that $A\lambda$ is the distance from any voltage minimum along the feeder.

TABLE 16

Standing-wave ratio $r = \frac{E_{max}}{E_{min}}$	Magnitude of reflection coefficient $\rho = \frac{r-1}{r+1}$	$\cos^{-1}\rho$ deg	$\frac{\cos^{-1}\rho}{720°} = A$	$\frac{d}{\lambda} = \frac{1}{4} - A$
1	0	90	0 125	0.125
2	0.333	70 55	0 098	0 152
2.5	0.427	64.7	0 0898	0.1602
5	0.667	48 2	0 067	0.183

Remarks: The fraction d/λ is the distance from the E_{max} location Thus for unity standing-wave ratio (flat distributions along the line), the stub has to be 0.125λ from the E_{max} location since this makes $E/I = Z_0$. Moreover, A would be the position from the voltage minimum.

Formula (666) gives a means of finding the length of the stub, since it is the expression for the susceptance B_x to be neutralized by an appropriate stub length l. Formula (666) can be greatly simplified, since at the required stub location, we find, from (667), the relation $\cos\theta = -\rho$. We have also $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1-\rho^2}$, and (666) degenerates to

$$B_x = -\frac{1}{Z_0}\frac{2\rho\sqrt{1-\rho^2}}{1+\rho^2+2\rho(-\rho)} = -\frac{2\rho}{Z_0}\frac{\sqrt{1-\rho^2}}{1-\rho^2} = \mp\frac{2\rho}{Z_0\sqrt{1-\rho^2}}.$$

The stub must then have a length such that its susceptance $B_l = -B_x$ in order to cancel the B_x effect at the stub location. Hence,

$$B_l = \pm\frac{2\rho}{Z_0\sqrt{1-\rho^2}} \qquad \text{mhos} \tag{673}$$

Open and short-circuited stubs, as short as about ¼ λ, act practically as pure cotangent and tangent functions, *i.e.*, as pure reactances

$$\left.\begin{aligned} X_l &= -Z_0 \cot(\beta l) && \text{for open-ended stub} \\ X_l &= +Z_0 \tan(\beta l) && \text{for short-circuited stub} \end{aligned}\right\} \tag{674}$$

When looking into the stub and $X_l = -1/B_l$

$$X_l = \mp\frac{Z_0\sqrt{1-\rho^2}}{2\rho} = \mp\frac{Z_0\sqrt{1-[(r-1)/(r+1)]^2}}{2(r-1)/(r+1)} = \mp Z_0\frac{\sqrt{r}}{r-1} \quad \text{ohms} \tag{675}$$

Looking into an open-ended stub of a physical length l smaller than ¼λ or $\beta l < 90$ deg, according to (674), makes the input impedance equal to a capacitive reactance $-jX_l$ and equal to an inductive reactance $+jX_l$ for l between ¼λ and ½λ. This holds for negligible attenuation. For a short-circuited stub, the input impedance is an inductive reactance $+jX_l$ for $l <$ ¼λ and a capacitive reactance $-jX_l$ for l between ¼λ and ½λ. Comparing (674) and (675), we note that cot βl and tan βl functions yield $\mp\sqrt{r}/(r-1)$. Inasmuch as the stub and line susceptances have to be equal and opposite at the stub-feeder junction, the $\pm(r-1)/(Z_0\sqrt{r})$ function

is instrumental in the determination of the correct stub length l. This is brought out in detail in the computation of Table 15.

The formulas for the stub length of Table 15 and its proper location can also be derived by means of the conjugate-impedance or the conjugate-admittance concept. This concept is based on the fact that a source of internal impedance $Z_1 = R_1 \pm jX_1$ transfers maximum power to a load impedance $Z_2 = R_1 \mp jX_1$, which is conjugate to Z_1, so that the reactances cancel out (neutralize each other's effect). Hence, if any line section with negligible attenuation is inserted between the source of internal impedance Z_1 and load impedance Z_2, at the generator terminals we must also have conjugate impedances looking toward the line and back into the generator. Also at the load terminals, conjugate impedances exist in opposite directions to the terminals in case of maximum power transfer from the source to the load Z_2.

With stubs, we deal with admittances, since the admittance looking into the stub is in *shunt* with the admittance at the stub joint looking into the feeder. Equation (650) is the expression for the input admittance for any stub loaded at the far end by an impedance of $1/Y_2$ ohms. Therefore, we can find the conditions for an open-ended stub ($Y_2 = 0$ mhos) and a short-circuited stub ($Y_2 = \infty$). For a characteristic stub admittance $Y_0 = 1/Z_0$ mhos, from (650), we find the input admittance

$$\left.\begin{aligned} Y &= +jY_0 \tan(\beta l) \qquad \text{for open-ended stub} \\ Y &= -jY_0 \cot(\beta l) \qquad \text{for short-circuited stub} \end{aligned}\right\} \tag{676}$$

Since in each case $Y = G - jB$ mhos with $G = 0$ conductivity for negligible attenuation, the susceptance B, when looking into such stubs, becomes

$$\left.\begin{aligned} B &= -Y_0 \tan(\beta l) \qquad \text{mhos} \qquad \text{for open-ended stub} \\ B &= +Y_0 \cot(\beta l) \qquad \text{mhos} \qquad \text{for short-circuited stub} \end{aligned}\right\} \tag{677}$$

Suppose we have as, in Fig. 79, a stub connected at the correct location x' from the load $Z_2 = 1/Y_2$ of a feeder. Suppose also that the stub length is likewise correct so as to inject an equal but opposite susceptance as would otherwise exist across the feeder joint of the stub in addition to the conductance $Y_0 = 1/Z_0$; then the load admittance $Y_2 = G_2 - jB_2$ is converted into Y_0 mhos at the distance x' from Y_2. According to (657) and (653), $Y_2/Y_0 = a - jb$ denotes the numerical load admittance. This means that $(Y_2/Y_0) = (G_2/Y_0) - j(B_2/Y_0)$. The numerical load admittance $a - jb$ is changed to a numerical admittance $1 + j0$ at the distance x' from the load.[1] Equation (650) will then give the numerical values

$$\frac{Y}{Y_0} = \frac{(Y_2/Y_0) + j\tan(\beta l)}{1 + j(Y_2/Y_0)\tan(\beta l)} \tag{678}$$

[1] For details, see A. H. Wing and J. Eisenstein, *Jour. Appl. Phys.*, **15**, 615, August, 1944.

The numerical susceptance of stubs is found from (677) also, by dividing by the characteristic admittance Y_0 of the stub, and we have

$$\left.\begin{aligned} \frac{B}{Y_0} &= -\tan(\beta l) \qquad \text{for open-ended stub} \\ \frac{B}{Y_0} &= +\cot(\beta l) \qquad \text{for short-circuited stub} \end{aligned}\right\} \tag{679}$$

We can now apply (678) and (679) to the terminals, where the stub joins the feeder line at x' from the load (Fig. 79). The admittance looking into this joint of the stub and the feeder toward the source must be equal to and conjugate to the admittance looking toward the load end. At x', the feeder is acting with Y_0 mhos and a numerical admittance $1 + j0$. For a closed stub, $1 + j0$ is to be added to $-j \cot \beta l$, according to (679), in order to bring about susceptance cancellation. The numeric ratio Y_2/Y_0 of (678) then yields $(1 + j0 - j \cot \beta l) = 1 - j \cot \beta l$ for a stub length l, while the distance l in $\tan \beta l$ refers to the distance x' from the load $1/Y_2$ ohms to the stub location. Hence (678) yields

$$\frac{Y}{Y_0} = \frac{[1 - j\cot(\beta l)] + j\tan(\beta x')}{1 + j[1 - j\cot(\beta l)]\tan(\beta x')}$$

a result that must be equal to the conjugate of the numerical load admittance $Y_2/Y_0 = a - jb$, or

$$\frac{[1 - j\cot(\beta l)] + j\tan(\beta x')}{1 + j[1 - j\cot(\beta l)]\tan(\beta x')} = a + jb \tag{680}$$

In order to find the desired formulas for determining the electrical lengths βl and $\beta x'$ for stub length and stub location with respect to the load location, we must solve for $\cot(\beta l)$ and $\tan(\beta x')$, which is readily done by separating real and imaginary terms in (680).

Putting $\tan(\beta x') = A$ and $\cot(\beta l) = B'$, we find for the left side of (680)

$$\frac{1 + j(A - B')}{1 + AB' + jA} = \frac{1 + j(A - B')}{C + jA} \quad \text{for } C = 1 + AB' \quad \text{and} \quad \frac{1 + j(A - B')}{C + jA} = a + jb$$

yields

$$1 + j(A - B') = aC - bA + j(aA + bC)$$

Separating real and imaginary terms gives

$$1 = aC - bA \qquad \text{and} \qquad A - B' = aA + bC$$

But $C = 1 + AB'$, and $1 = a(1 + AB') - bA$, $A - B' = aA + b(1 + AB')$. Solving each relation for A and equating,

$$\frac{1 - a}{aB' - b} = A = \frac{B' + b}{1 - a - bB'} \tag{681}$$

By cross multiplication, we find

$$\left.\begin{aligned} &1 - 2a + a^2 + B'(ab - b) = a(B')^2 + B'(ab - b) - b^2 \quad \text{or} \\ &\qquad a(B')^2 = (1 - a)^2 + b^2 \\ &\text{and} \\ &B' = \pm\sqrt{\frac{(1 - a)^2 + b^2}{a}} = \cot(\beta l) \end{aligned}\right\} \tag{682}$$

Inserting the value of B' in any of the two solutions of (681) yields

$$A = \frac{1 - a}{a \cot(\beta l) - b} = \tan(\beta x') \tag{683}$$

Formulas (682) and (683) are then basic for the simple expressions given in Fig. 73, since they give a means of finding the electrical length βl of a short-circuited stub and its electrical distance $\beta x'$ from the load end. The numerical values of a and b are given in (653) and represent the numeric load components of the numerical load admittance Y_2/Y_0, as given in (652).

The formulas given in Fig. 73 are based on the experimental determination of the standing-wave ratio $r = E_{max}/E_{min}$ and the position of a voltage maximum location, preferably the location closest to the load. At any voltage maximum, the line is, in effect, terminated into an ohmic resistance rZ_0 ohms or a conductance Y_0/r mhos. Hence, for such a location, $a = 1/r$ and $b = 0$. Then if we use a distance d measured from a voltage maximum instead of x' measured from the load, (682) and (683) simplify to

$$\cot(\beta l) = \pm\sqrt{\frac{\left(1 - \frac{1}{r}\right)^2 + 0}{\frac{1}{r}}} = \pm\frac{r - 1}{\sqrt{r}};$$

$$\tan(\beta d) = \frac{1 - \frac{1}{r}}{\frac{\cot(\beta l)}{r} - 0} = \pm\sqrt{r} \tag{684}$$

If the stub distance d is taken from a voltage minimum, we have $a = r = E_{max}/E_{min}$ and $b = 0$, since the feeder acts at such places like a conductance of rY_0 mhos. Using these substitutions in (682) and (683) leads to

$$\cot(\beta l) = \pm(r - 1)/\sqrt{r} \qquad \text{and} \qquad \tan(\beta d) = \mp\frac{1}{\sqrt{r}}$$

For the open-ended stub, according to (679), we must use $-\tan(\beta l)$ instead of $\cot(\beta l)$ in (380); and instead of (682) and (683), we find

$$\tan(\beta l) = \sqrt{\frac{(a-1)^2+b^2}{a}}; \qquad \tan(\beta x') = \frac{a-1}{a\tan(\beta l)+b} \tag{685}$$

which can again be used in connection with formulas (653) for a and b. We can also simplify matters by using the experimental standing-wave ratio $r = E_{max}/E_{min}$ and find, for the stub length l and stub distance d from any E_{max} location,

$$\tan(\beta l) = \pm\frac{r-1}{\sqrt{r}}; \qquad \tan(\beta d) = \mp\sqrt{r} \tag{686}$$

and the distance d from any E_{min} location

$$\tan(\beta d) = \pm\frac{1}{\sqrt{r}} \tag{687}$$

and βl as in (686).

With respect to double-stub matching, as in Fig. 76, with a stub spacing d for closed stubs, we simply add the numerical susceptance $\cot(\beta l_2)$ of the second stub to the susceptance component of the numerical admittance, and (680) changes to

$$\frac{[1-j\cot(\beta l_1)]+j\tan(\beta d)}{1+j[1-\cot(\beta l)_1]\tan\beta d} = a + j[b+\cot(\beta l_2)]$$

This relation separates again into real and imaginary terms. For respective stub lengths l_1 and l_2, we find

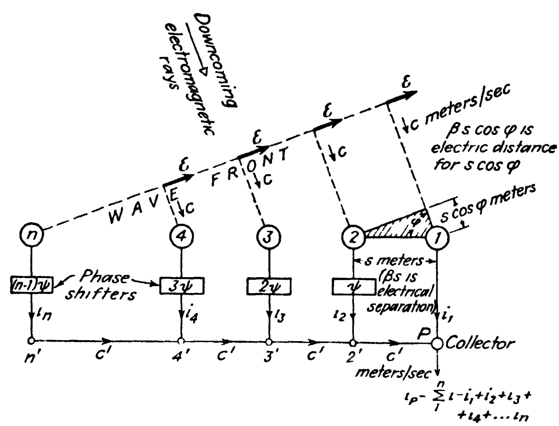
$$\cot(\beta l_1) = \pm\sqrt{\frac{\cot^2(\beta d)-a+1}{a}} - \cot(\beta d);$$
$$\cot(\beta l_2) = \pm\sqrt{a[\cot^2(\beta d)-a+1]} - \cot(\beta d) - b \tag{688}$$

These are the formulas given in (654). It will be noted that in these expressions, we also have a variable $\cot\beta d$. This quantity is generally suitably chosen, such as $\beta d = 90$ deg, making $\cot^2\beta d = 0$ or some other suitable electrical spacing. This is brought out in detail in Sec. 81.

85. Resultant Current of Receiving Antenna Arrays. With respect to Fig. 80, we have n like collector antennas, marked 1, 2, 3, 4, . . . , n. Inasmuch as the wave front is a place of equal field phases we have a phase difference

$$\left.\begin{aligned} \beta s\cos\varphi &= 6.28\frac{s}{\lambda}\cos\varphi \quad \text{radians} \\ \beta s\cos\varphi &= 360\frac{s}{\lambda}\cos\varphi \quad \text{deg} \end{aligned}\right\} \tag{689}$$

for an angle φ of an arriving plane electromagnetic wave with the array axis and a spacing of s meters (or s/λ numeric, if expressed in the free-space wavelength λ). Using i_1 as the reference instantaneous current $I\epsilon^{j\omega t}$ at collector point P, the current i_2 coming from the pickup antenna 2 has a relative phase due to the phase shifter ψ and to the separation between the pickup antennas 1 and 2. According to Fig. 80, this separation causes a relative phase shift that is partially due to an amount $\beta s \cos \varphi$, since the same wave front arrives at the pickup antenna 2 sooner than at the pickup antenna 1, because of the electric path differ-



$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{c} \text{ in free space and essentially so in air}$$

$$\beta' = \frac{2\pi}{\lambda'} = \frac{\omega}{c'}, \text{ along a feeder line}$$

FIG. 80. Linear end-on array.

ence $\beta s \cos \varphi$. In addition, we have a phase difference, since i_2 must flow over the additional path from $2'$ to P in order to reach the collector point P. The product $\beta' s$ is equal to this phase difference, because the phase constant $\beta' = \omega/c'$ of the line has to be used and not the phase constant $\beta = \omega/c$ holding essentially for air. If ψ stands for a positive phase shift, then $\beta' s$ is a negative phase shift, since it is a delay. The current i_2 arrives at the collector point in phase with the current i_1 when the injected phase advance is

$$\psi = \beta' s - \beta s = 6.28 f s \left(\frac{1}{c'} - \frac{\cos \varphi}{c} \right) \text{ radians}$$

$$= 360 f s \left(\frac{1}{c'} - \frac{\cos \varphi}{c} \right) \text{ deg} \quad (690)$$

This relation applies also to the phase shifters 2ψ, 3ψ, $\cdots$, $(n-1)\psi$ if all currents i_1, i_2, i_3, i_4, . . . , i_n arrive in phase at the collector point P. For any desired positive phase shift ψ in radians, we have the resultant collector current

$$\left.\begin{aligned} i_P &= \underbrace{I\epsilon^{j\omega t}}_{i_1} + \underbrace{I\epsilon^{j(\omega t+\delta)}}_{i_2} + \underbrace{I\epsilon^{j(\omega t+2\delta)}}_{i_3} + \underbrace{I\epsilon^{j(\omega t+3\delta)}}_{i_4} + \cdots \\ &\qquad + \underbrace{I\epsilon^{j[\omega t+(n-1)\delta]}}_{i_n} \\ \delta &= \psi - 6.28fs\left(\frac{1}{c'} - \frac{\cos\varphi}{c}\right) \quad \text{radians} \end{aligned}\right\} \tag{691}$$

where i_P is in amperes, f in cycles per second, ψ and δ in radians, s in meters, c and c' in meters per second, and $c = 3 \times 10^8$ m/sec The phase velocity c' along the paths, $2'P$, $3'P$, $4'P$, . . . $n'P$ is ω/β', where β' is computed from the β expression in (603). Equation (691) can also be transformed as follows:

$$i_P^{(\text{amp})} = I\epsilon^{j\omega t}[1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \cdots \epsilon^{j(n-1)\delta}] = I\epsilon^{j\omega t}\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}\epsilon^{j\frac{(n-1)\delta}{2}}$$

$$= \underbrace{I\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}}_{\substack{\text{direction factor}\\ D}}\epsilon^{j\left[\omega t + \frac{(n-1)\delta}{2}\right]} \tag{691a}$$

The detailed solution is as follows The series $1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \cdots$ represents a geometric progression $1 + x + x^2 + x^3 + \cdots + x^{n-1}$, which is equal to $(x^n - 1)/(x - 1)$ for $x = \epsilon^{j\delta}$. Hence,

$$1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \cdots = \frac{\epsilon^{jn\delta} - 1}{\epsilon^{j\delta} - 1}$$

The numerator yields

$$\epsilon^{jn\delta} - 1 = \cos(n\delta) + j\sin(n\delta) - 1 = \underbrace{\cos(n\delta) - 1}_{\text{real}} + j\sin(n\delta)$$

$$= \sqrt{[\cos(n\delta) - 1]^2 + \sin^2(n\delta)}\ \Big/ \tan^{-1}\frac{\sin(n\delta)}{\cos(n\delta) - 1} = \Phi_n$$

$$\epsilon^{jn\delta} - 1 = \sqrt{\cos^2(n\delta) - 2\cos(n\delta) + 1 + \sin^2(n\delta)}\ \underline{/\Phi_n} = 2\sqrt{\tfrac{1}{2}[1 - \cos(n\delta)]}\ \underline{/\Phi_n}$$

$$= 2\sin\left(\frac{n\delta}{2}\right)\underline{/\Phi_n}$$

The denominator gives, in the same way, $\epsilon^{j\delta} - 1 = 2 \sin (\delta/2)\underline{/\Phi}$, and

$$\frac{\epsilon^{jn\delta} - 1}{\epsilon^{j\delta} - 1} = \frac{\sin (n\delta/2)}{\sin (\delta/2)} \epsilon^{j(\Phi_n - \Phi)}$$

confirming (691a). Equation (691a) shows that maxima and zero effects can be obtained by properly choosing the parameters, determining δ as given (691). This is brought out in detail in connection with directivity patterns.

86. Remarks on Phasing Feeders for Transmitting Antennas. Figure 81a shows the principle of phasing by means of different electrical lengths

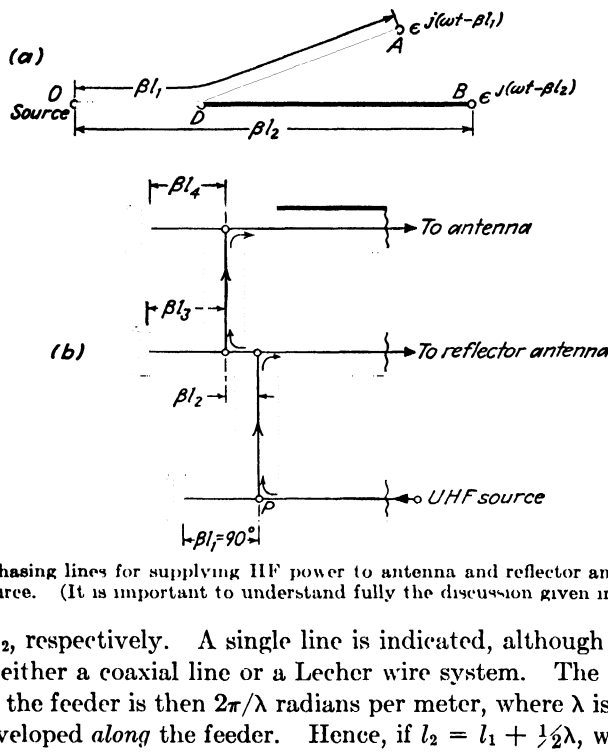


FIG. 81. Phasing lines for supplying HF power to antenna and reflector antenna from a common source. (It is important to understand fully the discussion given in Fig. 168.)

βl_1 and βl_2, respectively. A single line is indicated, although the actual case uses either a coaxial line or a Lecher wire system. The phase constant β of the feeder is then $2\pi/\lambda$ radians per meter, where λ is the wavelength developed *along* the feeder. Hence, if $l_2 = l_1 + \frac{1}{2}\lambda$, we have the difference in electric lengths

$$\beta l_2 - \beta l_1 = \beta \left(l_1 + \frac{1}{2}\lambda - l_1 \right) = \beta \frac{\lambda}{2} = \frac{2\pi}{\lambda} \frac{\lambda}{2} = \pi \qquad \text{radians}$$

and the current arriving at point B is 180 time degrees delayed with respect to the current arriving at point A. This means that the current at B must be in antiphase with the current at A. At branch point D, a suitable joint box is usually provided in order to meet the terminations

toward A and B as well as against the source. Figure 81*b* shows the case where a reflector antenna and an antenna are energized from a common transmitter. A quarter-wavelength transformer ($\beta l_1 = 90$ electrical degrees, or $l_1 = \frac{1}{4}\lambda$) can then be used in order to make the entrance impedance at feed point P very high. By choosing the other lengths l_2, l_3, and l_4 properly, a suitable energy transfer and current phases at the antenna and reflector antenna will be secured. Figure 82 shows a concentric-line feeder of a half-wave radiator with a quarter-wave impedance-matching transformer.

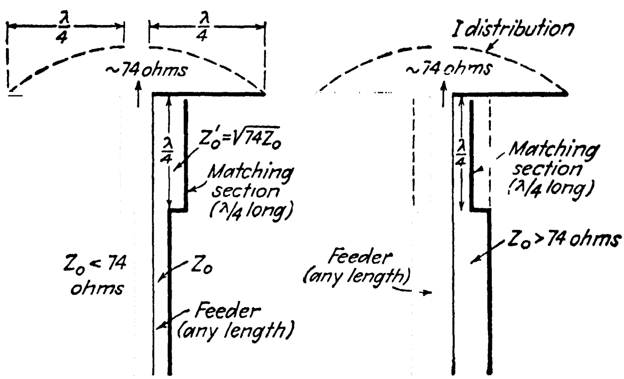


FIG. 82. Feeders with characteristic impedances which are less and more than the radiation resistance of a half-wave dipole.

87. Metallic Insulation Supports and Harmonic Suppressor Lines. According to Eq. (609), the impedance looking into a line that is short-circuited at the far end ($Z_2 = 0$) is $Z_1 = Z_0 \tanh (l\Gamma)$ complex ohms, if l meters is the length of the double line, and $\Gamma = \alpha + j\beta$ denotes the propagation constant in generalized radians per meter. The quantity Z_0 denotes the characteristic impedance of the line in ohms. For a properly designed HF double line, we can put $\alpha \cong 0$ and $Z_0 \cong \sqrt{L/C}$ resistive ohms. Therefore, $Z_1 = Z_0 \tanh (j\beta l) = jZ_0 \tan (\beta l)$ ohms. We know that for $\beta l = 0, 180, 360, 3 \times 180, 4 \times 180, \cdots$ deg, Z_1 vanishes. For $\beta l = 90, 3 \times 90 = 270, 5 \times 90, 7 \times 90, \cdots$ deg, Z_1 becomes theoretically $\pm \infty$, but measurements show that we may expect from 200*k* to 500*k* ohms when looking into a double line that is short-circuited at the far end and has such electric lengths. A line that is 90 electrical degrees means that 90 deg $= \beta l = 360°l/\lambda$, or $l = \lambda/4$ long. Hence, we can connect such a short-circuited double line across a feeder without disturbing much the current flow for a frequency that causes a quarter-wavelength distribution along such a double line. The same is true for a line that is $3\lambda/4$ in length and short-circuited at the far end and also for lengths

equal to $5\lambda/4$, $7\lambda/4$, $\cdots$. Now the same length that is $\lambda/4$, $3\lambda/4$, $5\lambda/4$, $\cdots$ for the exciting frequency f, given by $v = f\lambda$ meters per second, where v is the phase velocity along the double line, is a $\lambda/2$, $3\lambda/2$, $5\lambda/2$ line for the second harmonic (frequency $2f$ cycles per second). Let us first inspect $l = \lambda/2$, for which $\beta l = 360 l/\lambda = 180$ deg (electric degrees). But tan 180 = 0, and the $\lambda/2$ line that is short-circuited at the far end has an input impedance of zero ohms. Therefore, the quarter-wavelength shunt across a double line for the fundamental frequency must short out any double-frequency current in the feeder (a double-harmonic *suppressor* and a fundamental harmonic *insulator*). Inasmuch as the short-circuited $\lambda/4$ section in shunt with a feeder is a multiple of a half wavelength at any even harmonic (at $2f$ it is a $\lambda/2$ section, at $4f$

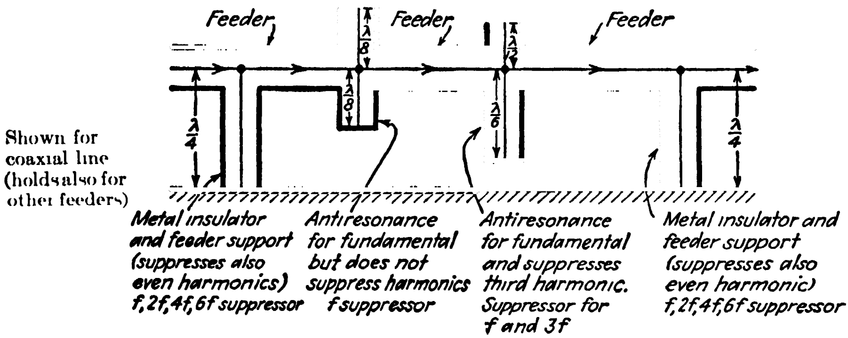
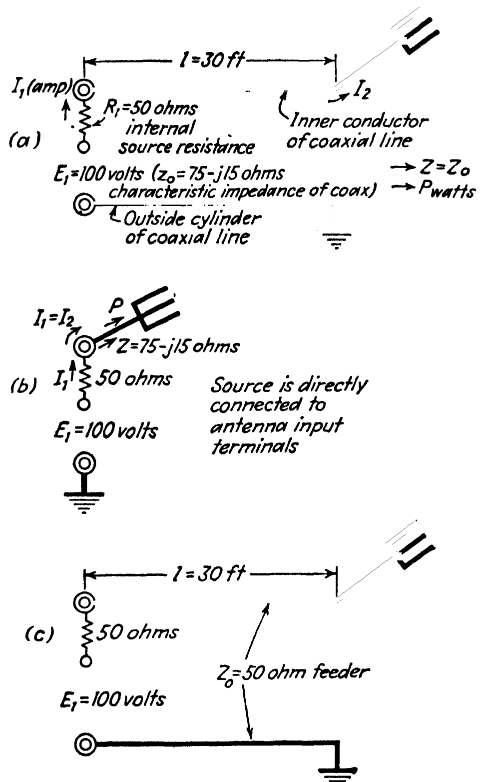


FIG. 83. Metal insulators and suppressors of currents of frequencies f $2f$ $3f$ $4f$, $6f$, $8f$, etc. All wavelengths λ refer to the fundamental frequency f, so that phase velocity = $\lambda^{(m)} f^{(\text{cycles/sec})}$.

it is a $2(\lambda/2)$ section, at $6f$ it is $3\lambda/2$ section, etc.), all *even* harmonics are suppressed by the $\lambda/4$ section for the fundamental frequency. Now, if we were to connect an open-ended double line in shunt with a feeder, we should find from Eq. (609) that $Z_1 = Z_0 \coth (l\Gamma) = Z_0 \coth (j\beta l) = -jZ_0 \cot (\beta l)$ ohms. The general open-ended line behaves then as a cotangent function that is *capacitive* as long as the value of cot (βl) is *positive*. The general short-circuited line was found to be a function $+jZ_0 \tan (\beta l)$, *i.e.*, an *inductive* reactance, as long as the value of tan (βl) is *positive*. Hence, if we connect across a feeder a *short*-circuited line that is $\lambda/8$ long, we have $Z_1 = jZ_0 \tan (\beta\lambda/8) = jZ_0 \tan [360\lambda/(8\lambda)] = jZ_0 \tan 45° = jZ_0$ inductive ohms. If we connect across the same feeder points an *open*-ended line that is likewise $\lambda/8$ long and has the same characteristic impedance, then $Z_1 = -jZ_0 \cot 45° = -jZ_0$ capacitive ohms will cause parallel resonance with $+jZ_0$ inductive ohms (antiresonance). Now, if we connect an open-ended line $\lambda/12$ long across the feeder, we have $\beta l = 360/12 = 30$ deg and $Z_1 = -jZ_0 \cot 30° = -jZ_0/$

$0.5774 = -j1.73Z_0$ ohms. For a short-circuited line that is $\lambda/6$ at the fundamental frequency, we have $\beta l = 360/6 = 60$ deg and $Z_1 = jZ_0 \tan 60° = j1.73Z_0$ ohms. We have, therefore, again parallel resonance for the common junctions. But there is a great advantage, since for the third harmonic ($3f$), a length of $\lambda/12$ is a quarter-wavelength distribution for the open-ended line, or $Z_1 = -jZ_0 \cot 90° = 0$ ohms.



(a) $I_1 = 0.79\epsilon^{j6.85°}$ amp
$I_2 = 0.753\epsilon^{-j66.15°}$ amp
$P = 42.6$ watts delivered to antenna (since the far end of the feeder is matched, the source looks into an impedance of $Z_0 = 75 - j15$ ohms)

(b) $I_1 = 0.79\epsilon^{j6.85°} = I_2$ amp
$P = 47$ watts

(c) $\left.\begin{array}{l} I_1 = 1.038\epsilon^{-j11.6°} \\ I_2 = 0.675\epsilon^{-j69.9°} \end{array}\right\}$ amp
$P = 34.4$ watts

FIG. 84. Conditions for UHF power transfer into a radiating system.

Therefore, the open-ended line suppresses the third harmonic but in conjunction with the short-circuited line behaves like an open-ended line across the feeder as far as the useful fundamental frequency current is concerned. These results are shown in Fig. 83.

88. Notes on the Power Transfer into an Antenna. When dealing with power transfer to an antenna system, we deal with *lumped* and *distributed* constants. Nevertheless, the input impedance into an antenna may, in equivalence, be taken as though consisting of a lumped resistance and a lumped reactance. It is not always possible ideally to

match a line into a load. As far as the surge (characteristic) impedance of a well-designed feeder is concerned, we can assume it resistive; but as far as the propagation constant plays a part, it seems better practice to assume it complex and of the form $\Gamma = \alpha + j\beta$. Maximum power transfer can occur only if the load impedance is *equal and conjugate* to the characteristic impedance of a line or of a source directly feeding it. The reason for this is that the $\pm jX$ reactive term of the load will neutralize the $\mp jX$ reactive term of the device which supplies the HF power. Such a termination, however, will leave some reflection losses where the power is being supplied.

With respect to such computations, we have then the following details: The discussion is to compare the power transfer from a given source when the source is matched between a coaxial feeder and the antenna but mismatched from the source to the coaxial line, when the same source is directly connected to the same antenna and mismatched, and when the same source is matched to the coaxial line but a mismatch exists from the coaxial line to the same antenna. Figure 84 gives the results of an applied voltage of 100 volts at 100 Mc/sec and a loss of 0.015 db/ft for the coaxial line. It is seen that the driving voltage in each case is $E_1 - 50I_1$ volts if the inscribed output current of the HF source is used. The same antenna is used in each case with an input impedance of $Z = 75 - j15$ ohms, yielding an absolute value of $\sqrt{75^2 + 15^2} = 76.5$ ohms with a phase angle of $\tan^{-1}(-15/75) = -11.3$ deg. The total feeder loss is $30 \times 0.015 = 0.45$ db. Since 1 db corresponds to 0.115 nepers, the total attenuation of the coaxial feeder is $\alpha l = 0.115 \times 0.45 = 0.0518$ neper, yielding for the Fig. 84*a* condition a ratio of $I_1/I_2 = \epsilon^{\alpha l} = \epsilon^{0.0518} = 1.05316$, which is not far from being unit value (meaning a very small feeder loss). We find the generator output current as

$$I_1 = \frac{E}{R_1 + Z_0} = \frac{100}{50 + 75 - j15} = \frac{100}{125 - j15} = \frac{100(125 + j15)}{125^2 + 15^2}$$

$$= \frac{100}{15{,}850}\sqrt{15{,}850}\ \Big/ \tan^{-1}\left(\frac{15}{125}\right) = 0.79\underline{/+6.85^\circ}$$

The current I_2 entering the antenna is then

$$I_2 = \frac{I_1}{1.05316} = \frac{0.79}{1.05316}\ \Big/ +6.85^\circ + \left(360^\circ \frac{l}{\lambda}\right)$$

where $\beta l = 360l/\lambda$ accounts for the fact that it takes a certain time for the generator current to reach the antenna input as current I_2. Hence, βl represents a time *delay* (must be taken negative). The phase constant β per unit length refers to $360/\lambda$ deg, if $\lambda = k\lambda_0$ is the operating wavelength along the coaxial feeder. It is usually somewhat less than the free-space[1] wavelength λ_0 and may be $\lambda = 0.95\lambda_0$. The value of β

[1] HUND, A., "High-frequency Measurements," 2nd Ed., McGraw-Hill Book Company, Inc., New York, 1951. Pages 496–497 and Figs. 340 and 342 give, for instance, a value of $k = 0.81$ and 0.87, since k is also the ratio of the phase velocity along the feeder to the free-space velocity.

can also be computed from (603). For $k = 0.95$, we have $\beta = 360/\lambda$ deg $= 360/(0.95\lambda_0) = 378.9/\lambda_0$. This means 378.9-deg phase delay for each free-space wavelength along the feeder. For 100 Mc/sec, we have $\lambda_0 = 300/100 = 3$ m $= 3.281 \times 3$ ft $= 9.843$ ft and a phase constant of $\beta = 378.9°/9.843 = 38.4$ deg/ft. If we had taken $k = 0.935$ instead of 0.95, then $\beta = 385°/\lambda_0 = 39.1$ deg/ft. For a total feeder length of $l = 30$ ft, we find then $\beta l = -30 \times 38.4 = -1{,}152$ deg for $k = 0.95$ and $-1{,}170$ deg for $k = 0.935$ as the phase factor. The phase of I_2 is, therefore, $6.85 - 1{,}152$ deg, which is equal to $-1{,}145.15$ deg for $k = 0.95$. It is *only* the angle $-(1{,}145.15 - \underbrace{1{,}080}_{3\times 360}) = -65.15$ deg that counts, and we find $I_2 = 0.753\epsilon^{-j65.15°}$ amperes. For $k = 0.935$, we find $I_2 = 0.753\epsilon^{-j83.15°}$ amperes. The power delivered to the antenna in Fig. 84*a* is then $P = RI_2^2 = 75 \times 0.753^2 = 42.6$ watts in either case, since the phase delay at the antenna input expresses only how much the arriving current is delayed with respect to the value at the generator end of the feeder. If we had no loss along the feeder, then I_2 would have been $0.79\epsilon^{j65.15°}$ in the $k = 0.95$ case, and the power output delivered to the antenna would be $P' = 75 \times 0.79^2 = 47$ watts, showing that $P' - P = 47 - 42.6 = 4.4$ watts, or $4.4 \times 100/47 = 9.37$ per cent is lost in the feeder. As mentioned already, a match at the antenna junction for *no* reflection loss does *not* mean optimum power transfer. Such transfer occurs only if a characteristic impedance $75 + j15$ ohms works into the conjugate antenna input impedance $75 - j15$ ohms, since then the inductive reactance $+j15$ ohms just cancels the capacitive reactance $-j15$ ohms. For a direct connection of the source to the antenna input as in Fig. 84*b*, we have $I_1 = I_2 = E/(R_1 + Z) = 100/(50 + 75 - j15) = 0.79\epsilon^{j6.85°}$ amperes. Hence, we find the same result, in effect, as for a feeder with no losses, except a different phase factor (since there can be no line delay). Hence, 47 watts must be delivered to the antenna.

For the case of Fig. 84*c*, matters are somewhat more involved, since we *no longer* look from the source end into a feeder that is matched at the far end, even though matched at the generator junction. Hence, the input impedance looking into the antenna-loaded feeder is now for $Z_0 = 50$ ohms, according to Eq. (609) on page 408, if divided by cosh Ω, where $\Omega = l\Gamma$,

$$Z_i = Z_0 \frac{Z + Z_0 \tanh (l\Gamma)}{Z_0 + Z \tanh (l\Gamma)} \qquad \text{ohms}$$

where $\Gamma = \alpha + j\beta$ is the propagation constant per unit length, α the attenuation, and β the phase constant per meter. The current delivered by the source is then

$$I_1 = \frac{E_1}{R_1 + Z_i} \quad \text{amps}; \qquad I_1 = I_2 \left[\cosh (l\Gamma) + \frac{Z}{Z_0} \sinh (l\Gamma)\right] \quad \text{amp}$$

Having I_1 from the left expression, we have the antenna entrance current

$$I_2 = \frac{I_1}{\cosh (l\Gamma) + (Z/Z_0) \sinh (l\Gamma)} \qquad \text{amp}$$

The antenna input power is computed from RI_2^2 watts as before, for $R = 75$ ohms. Since $\alpha l = 0.0518$ neper, we have $\alpha = \alpha l/l = 0.0518/30 = 0.001727$ neper/ft. For $k = 0.95$, we find $\beta = 6.28/(0.95\lambda_0) = 6.58/\lambda_0 = 6.58/\lambda_0 = 6.58/9.843 = 0.668$

radian/ft. Therefore,

$$l\Gamma = (\alpha + j\beta)l = (0.001727 + j0.668)l \qquad \text{generalized radians}$$

and we have $\Gamma \cong j0.668$ and $l\Gamma \cong j\beta l = j65.15$ deg.

When, however, the losses of the feeder line are more pronounced, so that such an approximation is no longer justified, we can use the substitution

$$l\Gamma = \alpha l + j\beta l = p + jq$$

and can compute sinh $(l\Gamma)$, cosh $(l\Gamma)$, and tanh $(l\Gamma)$ from

$$\sinh(l\Gamma) = \sinh p \cos q + j \cosh p \sin q = A\epsilon^{j\psi};$$

$$A = \sqrt{\tfrac{1}{2}[\cosh(2p) - \cos(2q)]} \qquad \psi = \tan^{-1}\left(\frac{\tan q}{\tanh p}\right)$$

$$\cosh(l\Gamma) = \cosh p \cos q + j \sinh p \sin q = B\epsilon^{j\varphi};$$

$$B = \sqrt{\tfrac{1}{2}[\cosh(2p) + \cos(2q)]} \qquad \varphi = \tan^{-1}(\tanh p \tan q)$$

$$\tanh(l\Gamma) = \frac{\sinh p \cos q + j \cosh p \sin q}{\cosh p \cos q + j \sinh p \sin q} = C\epsilon^{j\theta};$$

$$C = \sqrt{\frac{\cosh(2p) - \cos(2q)}{\cosh(2p) + \cos(2q)}} \qquad \theta = \tan^{-1}\left(\frac{\sin(2q)}{\sinh(2p)}\right)$$

For the $k = 0.95$ case, we find

$$p = -0.001727 \times 30 = -0.0518 \text{ neper}; \qquad q = -0.668 \times 30 = -20.04 \text{ circular radians}$$

For the $k = 0.935$ case, we find $q = -20.4$ circular radians. If we proceed with this value, we have $2q = -40.8$ radians $= -40 \times 57.3 = -2337.84$ deg This is equivalent to $-(2{,}337.84 - 6 \times 360) = -117.84$ deg, since only the number of degrees *within a complete cycle* (namely *within* 360 *deg*) are of practical concern. Hence, we have $q = \beta l = -117.84$ deg and since

$$\sin(-x) = -\sin x; \qquad \cos(-x) = \cos x; \qquad \sin(90^\circ + x) = \cos x; \qquad \cos(90^\circ + x) = -\sin x$$

we find

$$\sin(2q) = \sin(-117.84^\circ) = -\cos 27.84^\circ = -0.8843$$
$$\cos(2q) = \cos(-117.84^\circ) = -\sin 27.84^\circ = -0.4671$$

We have also

$$\sinh(-x) = -\sinh x; \qquad \cosh(-x) = \cosh x$$

$$\sinh(2p) = \sinh(-2 \times 0.0518) = -\sinh(0.1036) = -0.1002, \qquad \cosh(2p) = \cosh(0.1036) = 1.005$$

$$C = \sqrt{\frac{1.005 - (-0.4671)}{1.005 + (-0.4671)}} = 1.655; \ \tan\theta = \frac{-0.8843}{-0.1002} = +8.843; \ \theta = 83.5^\circ$$

$$\tanh(l\Gamma) = 1.655\epsilon^{j83.5^\circ} = 1.655(\cos 83.5^\circ + j\sin 83.5^\circ) = 0.186 + j1.64$$

$$Z_i = Z_0\frac{Z + Z_0\tanh(l\Gamma)}{Z_0 + Z\tanh(l\Gamma)} = 50\,\frac{75 - j15 + 50(0.186 - j1.64)}{50 + (75 - j15)(0.186 - j1.64)}$$

$$= 50\,\frac{84.3 - j97}{39.48 - j125.79}$$

$$= 50\,\frac{\sqrt{84.3^2 + 97^2}\,\Big/ -\tan^{-1}\left(\frac{97}{84.3}\right)}{\sqrt{39.48^2 + 125.79^2}\,\Big/ -\tan^{-1}\left(\frac{125.79}{39.48}\right)}$$

$$= 50\,\frac{128/-49^\circ}{131.7/-72.6^\circ}$$

$$= \frac{50\times 128/-49^\circ + 72.6^\circ}{131.7} = 49\epsilon^{j23.6^\circ}$$

$$= 49(\cos 23.6^\circ + j\sin 23.6^\circ) = 44.8 + j19.6 \text{ ohms}$$

$$I_1 = \frac{E_1}{R_1 + Z_i} = \frac{100}{50 + 44.8 + j19.6} = \frac{100}{94.8 + j19.6}$$

$$= \frac{100}{\sqrt{94.8^2 + 19.6^2}\,\Big/\tan^{-1}\left(\frac{19.6}{94.8}\right)}$$

$$= \frac{100}{96.7/+11.6^\circ} = 1.038\epsilon^{-j11.6^\circ} \quad \text{amp}$$

$$I_2 = \frac{I_1}{\cosh(l\Gamma) + (Z/Z_0)\sinh(l\Gamma)} = \frac{1.038\epsilon^{-j11.6^\circ}}{B\epsilon^{j\varphi} + A\epsilon^{j\psi}(75 - j15)/50}$$

$$A = \sqrt{\tfrac{1}{2}[\cosh(2p) - \cos(2q)]} = \sqrt{\tfrac{1}{2}[1.005 - (-0.4671)]} = 0.857$$

$$B = \sqrt{\tfrac{1}{2}[1.005 - 0.4671]} = 0.518$$

$$\psi = \tan^{-1}\left(\frac{\tan q}{\tanh p}\right) = \tan^{-1}\left[\frac{\tan(-58.92^\circ)}{\tanh(-0.0518)}\right] = \tan^{-1}\left[\frac{-\tan 58.92^\circ}{-\tanh 0.0518}\right]$$

$$= \tan^{-1} 32.6 = 88.25^\circ$$

$$\tan\varphi = \tan q \tanh p = 1.658 \times 0.051 = 0.0845; \qquad \varphi = 4.82^\circ$$

$$B\epsilon^{j\varphi} + \frac{75 - j15}{50}A\epsilon^{j\psi} = 0.518\epsilon^{j4.82^\circ} + (1.5 - j0.3)0.857\epsilon^{j88.25^\circ}$$

$$= 0.518(\cos 4.82^\circ + j\sin 4.82^\circ)$$
$$+ (1.28 - j0.256)(\cos 88.25^\circ + j\sin 88.25^\circ)$$

$$= 0.81 + j1.306 = \sqrt{0.81^2 + 1.306^2}\,\Big/\tan^{-1}\left(\frac{1.306}{0.81}\right)$$

$$= 1.54/+58.3^\circ = 1.54\epsilon^{j58.3^\circ}$$

Therefore,

$$I_2 = \frac{1.038\epsilon^{-j11.6^\circ}}{1.54\epsilon^{j58.3^\circ}} = 0.675\epsilon^{-j69.9^\circ} \quad \text{amp}; \qquad P = \frac{0.675^2}{2}75 = 17.2 \text{ watts}$$

89. Remarks on Radiation of Transmission Lines. Transmission lines consist of go and return conductors, which cannot coincide to make a power transfer possible, since we need a current to the load and its return from the load. Complete cancellation of respective external fields of the go and return is, therefore, *never* possible in the case of a *parallel-wire* system. A cancellation of the external field is also never possible for concentric lines unless *both ends* are perfectly *closed* and the outside conductor is a perfect conductor, so that no electromagnetic field can escape. At VHF's and UHF's, radiation must be expected, even though not so pronounced as for one linear radiator for customary spacing and use. There seems to be a belief that an aperiodic feeder, whether of the single-line or the double-line type, does not radiate. This is by no means correct. We know that an elementary radiator, which is, in effect, a Hertzian doublet, will radiate. Such elementary radiators have likewise fixed-current distributions as has an aperiodic elementary feeder length. Hence, we can integrate elementary radiators (sum up such radiators) and obtain a resultant radiation effect.

90. Radiation Effects of Linear Conductors with Traveling Waves Compared with Conductors with Standing-wave Distributions. Transmission-line theory shows that in lines loaded at the far end into an impedance Z, an applied harmonic voltage will cause a current as well as a potential distribution that is due to an outgoing and a returning (reflected) wavelike disturbance (page 410). For $Z = \infty$, we have the case of an open-ended conductor; and for $Z = 0$, we deal with a short-circuited far end. In either case, the reflection coefficient has unity value. For the open-ended case, for a length $l = \lambda/4, 3\lambda/4, 5\lambda/4, \cdots$, we have resonance effects with respect to the input end; while for $\lambda/2$, λ, $3\lambda/2, \ldots$, we have conditions known as "antiresonance" (parallel resonance) as far as the input end is concerned. This means that for the first case, the input impedance is very low and resistive while, for the second case, it is very high and resistive. Such conditions are known as "linear radiators" with standing-wave distributions, since we have fixed voltage and current nodes and fixed voltage and current antinodes, respectively, at certain places of the radiator, and the distributions can be verified experimentally by moving a probe along the conductor.

When a conductor is infinitely long or terminated at the far end into its characteristic impedance Z_0, then all the arriving energy is absorbed by the characteristic termination or is used up by an infinitely long conductor. Therefore, for either case, only a traveling wavelike disturbance passing from the input end outward along the conductor is possible. In the case of the characteristic termination $Z = Z_0$ resistive ohms, this then causes a radiation field due to a wave train, traveling toward the

TABLE 17

θ, deg	$\cos\theta$	$1-\cos\theta$	$180°(1-\cos\theta)$, deg	$\cos[180°(1-\cos\theta)]$	$1-\cos[180°(1-\cos\theta)] = a$	$\sqrt{a}$	$D_\theta = k\dfrac{\cos\theta\sqrt{1-\cos[180°(1-\cos\theta)]}}{1-\cos\theta}$
0	1 0000	0 0000	0 0000	1 0000	0 0000	0 00000	0,0 $\cong$ 2 22k
5	0 9962	0 0038	0 684	0 9999	0 0001	0 001	2 8k
10	0 9848	0 0152	2 74	0 99885	0 00115	0 034	2 2k
20	0 9387	0 0613	11 05	0 9816	0 0184	0 135	2 07k
30	0 866	0 134	24 1	0 9128	0 0872	0 295	1 91k
40	0 766	0 234	42	0 7431	0 2569	0 508	1 66k
50	0 6428	0 3572	67	0 3907	0 6093	0 78	1 41k
60	0 5	0 5	90	0 0000	1 0000	1 000	1 00k
70	0 342	0 658	118	−0 4695	1 4695	1 21	0 63k
80	0 1736	0 8264	148	−0 848	1 848	1 35	0 284k
90	0 0000	1 0000	180	−1 0000	2 0000	1 41	0 000k
100	−0 1736	1 1736	211	−0 9336	1 9336	1 38	−0 206k
110	−0 342	1 342	241	−0 6293	1 6293	1 27	−0 324k
120	−0 5000	1 5000	270	0 0000	1 0000	1 00	−0 333k
130	−0 6428	1 6428	295	0 4226	0 5774	0 755	−0 296k
140	−0 766	1 766	319	0 7547	0 2453	0 495	−0 214k
150	−0 866	1 866	335	0 9063	0 0937	0 306	−0 142k
160	−0 9387	1 9387	348	0 9781	0 0219	0 147	−0 071k
170	−0 9848	1 9848	355	0 9962	0 0038	0 0615	−0 0312k
175	−0 9962	1 9962	358	0 9994	0 0016	0 04	−0 0199k
180	−1 0000	2 0000	360	1 0000	0 0000	0 0000	0 0000k

far-end loading Z. The Beverage wave antenna, which is several wavelengths long, utilizes this principle for a pronounced unilateral field pickup system, with the characteristic loading pointing toward the wave arrival. It can be shown[1] that essentially zero effect then occurs when the receiver end of the line points toward the wave arrival and the length of the line is $l = 10\lambda$ or more long. The directivity can then be computed from

$$D_\theta = \cos\theta \sqrt{\frac{2\epsilon^{-\alpha l}\left\{\cosh(\alpha l) - \cos\left[\frac{\beta}{m} l(1 - m\cos\theta)\right]\right\}}{\alpha^2 + \left[\frac{\beta}{m} l(1 - m\cos\theta)\right]^2}};$$

$$\beta = \frac{2\pi}{\lambda}; \qquad m = \frac{c'}{c} \tag{692}$$

The pointing direction is then designated by $\theta = 0$ deg, while $\theta = 180$ deg is the direction from the characteristic termination toward the receiver side of the wave antenna (of the line). The quantity α denotes the attenuation of the line per meter length, and m is the ratio of the phase velocity c' along the line to the velocity of propagation in empty space. It is the phase velocity c' that accounts for the potential and current distributions along the line. Since for an operating wavelength λ along the line, we have $f\lambda = c'$ and for the free-space wavelength λ_0, we have $f\lambda_0 = c = 3 \times 10^8$ m/sec for the wavelength in meters and the frequency f in cycles per second, we also have $m = \lambda/\lambda_0$. If we have, for instance, a line length $l = \frac{1}{2}\lambda$, we can, for a conductor in air, neglect the attenuation ($\alpha \cong 0$) and $\beta l = (2\pi/\lambda)(\lambda/2) = \pi$, corresponding to 180 deg. Therefore,

$$D_\theta = \frac{1.41}{\pi}\cos\theta\,\frac{\sqrt{1 - \cos[180°(1 - \cos\theta)]}}{1 - \cos\theta} \tag{692a}$$

since $\cosh(\alpha l) \cong \cosh(0) = 1$ and $\epsilon^{-\alpha l} \cong \epsilon^{0} = 1$. This leads to the results of Table 17 for $k = 1.41/3.14 = 0.45$ and the directivity given in Fig. 85. We note in this figure only a small side lobe along the side of the $\frac{1}{2}\lambda$ conductor, where there is not the characteristic impedance termination. Table 17 shows that the $\theta = 5$ deg value yields $D_\theta = 2.8k$ and, if used in Fig. 85, would cause the extension AB, indicating that the directivity D_θ for $\theta = 0$ deg would be infinite. Such a condition cannot be physically correct. When the curve along AC is produced, it yields a value of about 2.22 for $\theta = 0$ deg, a result that seems more probable. It should be noted that formula (692a) assumes that the phase velocity c' along the $\frac{1}{2}\lambda$ conductor is essentially equal to the velocity c holding for vacuum. We know this assumption is not exactly true, since the length of a $\frac{1}{2}\lambda$ radiator is usually about 5 per cent shorter than would be the case if $\lambda = \lambda_0$. Hence, $\lambda \cong 0.95\lambda_0$ and $m = 0.95$ must be used in (692) and

[1] HUND, A., "Phenomena in High-frequency Systems," pp. 454–462, McGraw-Hill Book Company, Inc., New York, 1936. The Bruce rhombic (or diamond) antenna can be used over a wide range of frequencies without retuning the antenna. Neither the length nor the angles of the rhombus are very critical. For instance, if each side is 4 wavelengths long (based on highest frequency of band) for a rhombus, 40 ft above the ground, the gain is about 15 db. The angles of the four sides are usually made such that the major diagonal is about 1 wavelength shorter than the sum of the two sides. Optimum angles depend on the conditions of the earth crust and the angle with the horizon for receiving or transmitting. For a receiver antenna a termination of 800 ohms is desired. For details see BRUCE, E., *Proc. IRE*, **19**, 1406, August, 1931; BRUCE, E., A. C. BECK, and L. R. LOWRY, *Proc. IRE*, **23**, 24, January, 1935.

the ½λ case of formula (692a). This is needed all the more for small angles θ, since the square-root value of (692a) is for $\theta = 0$ deg for instance,

$$\sqrt{1 - \cos[180°(1 - \cos 0°)]} = 0$$

and the denominator is likewise zero, making D_θ an indetermined quantity. Using the more correct formula

$$D_\theta = \frac{1.41 \cos\theta \sqrt{1 - \cos\left[\frac{\beta l}{m}(1 - m\cos\theta)\right]}}{\frac{\beta l}{m}(1 - m\cos\theta)} \tag{693}$$

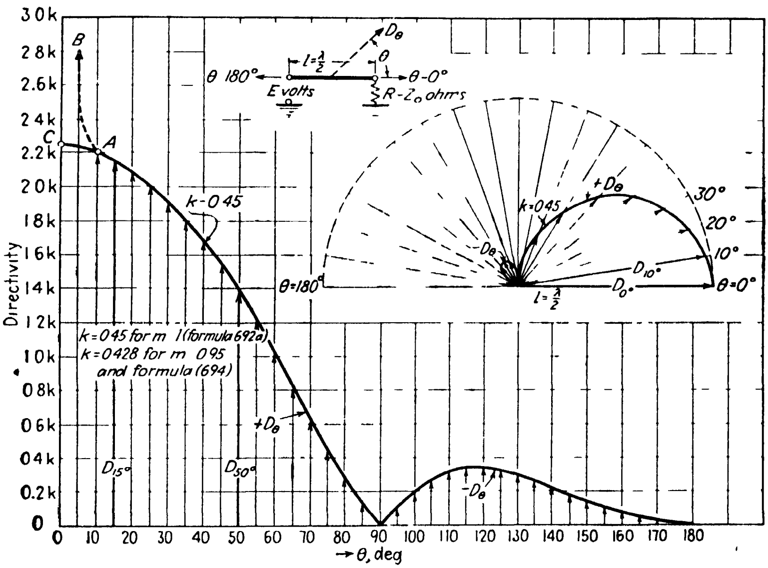


FIG. 85. Directivity of matched lines based on the computations of Table 17. (Assumes phase velocity c' along $\lambda/2$ line equal to the velocity $c = 3 \times 10^8$ m/sec holding for empty space.)

we find for $l = \frac{1}{2}\lambda$ and $m = 0.95$, the ratio $\beta l/m = \pi/m = \pi/0.95 = 1.05\pi$ radians, corresponding to $1.05 \times 180 \cong 190$ deg, $1.41/(\beta l/m) = 1.41/(1.05\pi) = 0.428$, and

$$D_\theta = 0.428 \frac{\cos\theta \sqrt{1 - \cos[190°(1 - 0.95\cos\theta)]}}{1 - 0.95\cos\theta} \tag{694}$$

Applying this result to the case of $\theta = 0$ deg, we find

$$1 - 0.95\cos 0° = 0.05$$
$$\cos[190°(1 - 0.95\cos 0°)] = \cos(0.05 \times 190°) = \cos 9.5° = 0.9863$$

and

$$D_{0°} = 0.428 \frac{\sqrt{1 - 0.9863}}{0.05} = \frac{0.428 \times 0.117}{0.05} = 1.0$$

If we compare this result with the value of the produced AC curve of Fig. 85, we find $D_{0°} = 2.22k = 2.22 \times 0.45 = 1.0$. Table 18 then gives the computations for $k_1 =$

TABLE 18

Phase velocity $c' = 0.95c$, $c = 3 \times 10^8$ m/sec; $k_1 = 0.428$									$c' = c$
θ, deg	$\cos\theta$	$m\cos\theta = 0.95\cos\theta$	$1 - 0.95\cos\theta$	α deg $= 190°(1 - 0.95\cos\theta)$	$\cos\alpha$	$1-\cos\alpha$	$\sqrt{1-\cos\alpha}$	$D_\theta = \frac{k_1\cos\theta\sqrt{1-\cos\alpha}}{1-0.95\cos\theta}$	D_θ from Fig. 85 with $k = 0.45$
0	1.0000	0.95	0.05	9.5	0.9863	0.0137	0.117	1 0000	1.0000
5	0.9962	0.945	0.055	10 4	0.9836	0.0164	0.128	1.000	0.9999
10	0.9848	0.935	0 065	12 6	0.9759	0.0241	0.155	1.00	0.99
20	0.9387	0.89	0.11	20.9	0.9342	0.0658	0.256	0.932	0.93
30	0.866	0.82	0 18	34.2	0.8271	0.1729	0.415	0.85	0 86
40	0.766	0.728	0 272	51.7	0.6198	0.3802	0.615	0.74	0.749
50	0.6428	0.61	0 39	74	0.2756	0.7244	0.85	0.6	0.644
60	0.5	0.475	0 525	100	−0.1736	1.1736	1.08	0.44	0.45
70	0.342	0.325	0.675	128	−0.6157	1.6157	1.27	0.275	0.284
80	0.1736	0.165	0.835	158.5	−0.9304	1.9304	1.38	0.123	0.128
90	0.0000	0.000	1.000	190	−0.9848	1.9848	1.4	0.0000	0.0000
100	−0.1736	−0.165	1.165	222	−0.7431	1.7431	1.32	−0.0845	−0.093
110	−0.342	−0.325	1.325	252	−0.309	1.309	1.14	−0.126	−0.146
120	−0.5000	−0.475	1.475	280	0.1736	0.8264	0.905	−0.132	−0.15
130	−0.6428	−0.61	1.61	305	0.5736	0.4264	0.65	−0.1115	−0.134
140	−0.766	−0.728	1.728	329	0.8572	0.1428	0.378	−0.0715	−0.096
150	−0.866	−0.82	1.82	345	0.9659	0.0341	0 184	−0.0374	−0.064
160	−0.9387	−0.89	1.89	359	0.9998	0.0002	0.0141	−0.00298	−0.032
170	−0.9848	−0.935	1.935	366	0.9945	0.0055	0 074	−0.0162	−0.014
180	−1.0000	−0.95	1.95	371	0.9816	0.0184	0.135	−0.0296	−0.0

0.428 using formula (694). In the last column are the results for a phase velocity along the ½λ-line antenna equal to $c = 3 \times 10^8$ m/sec, holding for vacuum. Figure 86 gives the patterns for $c' = 0.95c$ and $c' = c$. It is seen that both directivity patterns have about the same behavior, since the side lobes, if plotted in polar coordinates, are not very pronounced. Polar patterns for other systems as well as several

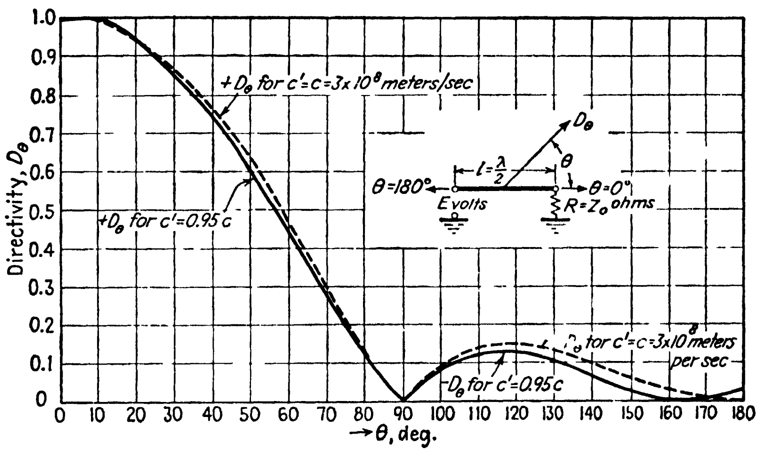


FIG. 86. Directivity patterns for matched loading of a line for ideal and actual phase velocities. Solid pattern for phase velocity $c' = 0.95 \times 3 \times 10^8$ m/sec, broken pattern for $c' = 3 \times 10^8$ m/sec.

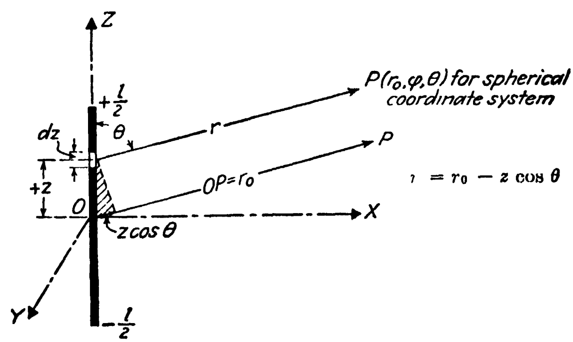


FIG. 87. Radiation effect of linear conductors. Linear radiator fed at mid-point O. For Cartesian system, distant field point P is at x, y, z, where $x = r_0 \sin\theta\cos\varphi$; $y = r_0 \sin\theta \sin\varphi$; $z = r_0 \cos\theta$.

wavelength distributions are given in A. Hund, "Phenomena in High-frequency Systems," pages 54–462.

91. Field Intensity Value and Radiation Resistance for Standing- and Traveling-wave Excitation along Linear Conductors. If I_z denotes the current value for any line element dz a distance z from the mid-point of a linear conductor (Fig. 87), we know from Sec. 35 that in the distant

radiation region, the field contributions $d\mathcal{E}_\theta$ volts per meter and dH_φ amperes per meter, at any point $P(x, y, z)$, for the Cartesian coordinates x, y, and z, as in the figure, due to a current element $I_z\, dz$, are given by

$$d\mathcal{E}_\theta = -j\,\frac{188.35}{r\lambda}\,I_z\,dz\epsilon^{jq}\sin\theta \quad \text{volts/m;} \qquad dH_\varphi = \frac{d\mathcal{E}_\theta}{Z_0} \quad \text{amp/m} \qquad (695)$$

where $Z_0 \cong \sqrt{\mu_0/\kappa_0} = 376.7$ ohms is the three-dimensional characteristic unbounded space impedance and $q = \omega t - \beta r$, $\beta = 2\pi/\lambda$ with λ meters equal to $300/f^{(\mathrm{Mc})}$, where r is in meters and $\omega/6.28$ in cycles per second. The quantity I_z represents, therefore, the amplitude time value at a certain conductor distance z. The radiation effect of all conductor elements $I_z\, dz$, taken over the limits $\pm\frac{1}{2}l$, then results in the total field contribution

$$\begin{aligned}\mathcal{E}_\theta^{(\mathrm{volts/m})} &= Z_0 H_\varphi^{(\mathrm{amp/m})} = -j\,\frac{188.35}{\lambda}\,\epsilon^{jq}\int_{-l/2}^{+l/2} I_z\,\frac{\sin\theta}{r}\,\epsilon^{j\beta z\cos\theta}\,dz \\ &= -j\,\frac{30I_0[\epsilon^{j0.5n\pi\cos\theta} - (-1)^n\epsilon^{-j0.5n\pi\cos\theta}]}{r_0\sin\theta} \qquad (696)\end{aligned}$$

for I_0 amperes at the current antinode for the I_z distribution along $\pm Z$. For integer multiples n of half wavelengths, we have then the results of pages 502, 506, and

$$\begin{aligned}\mathcal{E}_\theta^{(\mathrm{volts/m})} &= Z_0 H_\varphi^{(\mathrm{amp/m})} = -j60I_0\,\frac{\epsilon^{j(\omega t - \beta r_0)}}{r_0}\,\frac{\cos(0.5n\pi\cos\theta)}{\sin\theta} \\ &\qquad \text{for } n = 1, 3, 5, 7, \cdots \\ \mathcal{E}_\theta^{(\mathrm{volts/m})} &= Z_0 H_\varphi^{(\mathrm{amp/m})} = 60I_0\,\frac{\epsilon^{j(\omega t - \beta r_0)}}{r_0}\,\frac{\sin(0.5n\pi\cos\theta)}{\sin\theta} \\ &\qquad \text{for } n = 2, 4, 6, 8, \cdots\end{aligned} \qquad (697)$$

Hence, the first of Eqs. (697) refers to odd $\frac{1}{2}\lambda$ distributions like $l = \frac{1}{2}\lambda$, $3 \times \frac{1}{2}\lambda$, $5 \times \frac{1}{2}\lambda$, $7 \times \frac{1}{2}\lambda$, $\cdots$, and the second to even distributions as $l = \lambda$, 2λ, 3λ, 4λ, $\cdots$. Relations (695) to (697) then take the general line distributions into account with the far ends $\pm\frac{1}{2}l$ open and therefore *stationary* distributions, since zero current value must exist at $z = \pm\frac{1}{2}l$. This is possible only when the arriving current disturbance at the respective ends just cancels the current effect returning from these extremities. Hence, a reflected current disturbance in form of a reflected wave travels also back to the mid-point of the linear conductor.

For *pure* traveling waves (no stationary waves also present), either the distances $\frac{1}{2}l$ are infinitely long or the far ends are terminated through the characteristic impedance of the conductor so that all the energy arriving at respective far ends is being absorbed by the characteristic

termination. Derivation (696) then reads, for no reflections at the ends, $z = \pm\frac{1}{2}l$

$$\mathcal{E}_\theta^{(\text{volts/m})} = Z_0 H_\varphi^{(\text{amp/m})} = -j\,\frac{188.35 I_0 \sin\theta}{r_0\lambda}\,\epsilon^{jq}\int_{-l/2}^{+l/2}\epsilon^{j\beta(1-\cos\theta)z}\,dz \qquad (698)$$

since again, for $r \gg l$, the ratio $(\sin\theta)/r \cong (\sin\theta)/r_0$ can be taken before the integral sign, as $(\sin\theta)/r$ is a *slow variable* function of z and for the distant radiation region $r \cong r_0$, because r is practically parallel with r_0. The approximation $r \cong r_0$ is permissible, however, *only* as far as the $1/r_0$ effect is concerned (see page 488). The integration of (698) then yields

$$\mathcal{E}_\theta^{(\text{volts/m})} = -j60 I_0 \sin\theta\,\frac{\sin\left[\frac{\beta l}{2}(1-\cos\theta)\right]}{1-\cos\theta}\,\frac{\epsilon^{jq}}{r_0} \qquad (699)$$

The *phase* factor ϵ^{jq} is of no consequence; and if we call $\mathcal{E}_\theta$, H_φ, and I_0 the effective values, our relations on either side are just divided by $\sqrt{2}$, and we have

$$\left.\begin{aligned}
&\mathcal{E}_\theta^{(\text{volts/m})} = 376.7H_\varphi = -j30I_0\,\frac{\epsilon^{-j0.5n\pi\cos\theta} - (-1)^n\epsilon^{-j0.5n\pi\cos\theta}}{r_0\sin\theta}\\
&\qquad \textit{standing}\text{-wave distributions of practical importance for } n = 1, 2, 3, \cdots\\
&\mathcal{E}_\theta^{(\text{volts/m})} = 376.7H_\varphi = -j60I_0\,\frac{\sin\left[\frac{\beta l}{2}(1-\cos\theta)\right]}{1-\cos\theta}\,\frac{\sin\theta}{r_0}\\
&\qquad \textit{traveling}\text{ waves } (\textit{no}\text{ end reflections})
\end{aligned}\right\} \qquad (700)$$

For a half-wave radiator, we find, $\beta l = \beta(\lambda/2)$, and

$$\left.\begin{aligned}
&\mathcal{E}_\theta^{(\text{volts/m})} = 376.7H_\varphi^{(\text{amp/m})} = -j\,\frac{60I_0}{r_0}\,\frac{\cos(90^\circ\cos\theta)}{\sin\theta} \quad \text{for } \textit{standing} \text{ waves}\\
&\epsilon_\theta^{(\text{volts/m})} = 376.7H_\varphi^{(\text{amp/m})} = -j\,\frac{60I_0}{r_0}\,\frac{\sin[90^\circ(1-\cos\theta)]}{1-\cos\theta}\sin\theta\\
&\qquad \text{for } \textit{traveling} \text{ waves}
\end{aligned}\right\} \qquad (701)$$

$$\left.\begin{aligned}
&D_\theta = \frac{\cos(90^\circ\cos\theta)}{\sin\theta} \qquad \text{for } \textit{standing} \text{ waves}\\
&D_\theta = \frac{\sin[90^\circ(1-\cos\theta)]}{1-\cos\theta}\sin\theta \qquad \text{for } \textit{traveling} \text{ waves}
\end{aligned}\right\} \qquad (702)$$

Table 19 gives computations for the traveling-wave case, and Fig. 88 shows the corresponding Cartesian and polar patterns in comparison with the customary half-wave dipole radiator, the Hertzian doublet, and the isotropic point radiator. Both the isotropic point radiator with the directivity $D_\theta = 1$ and the Hertzian doublet (two-point)

TABLE 19

θ, deg	For traveling waves						For standing waves
	cos θ	1 − cos θ	90°(1 − cos θ), deg	sin [90°(1 − cos θ)] = A	sin θ	$D_\theta = \frac{A \sin\theta}{1 - \cos\theta}$	$D_\theta = \frac{\cos[90° \cos\theta]}{\sin\theta}$
0	1.0000	0 00000	0 0000	0 0000	0 0000	0 0000	0 0000
10	0 9848	0 0152	1 37	0 0244	0 1736	0 278	0 151
20	0 9387	0 0613	5 525	0 0958	0 342	0 531	0 281
30	0 866	0 134	12 05	0 2087	0 5	0 775	0 403
40	0 766	0 234	21	0 3584	0 6428	0 99	0 553
50	0 6428	0 3572	33.5	0 5519	0 766	1 185	0 691
60	0 5000	0 5000	45	0 7071	0 866	1 22	0 815
68	0 3746	0 6254	56 3	0 832	0 9272	1 24	0 9
70	0 342	0 658	59	0 8572	0 9397	1 245	0 915
80	0 1736	0 8264	74	0 9613	0 9848	1 115	0 975
90	0 0000	1 0000	90	1 0000	1 0000	1 000	1 000

radiator with the directivity $D_\theta = \sin\theta$ are representative only with respect to the *shape* of the pattern. The respective circles are altogether too large if we think in terms of the actual field amplitude, since

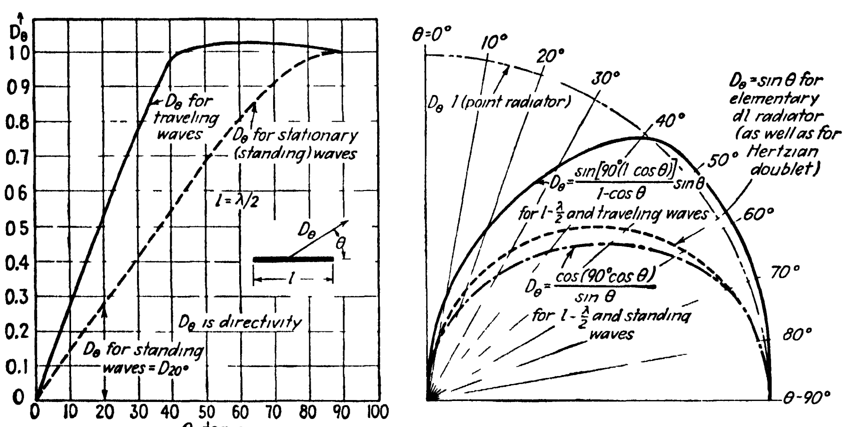


FIG. 88. Directivity patterns for stationary and traveling waves along a linear radiator.

the radiation resistance of the point source and of the doublet source is very small compared with the efficient and powerful ½λ radiator for both the standing-wave and the traveling-wave case. It is noted that the traveling-wave case has a maximum directivity in the neighborhood of $\theta \cong 70$ deg with respect to the conductor direction.

In the same way as described in Secs. 17, 26–28, 43 we obtain the electromagnetic energy flux per square meters in any direction r_0 from the origin of Fig. 87, by means of the real part of the complex Poynting vector $p = \frac{1}{2}\mathcal{E}_\theta H_\varphi^*$, where the square of the magnitude of H_φ is found by the multiplication of H_φ with its conjugate H_φ^*. For a customary $\frac{1}{2}\lambda$ radiator ($l = \frac{1}{2}\lambda$) with standing waves, we then have

$$p = \frac{0.5 \times 60 I_0^2}{\pi r_0^2} \frac{\cos^2 (90° \cos \theta)}{\sin^2 \theta}$$

where r_0 is the radius of the spherical wave-front surface reaching into the distant radiation region. The total radiation energy passing normally across this surface is found from

$$P^{(\text{watts})} = \int_0^{2\pi=\varphi} \int_0^{\pi=\theta} p r_0^2 \sin \theta \, d\theta \, d\varphi \tag{703}$$

as is shown in detail on page 75, 87. For a linear radiator of length $l = \frac{1}{2}\lambda$, we find then

$$\left.\begin{aligned} P^{(\text{watts})} &= 30 I_0^2 \int_0^\pi \frac{\cos^2 (90° \cos \theta)}{\sin \theta} \, d\theta && \text{for standing-wave excitation} \\ P^{(\text{watts})} &= 30 I_0^2 \int_0^\pi \frac{\sin^2 \theta \sin^2 [90°(1 - \cos \theta)]}{(1 - \cos \theta)^2} \sin \theta \, d\theta && \text{for traveling-wave excitation} \end{aligned}\right\} \tag{704}$$

As is shown on pages 511, 512, the value under the first integral can be transformed into $\frac{1}{2}(1/w)(1 - \cos w)\,dw$, for limits 0 to 2π for $l = \frac{1}{2}\lambda$, 0 to $2n\pi$ for $l = n\frac{1}{2}\lambda$, where $n = 1, 3, 5, 7, \cdots$. It is then easy to evaluate the integral, since the portion $\int_\eta^\infty \frac{\cos w}{w} \, dw = -\text{Ci}\,\eta$ is the integral cosine. In exactly the same way, we can transform the value under the second integral of (704) and find

$$\left.\begin{aligned} P^{(\text{watts})} &= 15 I_0^2 (A + B) \qquad \text{for } \textit{standing} \text{ waves} \\ P^{(\text{watts})} &= 30 I_0^2 (C + D + E + F) \qquad \text{for } \textit{traveling} \text{ waves} \\ A &= \log_\epsilon (6.28 \times 1.7811); \qquad B = -\text{Ci}(6.28); \qquad C = 1.415 \\ D &= \log_\epsilon \left(\frac{\beta l}{\pi}\right); \qquad E = -\text{Ci}(2\beta l); \qquad F = \frac{\sin (2\beta l)}{2\beta l}; \qquad \beta = \frac{2\pi}{\lambda} \end{aligned}\right\} \tag{705}$$

The factor 1.7811 is the Euler constant, and $\log_\epsilon (x) = 2.3026 \log_{10} (x)$.

The *radiation resistance* R_r, with respect to the effective current $I = I_0/\sqrt{2}$, is found from $I^2 R_r = P$ watts, $\beta l/\pi = 2l/\lambda = 1$, and $2\beta l = 2\pi$. Therefore,

$$\left.\begin{aligned} R_r^{(\text{ohms})} &= 30[2.3026 \log_{10} (11.2) - \text{Ci}(6.28)] \qquad \text{for } \textit{standing} \text{ waves} \\ R_r^{(\text{ohms})} &= 60\Big[1.415 + \underbrace{2.3026 \log_{10} (1)}_{=0} - \text{Ci}(6.28) + \underbrace{\frac{\sin (2\pi)}{6.28}}_{=0}\Big] \qquad \text{for } \textit{traveling} \text{ waves} \end{aligned}\right\} \text{Ci } 6.28 \text{ is nearly } -0.02 \tag{705a}$$

$$\left.\begin{aligned} R_r^{(\text{ohms})} &= 30 \times 2.41 + 30 \times 0.02 = 73.1 \text{ ohms} \quad \text{for } \textit{standing waves} \\ R_r^{(\text{ohms})} &= 60 \times 1.415 + 60 \times 0.02 = 85.7 \text{ ohms} \quad \text{for } \textit{traveling} \text{ waves} \end{aligned}\right\} \tag{706}$$

We have, for distributions such as $l = \lambda/2, \lambda, 1.5\lambda, 2\lambda, 2.5\lambda, \cdots,$ or $l = n\lambda/2$, for which $n = 1, 2, 3, 4, 5, \cdots,$

$$\left.\begin{aligned} R_r^{(\text{ohms})} &= 30[2.3026 \log_{10}(11.2n) - \text{Ci}(6.28n)] = 72.5 \\ &\quad + 69 \log_{10}(n) - \text{Ci}(6.28n) \quad \text{for } \textit{standing} \text{ waves} \\ R_r^{(\text{ohms})} &= 84.5 + 138 \log_{10}\left(2\frac{l}{\lambda}\right) + 60\,\frac{\sin\left(720°\frac{l}{\lambda}\right)}{12.56\frac{l}{\lambda}} \\ &\quad - 60\,\text{Ci}\left(12.56\frac{l}{\lambda}\right) \quad \text{for } \textit{traveling} \text{ waves} \end{aligned}\right\} \tag{707}$$

where $n = 1, 2, 3, 4, \cdots.$

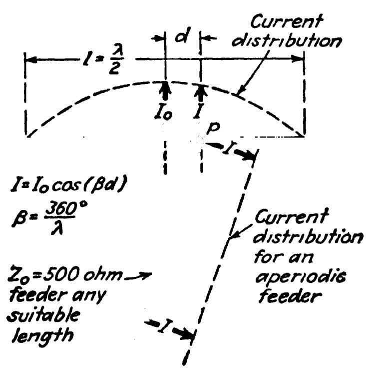


FIG. 89. Matched feeder connected to the appropriate point P of a half-wave radiator. (I is the fixed effective current distribution along the feeder, and I_0 is the effective current at the current loop of the half-wave radiator.)

An aperiodic single line is one that applies to the second relation. If the characteristic impedance of this feeder is $Z_0 = 500$ ohms and I is the effective current distribution along the feeder, then $I^2Z_0 = 500I^2$ is the watt input into the actual radiator of Fig. 89 at the match point P. If R_f is the radiation resistance of the feeder, as given by the feeder length l in terms of the operating wavelength λ by the ratio l/λ, then I^2R_f is the radiation loss of the feeder in watts, which may be an appreciable portion of the desired radiation due to the $\frac{1}{2}\lambda$ antenna. The radiation resistance is then computed from the second of Eqs. (707), which can be written as

$$\left.\begin{aligned} R_f &= 84.5 + a + b + c \quad \text{ohms} \\ a &= 138 \log_{10}\left(2\frac{l}{\lambda}\right); \quad b = 60\,\frac{\sin\left(720°\frac{l}{\lambda}\right)}{12.56\frac{l}{\lambda}}; \\ c &= -60\,\text{Ci}\left(12.56\frac{l}{\lambda}\right) \end{aligned}\right\} \tag{708}$$

as is done in Table 20 with the result of Fig. 90.

TABLE 20

$\frac{l}{\lambda}$	$2\frac{l}{\lambda}$	$\log_{10}\left(2\frac{l}{\lambda}\right)$	$12.56\frac{l}{\lambda}$	$a = 138 \times \log_{10}\left(2\frac{l}{\lambda}\right)$	$720°\frac{l}{\lambda}$, deg	$\sin\left(720°\frac{l}{\lambda}\right) = A$	$b = \frac{A}{12.56(l/\lambda)}60$	$\mathrm{Ci}\left(12.56\frac{l}{\lambda}\right)$	$c = -60 \times \mathrm{Ci}\left(12.56\frac{l}{\lambda}\right)$	$R_f = 84.5 + a + b + c$
0.1	0.2	−0.699	1.256	−96	72	0.9511	45.5	0.4345	−26	8
0.2	0.4	−0.3979	2.512	−54.8	144	0.5878	28	0.2827	−16.9	40.8
0.4	0.8	−0.0969	5.01	−13.3	288	−0.9511	−11.4	−0.19	−11.4	71.2
0.5	1.0	0.0000	6.3	0.00	360	0.0000	0.00	0.0000	0.000	84.5
0.6	1.2	0.0792	7.56	10.95	432	0.8829	7.00	0.1022	−6.12	96.33
0.8	1.6	0.2041	10.05	28.1	576	−0.5878	−3.51	−0.0496	2.98	112.07
1.0	2.0	0.301	12.56	41.5	720	0.00	0.00	−0.0662	3.97	129.97
1.25	2.5	0.3979	15.7	55	900	0.00	0.00	0.00447	−0.269	139.231
1.5	3.0	0.4771	18.9	66	1,080	0.00	0.00	−0.000098	0.000	150.5
1.75	3.5	0.5441	22	75	1,260	0.00	0.00	0.00164	−0.098	159.412
2.0	4.0	0.6021	25.1	83	1,440	0.00	0.00	−0.00287	0.172	167.672
2.25	4.5	0.6532	28.4	90.2	1,620	0.00	0.00	−0.00318	0.192	174.892
2.5	5.0	0.6990	31.5	96.5	1,800	0.00	0.00	0.00166	−0.0995	180.9
2.75	5.5	0.7404	34.6	102	1,980	0.00	0.00	−0.000395	Negligible	186.5
3.0	6.0	0.7782	37.9	107.5	2,160	0.00	0.00	0.00457	−0.275	191.725
3.5	7.0	0.8451	44.1	116.5	2,520	0.00	0.00	0.00215	−0.129	200.871
4.0	8.0	0.9031	50.2	124.5	2,880	0.00	0.00	−0.00169	0.1015	209.101
4.5	9.0	0.9542	56.8	132	3,240	0.00	0.00	0.00408	−0.245	216.255
5.0	10	1.0	63	138	3,600	0.00	0.00	0.0024	−0.144	222.36
6.0	12	1.0792	75.6	149	4,320	0.00	0.00	0.00248	−0.149	233.351
7.0	14	1.1461	88	158.5	5,040	0.00	0.00	0.00027	Negligible	243
8.0	16	1.2041	100.5	166	5,760	0.00	0.00	−0.000233		250.5
9.0	18	1.2553	113.8	173	6,480	0.00	0.00	Still more negligible		257.5
10	20	1.3010	125.6	180	7,200	0.00	0.00			264.5

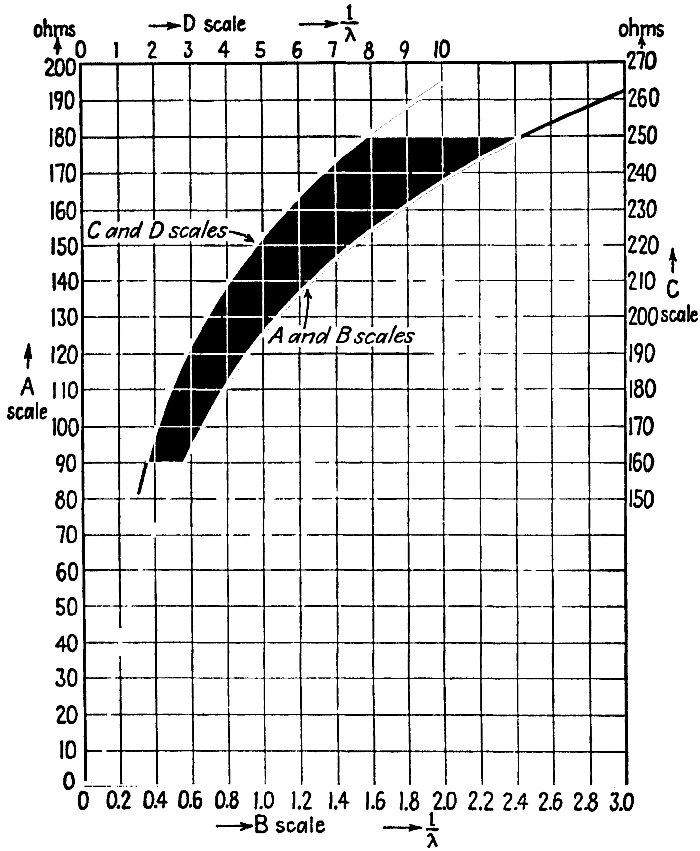


FIG. 90. Radiation resistance R ohms of an aperiodic feeder. R is the radiation resistance of the aperiodic feeder in ohms. In absence of the radiation effect of the feeder the power flowing to the radiator would be I^2Z_0 watts, and I^2R may be a large portion of I^2Z_0 watts.

Inasmuch as the Ci x tables that are available go to arguments only as high as $x = 100$, the asymptotic formulas

$$\left.\begin{aligned} &\text{Ci}\, x \cong P(x)\sin x + Q(x)\cos x; \qquad P(x) = \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \cdots \\ &Q(x) = \frac{1}{x^2} - \frac{3!}{x^4} + \frac{5!}{x^6} - \frac{7!}{x^8} + \cdots; \qquad 4! = 1 \times 2 \times 3 \times 4 \end{aligned}\right\} \tag{709}$$

can be used for computing Ci x. These formulas are satisfactory for values of $x > 30$ within 13 decimals. For $x = 100.5$, we find

$$P(100.5) = \frac{1}{100.5} - \frac{1 \times 2}{100.5^3} + \frac{1 \times 2 \times 3 \times 4}{100.5^5} - \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{100.5^7} + \cdots$$

If we had an even 100 for the argument x, then

$$P(100) = 10^{-2} - 2 \times 10^{-6} + 24 \times 10^{-10} - 144 \times 10^{-14} + \cdots$$
$$= 0.01 - 0.000002 + 0.0000000024 - 0.00000000000144 + \cong 0.01$$

Hence, it is necessary to use only the *first* term for arguments of 100 and higher, and $P(100.5) = 1/100.5 = 0.00995$. The $Q(x)$ term is likewise covered by the first term of (709), and we have

$$Q(100.5) = \frac{1}{100.5^2} = 0.000000988$$

Since 1 radian corresponds to 57.3° we have $x = 100.5$ radians equal to 5,758.65° or $15 \times 360° + 358.65°$, with only the 358.65 argument of interest We find then $\sin 358.65° = \sin (270 + 88.65)° = -\cos 88.65° = -0.0235$, and $\cos (270 + 88.65)° = \sin 88.65° = 0.9997$. Therefore,

$$\begin{aligned} \text{Ci}(100.5) &= P(100.5) \sin 358.65° + Q(100.5) \cos 358.65° \\ &= 0.00995(-0.0235) + 0.000000988(0.997) \\ &= -0.000234 + 0.000000985 = -0.000233015 \end{aligned}$$

It can be seen from Table 20 and Fig. 90 that even for a feeder length only one-tenth of a wavelength long, the radiation resistance of the aperi-

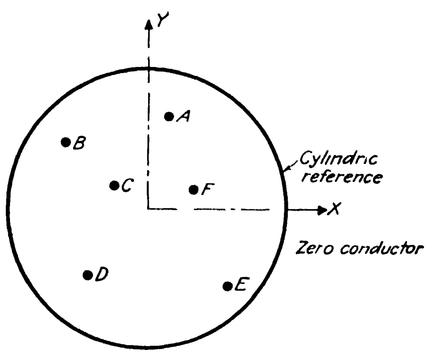


FIG. 91. Multiple-conductor transmission system. Cross section in XY plane; conductors along Z direction.

odic feeder is still 8 ohms. For the half-wave radiator of Fig. 89 the radiation resistance is about 74 ohms with respect to the effective loop value I_0. It should also be realized that for stationary waves, efficient radiation occurs only for $n = 1, 2, 3, 4, \cdots$ or $l = \frac{1}{2}\lambda, \lambda, 1.5\lambda, 2\lambda, \cdots$.

92. Electromagnetic Radiation along a Multiple-conductor System. This discussion has reference to a conductor system, as indicated in Fig. 91, where N parallel conductors with very small cross-sectional dimensions in comparison with the conductor length are enclosed by a zero conductor cylindric shell. The principle of superposition then holds for a cross-sectional area as is shown in the figure. Guided radiation along the Z direction is then characterized by $H_z = 0$ amp/m. The

energy flux can then be decomposed into a Poynting radiation p_z and $p_{x,y}$, respectively, for the electric field component $\mathcal{E}_z$ and the component $\mathcal{E}_{x,y}$ in volts per meter. We then have the relations

$$p_z = \mathcal{E}_{x,y} \times H_{x,y} \quad \text{watts/m}^2; \qquad p_{x,y} = \mathcal{E}_z \times H_{x,y} \quad \text{watts/m}^2 \tag{710}$$

where $p_{x,y}$ is negligible for good conductors since the Poynting vector p is the *cross* product, which accounts for the energy flux crossing the unit area perpendicular to p during 1 sec of time. For the vector $p = \mathcal{E} \times H$, we have the energy density $\frac{1}{2}(\kappa\mathcal{E}^2 + \mu H^2)$ in joules per cubic meter, representing a scalar. Inasmuch as for a large specific conductivity ($\sigma = 5.8 \times 10^7$ mhos/m for copper conductors), the resultant $\mathcal{E}$ lines are essentially normal to the conductor surfaces, we have, at best, only a very small $\mathcal{E}_z$ component in comparison with the normal $\mathcal{E}_{x,y}$ components. Assuming ideal conductivity, we then have both $\mathcal{E}_z = 0$ and $H_z = 0$, and the Maxwellian curl $\mathcal{E}$ and curl H components of Eq. (1) on page 11 and Eq. (137) on page 148, read

$$\overbrace{\frac{\partial \mathcal{E}_z}{\partial y}}^{=0} - \frac{\partial \mathcal{E}_y}{\partial z} = -\frac{\partial(\mu H_x)}{\partial t}; \qquad \frac{\partial \mathcal{E}_x}{\partial z} - \overbrace{\frac{\partial \mathcal{E}_z}{\partial x}}^{=0} = \frac{\partial(\mu H_y)}{\partial t};$$

$$\text{hence,} \quad \overbrace{\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}}^{=0} = -\overbrace{\frac{\partial(\mu H_z)}{\partial t}}^{=0}$$

All components for curl $\mathcal{E} = -[\partial(\mu H)]/\partial t$.

$$\overbrace{\frac{\partial H_z}{\partial y}}^{=0} - \frac{\partial H_y}{\partial z} = \sigma\mathcal{E}_x + \frac{\partial(\kappa\mathcal{E}_x)}{\partial t}; \qquad \frac{\partial H_x}{\partial z} - \overbrace{\frac{\partial H_z}{\partial x}}^{=0} = \sigma\mathcal{E}_y + \frac{\partial(\kappa\mathcal{E}_y)}{\partial t};$$

$$\text{hence,} \quad \overbrace{\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}}^{=0} = \overbrace{\sigma\mathcal{E}_z + \frac{\partial(\kappa\mathcal{E}_z)}{\partial t}}^{=0}$$

All components for curl $H = \sigma\mathcal{E} + [\partial(\kappa\mathcal{E})]/\partial t$, where the specific conductivity σ in mhos per meter, the permeability μ in henrys per meter, and the dielectric constant κ in farads per meter refer to the medium in which the N parallel perfect conductors are embedded. In most of our work, we shall deal with a medium that is practically a perfect dielectric. The conduction current densities $J_x = \sigma\mathcal{E}_x$, $J_y = \sigma\mathcal{E}_y$, and $J_z = \sigma\mathcal{E}_z$ in amperes per square meter can then be neglected, and the above curl relations reduce to

$$\left.\begin{aligned}\frac{\partial \mathcal{E}_x}{\partial z} &= -\mu \frac{\partial H_y}{\partial t} \quad (711)\\ \frac{\partial \mathcal{E}_y}{\partial z} &= \mu \frac{\partial H_x}{\partial t} \quad (712)\end{aligned}\right\} \text{define completely curl } \mathcal{E} = -\mu \frac{\partial H}{\partial t} \text{ in our example}$$

$$\left.\begin{aligned}\frac{\partial H_x}{\partial z} &= \kappa \frac{\partial \mathcal{E}_y}{\partial t} \quad (713)\\ \frac{\partial H_y}{\partial z} &= -\kappa \frac{\partial \mathcal{E}_x}{\partial t} \quad (714)\end{aligned}\right\} \text{define completely curl } H = \kappa \frac{\partial \mathcal{E}}{\partial t}$$

since for isotropic mediums also μ and κ are fixed everywhere in the medium. Since H_z along the conductor directions is zero, we have no magnetic displacement current $\partial B_z/\partial t$ volts per square meter, for $B_z = \mu H_z$ normal to the cross section shown in Fig. 91. Consequently, the electric voltage circulation vanishes also for the entire cross section, and we can express the electric field intensity by a potential V volts, holding for the cross section, and

$$\mathcal{E}_x = -\frac{\partial V}{\partial x} \text{ volts/m}; \qquad \mathcal{E}_y = -\frac{\partial V}{\partial y} \text{ volts/m} \tag{715}$$

Similarly to the derivations on page 301, we can also find the magnetic field components from a flow or *current* function ψ in amperes by means of

$$H_x = \frac{\partial \psi}{\partial y} \text{ amp/m}; \qquad H_y = -\frac{\partial \psi}{\partial x} \text{ amp/m} \tag{716}$$

Using these expressions for respective electric and magnetic field components in (711) to (714), we find

$$\frac{\partial \mathcal{E}_x}{\partial z} = -\frac{\partial^2 V}{\partial x\,\partial z}; \quad \frac{\partial \mathcal{E}_y}{\partial z} = -\frac{\partial^2 V}{\partial y\,\partial z}; \quad \frac{\partial \mathcal{E}_x}{\partial t} = -\frac{\partial^2 V}{\partial x\,\partial t}; \quad \frac{\partial \mathcal{E}_y}{\partial t} = -\frac{\partial^2 V}{\partial y\,\partial t}$$

$$\frac{\partial H_x}{\partial z} = \frac{\partial^2 \psi}{\partial y\,\partial z}; \quad \frac{\partial H_y}{\partial z} = -\frac{\partial^2 \psi}{\partial x\,\partial z}; \quad \frac{\partial H_x}{\partial t} = \frac{\partial^2 \psi}{\partial y\,\partial t}; \quad \frac{\partial H_y}{\partial t} = -\frac{\partial^2 \psi}{\partial x\,\partial t}$$

Therefore,

$$\left.\begin{aligned}&\left.\begin{aligned}\kappa \frac{\partial^2 V}{\partial x\,\partial t} &= -\frac{\partial^2 \psi}{\partial x\,\partial z}\\ \kappa \frac{\partial^2 V}{\partial y\,\partial t} &= -\frac{\partial^2 \psi}{\partial y\,\partial z}\end{aligned}\right\} \text{satisfied by } \kappa \frac{\partial V}{\partial t} = -\frac{\partial \psi}{\partial z}\\ &\left.\begin{aligned}\mu \frac{\partial^2 \psi}{\partial x\,\partial t} &= -\frac{\partial^2 V}{\partial x\,\partial z}\\ \mu \frac{\partial^2 \psi}{\partial y\,\partial t} &= -\frac{\partial^2 V}{\partial y\,\partial z}\end{aligned}\right\} \text{satisfied by } \mu \frac{\partial \psi}{\partial t} = -\frac{\partial V}{\partial z}\end{aligned}\right\} \tag{717}$$

with the following details:

Dealing at first with the solution of the κ relations, we find

$$\underbrace{\kappa \frac{\partial\left(\frac{\partial V}{\partial t}\right)}{\partial x} = \kappa \frac{\partial^2 V}{\partial x\,\partial t}}_{\text{left side of first equation}}; \quad \underbrace{\kappa \frac{\partial\left(\frac{\partial V}{\partial t}\right)}{\partial x} = -\frac{\partial\left(\frac{\partial \psi}{\partial z}\right)}{\partial x} = -\frac{\partial^2 \psi}{\partial x\,\partial z}}_{\text{right side of first equation}} \quad \text{or} \quad \kappa \frac{\partial^2 V}{\partial x\,\partial t} = -\frac{\partial^2 \psi}{\partial x\,\partial z}$$

which checks the first equation. Similarly, we find for the other κ relations

$$\kappa \frac{\partial\left(\frac{\partial V}{\partial t}\right)}{\partial y} = \underbrace{\kappa \frac{\partial^2 V}{\partial y\,\partial t}}_{\text{left side}}; \quad \kappa \frac{\partial\left(\frac{\partial V}{\partial t}\right)}{\partial y} = -\frac{\partial\left(\frac{\partial \psi}{\partial z}\right)}{\partial y} = \underbrace{-\frac{\partial^2 \psi}{\partial y\,\partial z}}_{\text{right side}}$$

For the μ relations, we find in the same way

$$\mu \frac{\partial\left(\frac{\partial \psi}{\partial t}\right)}{\partial x} = \underbrace{\mu \frac{\partial^2 \psi}{\partial x\,\partial t}}_{\text{left side}}; \quad \mu \frac{\partial\left(\frac{\partial \psi}{\partial t}\right)}{\partial y} = \underbrace{\mu \frac{\partial^2 \psi}{\partial y\,\partial t}}_{\text{left side}}$$

$$\mu \frac{\partial\left(\frac{\partial \psi}{\partial t}\right)}{\partial x} = -\frac{\partial\left(\frac{\partial V}{\partial z}\right)}{\partial x} = \underbrace{-\frac{\partial^2 V}{\partial x\,\partial z}}_{\text{right side}}; \quad \mu \frac{\partial\left(\frac{\partial \psi}{\partial t}\right)}{\partial y} = -\frac{\partial\left(\frac{\partial V}{\partial z}\right)}{\partial y} = \underbrace{-\frac{\partial^2 V}{\partial y\,\partial z}}_{\text{right side}}$$

By means of the two solutions given in (717), we can find a combined solution by the elimination of the current function ψ. We find then, by differentiating $\kappa(\partial V/\partial t) = -\partial\psi/\partial z$ with respect to t and $\mu(\partial\psi/\partial t) = -\partial V/\partial z$ with respect to z,

$$\kappa \frac{\partial^2 V}{\partial t^2} = -\frac{\partial^2 \psi}{\partial z\,\partial t}; \quad \mu \frac{\partial^2 \psi}{\partial z\,\partial t} = -\frac{\partial^2 V}{\partial z^2} \quad \text{or} \quad \frac{\partial^2 \psi}{\partial z\,\partial t} = -\kappa \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\mu}\frac{\partial^2 V}{\partial z^2}$$

Therefore,

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0; \qquad v = \frac{1}{\sqrt{\mu\kappa}} \tag{718}$$

This equation is well known. It can generally be satisfied by

$$V = V_0(x, y)\left[F\left(t - \frac{z}{v}\right) + G\left(t + \frac{z}{v}\right)\right] \quad \text{volts} \tag{719}$$

where $V_0(x, y)F[t - (z/v)]$ is a wavelike function moving along the $+Z$ direction, which is the axis to which all N conductors are parallel. The

other function $V_0(x, y)G[t + (z/v)]$ moves with the same velocity $v/\sqrt{\mu\kappa}$ meters per second in the $-Z$ direction. Hence, for air as the dielectric medium in which the N conductors are embedded, v would be the velocity $c = 3 \times 10^8$ m/sec. This is based on the assumption that we have infinite conductivity for the conductors, and no electromagnetic field is, therefore, possible in the space occupied by the conductors. The potential function $V_0(x, y)$ depends only on the geometric relative disposition of the N parallel conductors, but *not* on z. Unlike unobstructed space radiation, we generally have here oppositely moving wave trains, where F and G depend upon the terminal conditions. For the current function ψ, we find, from the first solution of (717),

$$\frac{\partial\psi}{\partial z} = -\kappa\frac{\partial V}{\partial t} = -\kappa V_0(x, y)\frac{\partial\left[F\left(t - \frac{z}{v}\right) + G\left(t + \frac{z}{v}\right)\right]}{\partial t} \tag{720}$$

with the following details:

For $F_1 = F/v$ and $G_1 = G/v$, we find

$$\frac{\partial\left[F\left(t - \frac{z}{v}\right) + G\left(t + \frac{z}{v}\right)\right]}{\partial t} = \frac{\partial\left[F\left(t - \frac{z}{v}\right)\right]}{\partial t} + \frac{\partial\left[G\left(t + \frac{z}{v}\right)\right]}{\partial t} = \frac{\partial[F_1(vt - z)]}{\partial t} + \frac{\partial[G_1(vt + z)]}{\partial t} = vF_1'(vt - z) + vG_1'(vt + z)$$

$$\frac{\partial\psi}{\partial z} = -v\kappa V_0(x, y)[F_1'(vt - z) + G_1'(vt + z)]$$

$$\psi = -v\kappa V_0(x, y)[-F_1(vt - z) + G_1(vt + z)] = \frac{\kappa V_0(x, y)}{\sqrt{\mu\kappa}}[F_1(vt - z) - G_1(vt + z)]$$

$$\psi = \sqrt{\frac{\kappa}{\mu}}\,V_0(x, y)\left[F\left(t - \frac{z}{v}\right) - G\left(t + \frac{z}{v}\right)\right] \quad \text{amp} \tag{721}$$

For the current function, we find then

$$\psi = \psi_0(x, y)\left[F\left(t - \frac{z}{v}\right) - G\left(t + \frac{z}{v}\right)\right] \quad \text{amp};$$

$$\psi_0(x, y) = \frac{V_0(x, y)}{Z_0} \quad \text{amp}; \qquad Z_0 = \sqrt{\frac{\mu}{\kappa}} \quad \text{ohms} \tag{722}$$

showing that for any cross-sectional area, as indicated in Fig. 91, the potential function $V_0(x, y)$ in volts and the current function $\psi_0(x, y)$ in amperes are proportional to each other. Therefore, electric equipotential lines and magnetic lines of force must coincide.

Moreover, the arrangement of Fig. 91 can be applied to many line systems, since, for instance, for an axial conductor with the cylindrical reference shell as the return, we deal with the coaxial transmission line and the principal wave solution (TEM wave) just given applies in the case of perfect conductivity for the conductors and a perfect dielectric as the wave medium. If we have a parallel-wire transmission line with both wires parallel to ground, we can take the ground for quite a frequency range and especially for ocean surface as the zero reference con-

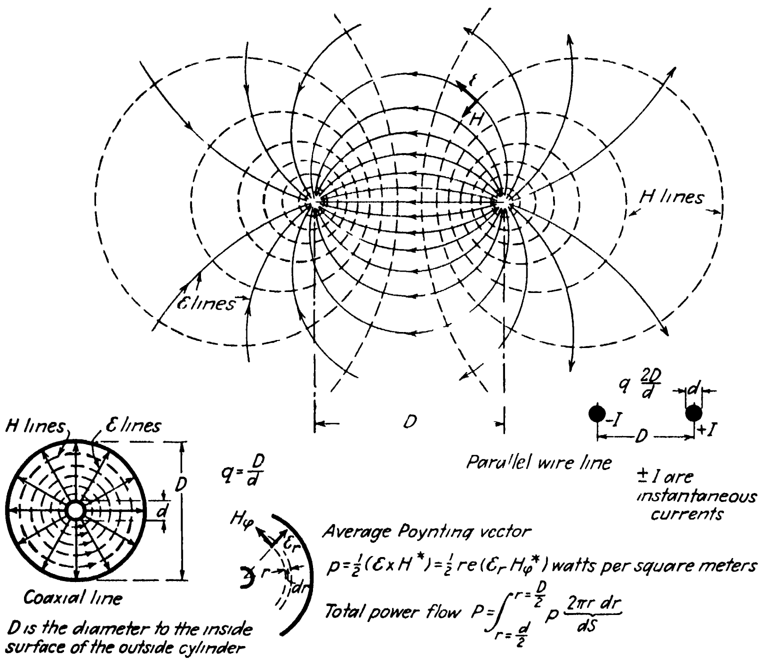


FIG. 92. ℰ and H lines for customary double lines.

ductor. For such a symmetrical disposition, we have then $V_1 = -V_2$ volts and the voltage $E = V_1 - V_2$, so that $V_1 = \frac{1}{2}E = -V_2$. Since the respective currents flow in opposition, we have $I_1 = I$ and $I_2 = -I$ amperes. The ground current vanishes for such a symmetrical arrangement. The power of an outgoing wave train is then E_eI_e watts, which is also equal to $Z_0I_e^2$ watts if we deal with perfect conductivity and Z_0 denotes the wave resistance, and the effective values $E_e = E/\sqrt{2}$ and $I_e = I/\sqrt{2}$. In this case, Z_0 is the guided radiation resistance. This is dealt with in detail on page 73.

93. Principal Wave Propagation along a Parallel-wire System and along a Concentric Line. From Fig. 92 we see that for TEM waves, the

electric $\mathcal{E}$ lines and the magnetic H lines are orthogonal to each other and both normal to the direction of energy transfer. If we had a single line in unobstructed space, a current flow in the line would cause magnetic lines that are concentric circles about the wire axis, while the electric lines of force are normal to the cylindrical surface of a perfect conductor. For a parallel-wire system as indicated in Fig. 92, the electric $\mathcal{E}$ lines leave one conductor normal to its surface and enter the other conductor normally, while the magnetic H lines still surround each conductor. It can be shown that they are still circles but eccentric with respect to the axis of the conductor that they surround. The electric $\mathcal{E}$ lines are likewise circular and orthogonal to the H lines. For the concentric line, the H lines are concentric circles between the inner and outer conductor while the electric $\mathcal{E}$ lines are radial. The difference between the parallel-wire and the concentric-line transmission system is that in the former, we have, as a rule, two like cylindrical conductors, one serving as a go and the other as a return path for the current flowing around the load at the far end. For the coaxial transmisson line, the inside cylinder is one path and the outside cylinder is the other path with a current flow in opposite directions with respect to the corresponding section of the other cylindrical path. At ultrahigh frequencies, the skin effect is so pronounced that only the inner skin of the outside conductor and the outer skin of the inside cylindrical conductor carry the current; and if both ends of the concentric line are closed, clearing the axial conductor, with a good conducting disk, there will be no external field (also no external radiation). For the parallel-wire system there will always be some radiation, since neither conductor can act as a screen for the field effects.

For both the parallel-wire system and the coaxial line, the lines of force can be accounted for by the Laplace equation, since, for instance, we have, for the line along the Z direction, $(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = 0$ as the result of $\nabla^2 V = 0$. We have then generally $\mathcal{E}_{x,y} = -\partial V/\partial(x, y)$, since *circular symmetry* prevails for a coaxial line. The X and Y directions, respectively, can be along any radial directions (as long as X is normal to Y), in a plane perpendicular to the axis (Z axis, along the line). The magnetic H lines of force are concentric circles about the Z axis (length dimension of concentric line). We use then cylinder coordinates r, φ, and z. The H field is then tangent to these concentric circles and called H_φ amperes per meter, and the $\mathcal{E}$ field is radial and designated $\mathcal{E}_r$ volts per meter. For TEM wave propagation along the Z axis, we must also have $\mathcal{E}_r/H_\varphi = Z_0$ if air is used as the dielectric, where $Z_0 = \sqrt{\mu_0/\kappa_0} = 376.7$ ohms, and we think in terms of plane waves. For any other perfect dielectric medium in the space between the $r = d/2$ and $r = D/2$ walls of the concentric line, we have $\sqrt{\mu/\kappa}$ as the ratio

of the radial to tangential field associations. We have then for any perfect dielectric wave medium,

$$\mathcal{E}_r = \frac{\mathcal{E}_0}{r}\epsilon^{j\omega t - z\Gamma}; \qquad H_\varphi = \frac{H_0}{r}\epsilon^{j\omega t - z\Gamma}; \qquad \mathcal{E}_0 = \sqrt{\frac{\mu}{\kappa}}\,H_0;$$

$$\Gamma = j\omega\sqrt{\mu\kappa} \quad (723)$$

for the propagation constant Γ along the dielectric medium of the concentric line. The above relations then account for a TEM wave traveling with the velocity $v = 1/\sqrt{\mu\kappa}$ meters per second along the $+Z$ direction. The larger the radius r is chosen the weaker both $\mathcal{E}_r$ and H_φ field associations become. Since the outer diameter of the inner conductor is d meters and the inner diameter of the outer conductor is D meters, we must have the weakest fields right next to the inner skin of the outer conductor (next to $r = 0.5D$), while the largest field associations occur next to the inner conductor ($r = 0.5d$). Hence, when we use a very thin inner conductor, a very strong field condition must prevail next to its surface. We also note that if the inner conductor vanishes altogether, *i.e.*, $d = 0$ and $r = 0.5d = 0$, we can have no longer a principal wave, since this would result in $\mathcal{E}_r$ and H_φ values that are infinite. The functions of (723) are, therefore, no longer analytical for $r = 0$, and a hollow metallic guide cannot be excited with the above-mentioned TEM wave (principal wave). Relations (723) represent not only a plane TEM wave being propagated with the constant Γ along the Z direction but also such a wave that satisfies the boundary condition of a concentric line for $r = 0.5d$ and $0.5D$, respectively.

We have to distinguish between the medium constants of the *conductors* and the medium constants of the *intervening space* for which the $\mathcal{E}_r$ and H_φ relations of (723) hold. Since we assume very high conductivity (σ mhos per meter) for the conductors and the permeability of nonferromagnetic metals is μ_0, which holds for vacuum, and since it can also be assumed that the dielectric constant is also the κ_0 value for vacuum, we have, for such a conductor, a propagation constant $\Gamma = j\omega\sqrt{\mu_0\kappa_e}$, where the equivalent dielectric constant $\kappa_e = \kappa_0 - j\sigma/\omega$. Now $\kappa_0 = 8.854 \times 10^{-12}$ farad/m; and for copper concentric cylinders, $\sigma = 5.8 \times 10^7$ mhos/m at room temperature, we should have, even at a frequency as high as 100,000 Mc/sec, $\omega = 6.28 \times 10^5 \times 10^6 = 6.28 \times 10^{11}$ and, therefore, $\sigma/\omega = 5.8 \times 10^7/(6.28 \times 10^{11}) = 9.3 \times 10^{-5}$ farad/m as against a value $\kappa_0 = 8.854 \times 10^{-12}$ farad/m. Therefore, even at such a high frequency for which we could also hardly use a coaxial line, the κ_0 value is negligible in comparison with the $j\sigma/\omega$ term. We can then safely assume that for good conductors, the propagation constant within the conductor is $\Gamma = j\omega\sqrt{-j(\sigma/\omega)\mu}$ with the following details:

A minus j factor in $-j\mu\sigma/\omega$ means that the quantity $\mu\sigma/\omega$ is along the negative imaginary axis or the absolute value $\mu\sigma/\omega$ under the square root has a relative phase angle of -90 deg. But -90 deg under the square root means -45 deg for the value

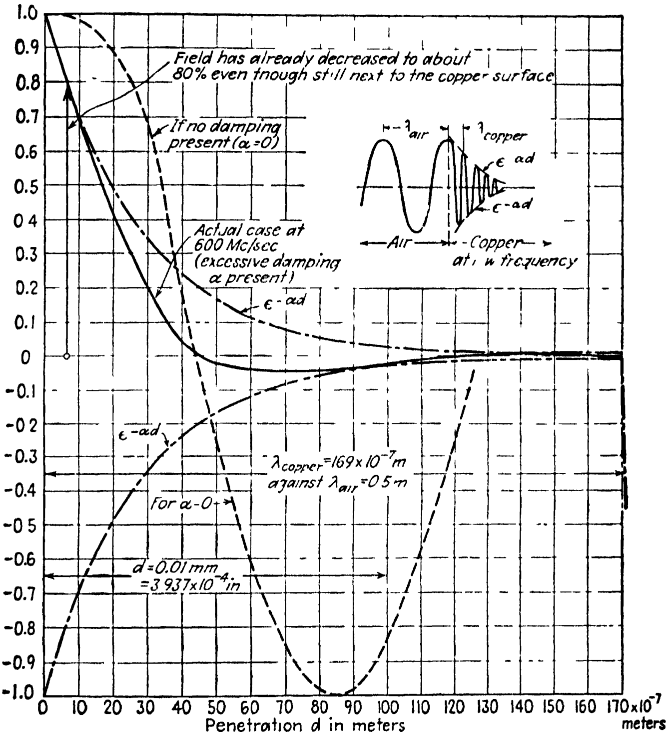


FIG. 93. Rapid field amplitude decay for an electromagnetic wave penetrating into a medium of copper at 600 Mc/sec (practically no penetration). The oscillations within the copper medium have a greatly reduced distance scale.

out of the square root. This implies that $\sqrt{-j} = (1/\sqrt{2}) - j(1/\sqrt{2})$, since this conversion gives a phase angle $\tan^{-1}(-\sqrt{2}/\sqrt{2}) = -45$ deg. We find

$$\Gamma = j\omega\sqrt{\frac{-j\mu\sigma}{\omega}} = j\omega\sqrt{-j}\sqrt{\frac{\mu\sigma}{\omega}} = \frac{j\omega(1-j)}{\sqrt{2}}\sqrt{\frac{\mu\sigma}{\omega}}$$

$$= \frac{j(1-j)}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

The propagation constant for a very good conducting medium, for $\mu = \mu_0$, is

$$\Gamma = \sqrt{0.5\omega\mu\sigma} + j\sqrt{0.5\omega\mu\sigma} = \alpha + j\beta; \qquad \alpha = \sqrt{0.5\omega\mu\sigma} = \beta \quad (724)$$

showing that the attenuation as well as the phase constant *within* very good metallic conductors are *numerically* equal.

Let us quickly compare this propagation constant with the value $\Gamma = j\omega\sqrt{\mu\kappa}$ holding for a pure dielectric medium next to conductors for TEM waves. Suppose, we have *air* for the medium in which the conductors are embedded. Then $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m and $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m. We have then in the dielectric space of the very good conducting coaxial line, a phase distribution given by $\Gamma = \alpha + j\beta_0 = j\omega\sqrt{\mu_0\kappa_0}$, since the attenuation constant $\alpha = 0$. For a 600-Mc TEM wave, this would correspond to a wavelength $\lambda_0 = 300/600 = 0.5$ m, since we have for such assumptions a phase velocity $c' = \omega/\beta_0 = \omega/(\omega\sqrt{\mu_0\kappa_0}) = 1/\sqrt{\mu_0\kappa_0} = 3 \times 10^8$ m/sec. Now suppose that a TEM wave travels along the Z direction, as given by the $\mathcal{E}_r$ and H_φ relations of (723) and that at the end of the coaxial line is a copper disk with a thickness of $\frac{1}{8}$ in. closing up the coaxial line. The TEM wave will attempt then to pierce its way into this disk. If the disk had infinite conductivity, both the attenuation constant α and phase constant β as given by (724) would also be infinite. Hence, the phase velocity $c' = \omega/\beta = \omega/\infty = 0$ m/sec and $\alpha = \infty$ nepers per meter. No wave energy could ever penetrate such a perfect electromagnetic obstacle. As we know, 100 per cent reflection would take place at the very surface of the disk. However, the specific conductivity is finite, and $\sigma = 5.8 \times 10^7$ mhos/m for copper. We find then for a 600-Mc TEM wave, arriving squarely at the disk, for the copper medium $\omega/2 = 6.28 \times 300 \times 10^6 = 1.884 \times 10^9$, $\omega\mu\sigma/2 = \omega\mu_0\sigma/2 = 1.884 \times 10^9 \times 1.257 \times 10^{-6} \times 5.8 \times 10^7 = 1.37 \times 10^{11}$, and $\alpha = \beta = \sqrt{13.7 \times 10^{10}} = 3.7 \times 10^5$. We have then only a phase velocity $c' = \omega/\beta = 3.768 \times 10^9/(3.7 \times 10^5) = 1.02 \times 10^4$ m/sec as against 3×10^8 m/sec before the wave energy enters the copper disk. The phase velocity in the copper medium is, therefore, only $1.02 \times 10^4/(3 \times 10^8) = 3.4 \times 10^{-5}$ of the value that it had before the TEM wave pierced the disk. This means, in other words, that the wavelength in the copper disk is only 3.4×10^{-5} of the wavelength along the concentric line. This is shown with an enlarged penetration-distance scale in Fig. 93 in order to show it at all, since the effect in copper compared with air is so pronounced. Not only the wavelength in the obstructing medium but also the attenuation α must be taken into account, since the purpose of the copper disk is to keep the electromagnetic field within the coaxial line. Since the attenuation constant α is numerically equal to the phase constant β in the above case for copper as the medium of wave propagation, we have now an attenuation constant as high as $\alpha = 3.7 \times 10^5$ nepers/m. This means that for a thickness of the disk of $d = \frac{1}{8}$ in. $= 2.54/8$ cm $= 2.54 \times 10^{-2}/8$ m or 3.175×10^{-3} m, a field amplitude $\mathcal{E}_0$, arriving at the inside of the disk, has a value of only $\mathcal{E}_0\epsilon^{-\alpha d}$. We find then $\alpha d = 3.7 \times 10^5 \times 3.175 \times 10^{-3} \cong 1.18 \times 10^3$ nepers and $\epsilon^{-1.18\times10^3} = \log_{10}^{-1}(-0.4343 \times 1.18 \times 10^3) = \log_{10}^{-1}(-511) = 1/[\log_{10}^{-1}(+511)]$. But $\log^{-1}$ of 511 is an inconceivably *large* numerical value, as can be realized from the following steps: We have $\log 10^1 = 1$, $\log 10^2 = 2$, $\log 10^3 = 3$, and $\log 10^{511} = 511$ with the inverse functions $\log^{-1}(1) = 10$, $\log^{-1}(2) = 100 = 10^2$, $\log^{-1}(3) = 1{,}000$, and $\log^{-1}(511) = 10^{511}$ which is an inconceivably large value, and therefore, $\epsilon^{-1.18\times10^3} = 1/[\log^{-1}(511)] \cong 0$, and also $\mathcal{E}_0\epsilon^{-1.18\times10^3} \cong 0$. The $\frac{1}{8}$-in. copper disk is, therefore, considerably thicker than needed for an electromagnetic barrier. If we call a field amplitude that has fallen in value to 1 per cent of the value it had before entering the copper disk satisfactory screening, then we have $\mathcal{E}_0\epsilon^{-4.6} = 0.01\mathcal{E}_0$, where $\mathcal{E}_0$ volts per meter is the electric field value when entering the disk. This would correspond to a thickness d in meters, given by $4.6 = \alpha d =$

$3.7 \times 10^5 d$ or $d = 4.6 \times 10^{-5}/3.7 = 1.24 \times 10^{-5}$ m $= 1.24 \times 10^{-3}$ cm $= 1.24 \times 10^{-3}/2.54$ in., *i.e.*, 4.9×10^{-4} in., which is a very thin skin of copper indeed. Now let us compare this result with the wavelength in copper. Since $\beta = 3.7 \times 10^5 = 6.28/\lambda$, we have $\lambda = 6.28/(3.7 \times 10^5) = 1.69 \times 10^{-5}$ m as against 0.5 m in air. This means that during the duration of 1 cycle of the 600-Mc wave, the wave would have moved through a distance of only 1.69×10^{-5} m instead of 0.5 m as in air. According to the attenuation effect computed above, already at a distance d of 1.24×10^{-5} m, the field amplitude has decreased to 1 per cent of its original value. Therefore, less than 1 wavelength can be developed in the disk before we have only 1 per cent of the original field amplitude. We can say that in less than 1 cycle's time, the wave is essentially extinguished. Table 21 gives the values for the amplitude decay in the disk

TABLE 21

Penetration d, m	αd, nepers	Amplitude factor $\epsilon^{-\alpha d}$	Per cent of original amplitude
0.0000000	0.0000000	1.000	100
2.7×10^{-7}	0.1	0.9	90
5.95×10^{-7}	0.22	0.8	80
9.7×10^{-7}	0.36	0.7	70
1.38×10^{-6}	0.51	0.6	60
1.86×10^{-6}	0.69	0.5	50
2.46×10^{-6}	0.915	0.4	40
3.25×10^{-6}	1.205	0.3	30
4.32×10^{-6}	1.6	0.2	20
6.22×10^{-6}	2.3	0.1	10
7×10^{-6}	2.59	0.075	7.5
8.1×10^{-6}	2.99	0.05	5
8.7×10^{-6}	3.22	0.04	4
9.5×10^{-6}	3.51	0.03	3
1.06×10^{-5}	3.91	0.02	2
1.24×10^{-5}	4.6	0.01	1

Remarks: $\alpha = \sqrt{\frac{1}{2}\omega\mu\sigma} = 3.7 \times 10^5$ nepers/m for $f = 600$ Mc/sec, $\omega = 6.28 \times 600 \times 10^6 = 3.768 \times 10^9$ radians/sec, $\sigma = 5.8 \times 10^7$ mhos/m, $\mu = \mu_0 = 1.257 \times 10^{-6}$ henry/m. 1 m = 3.281 ft = 39.37 in. holds for copper. Original amplitude is the amplitude at the surface of the copper.

Figure 93 and Table 21 show what happens at 600 Mc/sec in a medium of copper. It is seen that the amplitude falls off to more than 1 per cent of the surface value before the wave has advanced even 1 wavelength and that the penetration is practically zero. The scale for the penetration has been drawn very exaggeratedly in order to bring out these facts at all. Even the portion of the fractional wavelength shown is extremely distorted, for which reason the wavelength distribution is also given for the case of no attenuation. In the same figure, the penetration at a much lower frequency of an advancing wave is also given and again with the penetration scale well drawn out.

When dealing with transmission lines as well as wave guides, we are also interested in metallic absorption due to *tangential* components. The attenuation can then be found for transmission systems (feeders) from the tangential components of the electric and magnetic fields that are effective over the total conductive surface. If Z denotes the direction of energy transmission along the feeder α, the attenuation constant in nepers per meter, the $\mathcal{E}$ as well as the H field is attenuated according to an $\epsilon^{-\alpha z}$ function, causing a power diminution dP according to an $\epsilon^{-2\alpha z}$ law, and we have $dP/dz = -2\alpha P$, where $-dP/dz$ is the energy decrease per unit length of the line. This means that it represents $\int p\, dS$, where p denotes the real part of the average Poynting vector taken over the total metal surface per unit length. The factor 2 appears in $2\alpha z$, since in $p = \frac{1}{2}$ Re $(\mathcal{E} \times H^*)$, the electric field vector $\mathcal{E}$ attenuates according to an $\mathcal{E}_0\epsilon^{-\alpha z}$ law and the conjugate magnetic vector H^* according to an $H^*\epsilon^{-\alpha z}$ law. For the ideal line ($\sigma = \infty$), we have $\alpha = 0$. For the concentric line shown in Fig. 92, we have the radial $\mathcal{E}_r$ and the tangential H_φ field associations and the following details:

As is shown on page 210, when we are interested in magnitudes of complex quantities such as $a + jb = A\epsilon^{j\theta} = \underset{.}{A}$, we obtain the real absolute magnitude A of vector $\underset{.}{A}$ by extracting the square root out of the product $\underset{.}{A}\underset{.}{A}^*$, where $\underset{.}{A}^*$ stands for the *conjugate* vector $\underset{.}{A}^* = a - jb = A\epsilon^{-j\theta}$. We find then $\sqrt{(a + jb)(a - jb)} = A$, because $(a + jb)(a - jb) = a^2 + b^2 = A\epsilon^{j\theta}A\epsilon^{-j\theta} = A^2\epsilon^{j(\theta-\theta)} = A^2 = \underset{.}{A}\underset{.}{A}^*$. In these relations, $a = A \cos \theta$ and $b = A \sin \theta$, since a is along the real and b along the imaginary axis and vector $\underset{.}{A}$ is turned by an angle $+\theta$ and $\underset{.}{A}^*$ by an angle $-\theta$ out of the plus real direction. Now as far as the power transfer along the Z direction is concerned, the radial $\mathcal{E}_r$ and the tangential H_φ field associations of Fig. 92 give the Poynting vector $p = \frac{1}{2}$ Re $(\mathcal{E}_r H_\varphi^*)$ watts per square meter, since both $\mathcal{E}_r$ and H_φ are normal to each other and also normal to the direction of energy transfer. Since in the p expression, $\mathcal{E}_r$ and H_φ and, therefore, also H_φ^* denote the maximum field values, we have to use also the maximum values $\mathcal{E}_0/r$ and H_0/r, and we have, for the real component of the average Poynting vector, according to (210),

$$p = \frac{1}{2} \text{Re } (\mathcal{E}_r H_\varphi^*) = \frac{1}{2r^2} \mathcal{E}_0 \cdot H_0 = \frac{1}{2r^2}\sqrt{\frac{\mu}{\kappa}} H_0^2 \quad \text{watts/m}^2 \tag{725}$$

We could also have expressed this result in terms of the electric field amplitude $\mathcal{E}_0$ by writing $p = \frac{1}{2}\mathcal{E}_0^2\sqrt{(\kappa/\mu)}/r^2$. Expression (725) seems, however, more direct, since when applying p to the computation of the joulean heat losses in the respective conductors, we have to use the *tangential* components and H_0 is such a component for $r = \frac{1}{2}d$ and $\frac{1}{2}D$, respectively. Inasmuch as the time average value p, for the Poynting vector, stands for the field energy transfer in watts per unit area, the total energy transfer P in watts is the sum of all p effects taken over the cross-sectional area $S = \frac{1}{4}\pi(D^2 - d^2)$ square meters of the dielectric space of the concentric line. But according to (723), both $\mathcal{E}_r$ and H_φ, *i.e.*, also $\mathcal{E}_0$ and H_0, are inverse functions of the distance r of the field point from the axis of the concentric line. For this reason, the summation over the total area S is to be made by integration, using the p effects

on elementary areas $dS = 2\pi r\,dr$, with the solution

$$P = \int_{r=0.5d}^{r=0.5D} p\,dS = \int_{0.5d}^{0.5D} \frac{1}{2r^2}\sqrt{\frac{\mu}{\kappa}}\,H_0^2 2\pi r\,dr = \int_{0.5d}^{0.5D} \frac{\pi\sqrt{\mu/\kappa}}{r}\,H_0^2\,dr$$

$$= \pi\sqrt{\frac{\mu}{\kappa}}\,H_0^2 \left|\log_\epsilon r\right|_{r=0.5d}^{r=0.5D} = \pi\sqrt{\frac{\mu}{\kappa}}\,H_0^2 \underbrace{\left[\log_\epsilon\left(\frac{D}{2}\right) - \log_\epsilon\left(\frac{d}{2}\right)\right]}_{=\log_\epsilon (D/d)}$$

$$= \overbrace{2.3026\pi}^{=7.2}\sqrt{\frac{\mu}{\kappa}}\,H_0^2 \log_{10}\left(\frac{D}{d}\right) \quad \text{watts} \qquad (726)$$

If we use air as the dielectric of the coaxial line, then $\mu \cong \mu_0$, $\kappa \cong \kappa_0$, and $7.2\sqrt{\mu/\kappa} = 7.2 \times 376.7 = 2{,}734.4$. The total energy transfer is then

$$P = 2.734 \times 10^3 H_0^2 \log_{10}\left(\frac{D}{d}\right) \quad \text{watts} \qquad (727)$$

where H_0 is the amplitude value in amperes per meter, and we are dealing with the principal transverse electromagnetic energy propagation, which is possible for all frequencies (no cutoff limitation). Using a ratio $D/d = 3.6$, for which the concentric line gives minimum attenuation effect, we have $\log_{10} 3.6 = 0.5563$, and the total power transfer yields $1.52 \times 10^3 H_0^2$ watts. Hence, we have 1.52-kw power transfer per 1 amp/m magnetic field effect.

The attenuation α in nepers per meter length of the concentric line is found from the field energy transfer to the conductor surfaces 1 m along the Z direction. These surfaces are πD and πd square meters for the outside and the inside conductor, respectively, since π times the diameter is one side of this surface and unity the other side. Inasmuch as the field energy absorption due to the finite conductivity of the conductor is computed by the time average value of the Poynting vector that is normal to the conductor surface, the tangential components of $\mathcal{E}$ and H have to be used. According to Fig. 92, H_φ is tangent to conductor surfaces for $r = 0.5d$ and $0.5D$. For a finite conductivity σ mhos per meter of the conductors, we have, in addition to the radial $\mathcal{E}_r$ component shown in the figure, also a small tangential component, since the resultant $\mathcal{E}$ vector is somewhat less than 90 space degrees with respect to the conductor surfaces. The tangential electric field component is $Z_0 H_{0.5D}$ and $Z_0 H_{0.5d}$, respectively, where Z_0 is the ratio of the tangential components of the electric and magnetic fields at respective conductor surfaces facing the field space of the coaxial line. In this case, Z_0 is *not* the ratio $\sqrt{\mu/\kappa}$ ohms, but $\sqrt{\mu/\kappa_e}$ complex ohms, since the conductor has pronounced conductivity $\sigma = 5.8 \times 10^7$ mhos/m in the case of copper, for instance,

and the equivalent dielectric constant $\kappa_e = \kappa - j\sigma/\omega$ has to be used. As was shown in connection with the propagation constant [formulas (724)], the actual κ effect is negligible in comparison with the $j\sigma/\omega$ effect, and we find

$$Z_0 = \sqrt{\frac{\mu}{\kappa_e}} \cong \sqrt{\frac{\mu}{-j\sigma/\omega}} = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}} + j\sqrt{\frac{\omega\mu}{2\sigma}} \quad \text{complex ohms} \quad (728)$$

It is the *real* component of Z_0 that causes the joulean heat losses, and the Poynting vectors for the tangential components become

$$p = \tfrac{1}{2}\,\mathrm{Re}\,(\mathcal{E} \times H) = \tfrac{1}{2}\,\mathrm{Re}\,(Z_0 H_{0.5d} H^*_{0.5d}) \quad \text{watts/m}^2 \text{ for the inner conductor}$$
$$p = \tfrac{1}{2}\,\mathrm{Re}\,(Z_0 H_{0.5D} H^*_{0.5D}) \quad \text{watts/m}^2 \text{ for the outer conductor}$$
$$\mathrm{Re}\,(Z_0) = \sqrt{\frac{\omega\mu}{2\sigma}} \quad \text{ohms}$$

But $H_{0.5d} = H_0/d$ and $H_{0.5D} = H_0/D$. Therefore,

$$\left.\begin{aligned} p &= \sqrt{\frac{\omega\mu}{2\sigma}}\,\frac{H_0^2}{d^2} \quad \text{watts/m}^2 \text{ for the inner conductor} \\ p &= \sqrt{\frac{\omega\mu}{2\sigma}}\,\frac{H_0^2}{D^2} \quad \text{watts/m}^2 \text{ for the outer conductor} \end{aligned}\right\} \quad (729)$$

It is, therefore, necessary only to multiply these values with respective surfaces πd and πD square meters per meter length, in order to obtain the *total* conductor loss P_c in watts. We then find

$$P_c = \pi d\,\frac{H_0^2}{d^2}\sqrt{\frac{\omega\mu}{2\sigma}} + \pi D\,\frac{H_0^2}{D^2}\sqrt{\frac{\omega\mu}{2\sigma}} = \pi\left(\frac{1}{d} + \frac{1}{D}\right) H_0^2 \sqrt{\frac{\omega\mu}{2\sigma}} \quad \text{watts per meter length} \quad (730)$$

where the diameters d and D are in meters.

Inasmuch as the transverse electromagnetic field energy P [Eq. (727)] is the outcome of the transverse $\mathcal{E}_r$ and H_φ relations given in Eq. (724), we have, as already discussed on page 136, an exponential energy decrease along the coaxial line, according to $\epsilon^{-2\alpha}$, defined by

$$\left.\begin{aligned} &\text{Power dissipated per meter distance} = -\frac{dP}{dz} = 2\alpha P \\ &\text{Since } \frac{dy}{dz} + ny = 0 \text{ for } n = 2\alpha;\ y = P, \text{ we have} \\ &y = A\epsilon^{-nz} \quad \text{and} \quad P = P_0\epsilon^{-2\alpha z} \end{aligned}\right\} \quad (731)$$

The result of (730) gives the power lost per unit distance as P_c, since there is no radiation loss in a closed concentric line, and the only field energy subtracted from the useful field power traveling along the Z direction

can be contributed to the copper losses (generally to the conductor dissipation). Substituting P_c for $-dP/dz$ in (731) and solving for α yield

$$\alpha = \frac{P_c}{2P} = \frac{\pi\left(\frac{1}{d}+\frac{1}{D}\right)H_0^2\sqrt{\frac{\omega\mu_1}{2\sigma}}}{2 \times 2.303\pi H_0^2\sqrt{\frac{\mu_2}{\kappa}}\log_{10}\left(\frac{D}{d}\right)} = \frac{\left(\frac{1}{d}+\frac{1}{D}\right)\sqrt{\frac{\omega\mu_1}{\sigma}}}{6.5\sqrt{\frac{\mu_2}{\kappa}}\log_{10}\left(\frac{D}{d}\right)} \quad \text{nepers/m} \qquad (732)$$

where μ_2 henrys per meter and κ farads per meter signify the properties of the pure dielectric medium through which the principal transverse electromagnetic field power is propagated, while μ_1 is the permeability of the respective conductors. In most cases, we use conductors that are not ferromagnetic and have $\mu_1 = \mu_0 = 1.257 \times 10^{-6}$ henry/m. For the most useful mediums through which the field energy passes down the concentric line, we have likewise $\mu_2 = \mu_0$, simplifying the above expression to

$$\alpha = \frac{[(1/d) + (1/D)]\sqrt{\omega\kappa\sigma}}{6.5\log_{10}(D/d)} \quad \text{nepers/m}$$

which, for the frequency f in megacycles per second and d and D in meters, reads

$$\alpha = 386\frac{(1/d) + (1/D)}{\log_{10}(D/d)}\sqrt{\frac{f\kappa}{\sigma}} \quad \text{nepers/m} \qquad (732a)$$

where σ is the conductivity in mhos per meter for the conductors and κ in farads per meter is the dielectric constant of the dielectric between the conducting surfaces. If we use air as the dielectric and copper conductors, then $\kappa \cong \kappa_0 = 8.854 \times 10^{-12}$ farad/m and $\sigma = 5.8 \times 10^7$ mhos/m, simplifying the above relation to

$$\alpha = 1.51 \times 10^{-6}\sqrt{f}\frac{(1/d) + (1/D)}{\log_{10}(D/d)} \quad \text{nepers/m} \qquad (732b)$$

If we choose the optimum ratio $D/d = 3.6$, we have $\log(D/d) = 0.5563$ and the formulas

$$\alpha = 2.72 \times 10^{-6}\sqrt{f}\left(\frac{1}{d}+\frac{1}{D}\right) = 2.72 \times 10^{-6}\frac{\sqrt{f}}{d}\left(1+\frac{1}{3.6}\right) \quad \text{nepers/m} \qquad (732c)$$

showing that the inner conductor with the outside diameter d causes an attenuation of $2.72 \times 10^{-6}\sqrt{f}/d$ nepers per meter, while the outside conductor of inside diameter D meters causes an attenuation that is only $100/3.6 = 27.8$ per cent of the inside conductor surface attenuation.

In all formulas for α, the attenuation loss is mostly due to the inside conductor and all the more so the larger the ratio D/d. This is evident, since the respective tangential fields around the inside conductor surface are much stronger (H_0^2/d^2 is operative) than the tangential fields around the active surface of the outside conductor (H_0^2/D^2 is operative). The ratio $D/d = 3.6$ happens also to give a ratio $E/I = \sqrt{L/C}$ of about 74 ohms, which is a characteristic impedance often used. By E/I is then meant the voltage-to-current ratio of an advancing wavelike disturbance.

At this point, we shall briefly call attention to customary transmission formulas, where we usually employ fundamental expressions in terms of a unit length section (1 m length of go and return conductors) and have for instance,

$$\left.\begin{aligned}
C^{(\text{farads/m})} &= \frac{\pi\kappa}{\log_\epsilon (q)} && \text{for } \textit{parallel}\text{-wire line; } q = 2D/d;\ d \text{ is diameter of wires; } D \text{ is spacing between centers of wires}\\
C^{(\text{farads/m})} &= \frac{2\pi\kappa}{\log_\epsilon (q)} && \text{for } \textit{concentric} \text{ line; } q = D/d;\ d \text{ is } \textit{outside} \text{ diameter of } \textit{inside} \text{ conductor; } D \text{ is } \textit{inside} \text{ diameter of } \textit{outside} \text{ conductor}\\
L^{(\text{henrys/m})} &= \frac{\mu}{\pi} \log_\epsilon (q) && \text{for } \textit{parallel} \text{ wires}\\
L^{(\text{henrys/m})} &= \frac{\mu}{2\pi} \log_\epsilon (q) && \text{for } \textit{concentric} \text{ line}
\end{aligned}\right\} \quad (733)$$

The parallel-wire expressions assume that the diameter d of the wires is small compared with the spacing D between centers of go and return wires. The formulas are the well-known static solutions, which are given in more detail in Fig. 63 of Sec. 74. Employing the well-known transmission-line relations $v = 1/\sqrt{CL}$ meters per second for the phase velocity along the HF line and $Z_0 = \sqrt{L/C}$ ohms for the characteristic impedance of the HF line, we find

$$\left.\begin{aligned}
Z_0^{(\text{ohms})} &= \sqrt{\frac{\mu}{\kappa}}\frac{\log_\epsilon (q)}{\pi} && \text{for parallel-wire line}\\
Z_0^{(\text{ohms})} &= \sqrt{\frac{\mu}{\kappa}}\frac{\log_\epsilon (q)}{2\pi} && \text{for concentric line}\\
v &= \frac{1}{\sqrt{\mu\kappa}} \ \text{m/sec} && \text{for parallel-wire and concentric line and TEM waves}
\end{aligned}\right\} \quad (734)$$

We note that the lumped-line constants C and L per meter length, when applied to the $v = 1/\sqrt{CL}$ formula holding for no attenuation ($\alpha = 0$),

confirm the field velocity $v = 1/\sqrt{\mu\kappa}$ for both the parallel-wire system and the concentric line, even though the formulas for C and L are not based on Maxwell's field relations and the boundary conditions to be satisfied. *Neither* for the parallel-wire line nor for the concentric line do the expressions for the characteristic line impedance Z_0, holding for the ratio of $E^{(\text{volts})}/I^{(\text{amp})}$ of an advancing or reflected wave train, confirm the ratio $\sqrt{\mu/\kappa}$ ohms, which would be expected from the electromagnetic field theory in the case of TEM waves where there is no attenuation along the line. Therefore, the characteristic impedance formulas of (734) check only the E/I relation as defined above and can be called the "line-guided" relations for the principal TEM wave in case of zero attenuation along the line.

Relations (723) for the radial $\mathcal{E}_r$ electric field component and the tangential H_φ magnetic field association can also be used to account for the voltage difference across the line and the go and return currents. We find then the voltage E volts across a concentric line by integration of the corresponding electric field $\mathcal{E}_r$, which is in volts per meter, yielding the solution

$$E^{(\text{volts})} = \int_{r=0.5d}^{r=0.5D} \frac{\mathcal{E}_0 \epsilon^{j\omega t - z\Gamma}}{r}\, dr = \mathcal{E}_0 \epsilon^{j\omega t - z\Gamma} \Big|\log_\epsilon (r)\Big|_{0.5d}^{0.5D}$$
$$= \mathcal{E}_0 \epsilon^{j\omega t - z\Gamma}\left[\log_\epsilon\left(\frac{D}{2}\right) - \log_\epsilon\left(\frac{d}{2}\right)\right] = 2.303\mathcal{E}_0 \epsilon^{j\omega t - z\Gamma} \log_{10}\left(\frac{D}{d}\right) \quad (735)$$

Inasmuch as the current I in the go and the current $-I$ in the return conductor are in amperes, we likewise find, upon integration of the H component, the surface current acting around the $r = 0.5d$ and $r = 0.5D$ circumferences. We find, then, since H_φ is constant around the circular path $2\pi r$,

$$I^{(\text{amp})} = \frac{H_0 \epsilon^{j\omega t - z\Gamma}}{r} 2\pi r = 6.28 H_0 \epsilon^{j\omega t - z\Gamma} \quad (736)$$

proving that the outer surface of the inner conductor and the inner surface of the outer conductor carry the same currents (positive and negative instantaneous polarity), *since* the radius r *cancels* out. The integration is actually the outcome of $\int H_\varphi r\, d\varphi$, where $r\, d\varphi$ is a line element of the circular path of radius r, and the integration is, on account H_φ = constant, along such a path

$$\int_{\varphi=0}^{\varphi=2\pi} H_\varphi r\, d\varphi = r H_\varphi \int_0^{2\pi} d\varphi = 2\pi r H_\varphi = 2\pi r \frac{H_0 \epsilon^{j\omega t - z\Gamma}}{r} = 6.28 H_0 \epsilon^{j\omega t - z\Gamma}$$

With respect to the quality Q of a transmission line, we have the following details:

The propagation constant, as found by the classic line theory for R ohms per meter, L henrys per meter, G mhos per meter, and C farads per meter, is

$$\Gamma = \alpha + j\beta = \pm\sqrt{(R + j\omega L)(G + j\omega C)} = \pm\sqrt{j\omega L\frac{R + j\omega L}{j\omega L}\, j\omega C\frac{G + j\omega C}{j\omega C}}$$

$$= \pm j\omega\sqrt{CL}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} = j\omega\sqrt{CL}\,(1 + a_1)^{\frac{1}{2}}(1 + a_2)^{\frac{1}{2}}$$

$$a_1 = \frac{R}{j\omega L}; \qquad a_2 = \frac{G}{j\omega C}$$

Since in most well-designed HF feeders, the quantities a_1 and a_2 are small compared with unity, we can apply the approximation $(1 + a)^{\frac{1}{2}} = 1 + \frac{1}{2}a$ and find

$$\alpha + j\beta = \pm j\omega\sqrt{CL}\left(1 + \frac{R}{j2\omega L}\right)\left(1 + \frac{G}{j2\omega C}\right)$$

$$= \pm j\omega\sqrt{CL}\left(1 + \frac{R}{j2\omega L} + \frac{G}{j2\omega C} - \frac{GR}{4\omega^2 CL}\right)$$

$$= \pm\omega\sqrt{CL}\left[\frac{R}{2\omega L} + \frac{G}{2\omega C} + j\left(1 - \frac{GR}{4\omega^2 CL}\right)\right]$$

$$\alpha = \sqrt{CL}\left(\frac{R}{2L} + \frac{G}{2C}\right) = 0.5R\sqrt{\frac{C}{L}} + 0.5G\sqrt{\frac{L}{C}} \qquad \text{nepers/m}$$

$$\beta = \omega\sqrt{CL} - \frac{GR}{4\omega\sqrt{CL}} \qquad \text{circular radians/m}$$

Neglecting the effect of G mhos per meter across the unit length of the line, we find

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad \text{nepers/m}; \qquad \beta = \omega\sqrt{CL} \qquad \text{circular radians/m};$$

$$\frac{\beta}{\alpha} = \frac{2\omega L}{R} = 2Q$$

From the foregoing derivation, we note that the Q value of a transmission line that is well designed is, at high frequencies, given by

$$Q = \frac{\beta}{2\alpha} \qquad \text{numeric} \tag{737}$$

Applying this result to a concentric line, we have α as in (732) for the *general case* and $\beta = \omega\sqrt{\mu_2\kappa}$, where μ_2 and κ give the properties of the pure dielectric medium between the conductors of conductivity σ and permeability μ_1. We then find

$$Q = \frac{\omega \sqrt{\mu_2 \kappa}}{2 \dfrac{[(1/d) + (1/D)] \sqrt{\omega\mu_1/\sigma}}{6.5 \sqrt{\mu_2/\kappa} \log_{10} (D/d)}} = 3.25 \frac{\mu_2 \sqrt{\omega\sigma/\mu_1}}{(1/d) + (1/D)} \log_{10} (D/d) \tag{738}$$

Inasmuch as for air as the dielectric, we have $\mu_2 = \mu_0$ and for nonferromagnetic conductors also $\mu_1 = \mu_0 = 1.257 \times 10^{-6}$ henry/m. and if we express the frequency in megacycles per second, we find

$$Q = 3.25 \frac{\sqrt{6.28 \times 10^6 f \times 1.257 \times 10^{-6} \sigma}}{(1/d) + (1/D)} \log_{10} \left(\frac{D}{d}\right) = 9.12 \frac{\sqrt{f\sigma} \log_{10} (D/d)}{(1/d) + (1/D)} \tag{739}$$

For copper conductors, we have $\sigma = 5.8 \times 10^7$ mhos/m, and

$$Q = 6.93 \times 10^4 \frac{\log_{10} (D/d)}{(1/d) + (1/D)} \sqrt{f} = 6.93 \times 10^4 \frac{dq \log_{10} (q)}{1 + q} \sqrt{f} \tag{740}$$

Both (739) and (740) show that the quality Q of the concentric line increases with the square root of the frequency and is, for a fixed ratio $q = D/d$, also directly proportional to the diameter d of the inside conductor. This has already been indicated in the expression for the attenuation constant α, from which it is seen that the smaller the inner conductor the larger is the α value, *i.e.*, the neper loss per unit distance. For $q = 3.6$, we have then

$$Q = 6.93 \times 10^4 \frac{3.6 \times 0.5563 d}{4.6} \sqrt{f} = 3.02 \times 10^4 d \sqrt{f} \tag{741}$$

where the outside diameter d of the inside conductor is in meters and f is in megacycles per second. For such an ideal optimum condition, $\alpha = \alpha_{min}$, we would have for an AWG No. 14 wire as the inside conductor, $d = 0.0641$ in. $= 2.54 \times 0.0641$ cm $= 0.0641 \times 2.54 \times 10^{-2}$ m $= 1.628 \times 10^{-3}$ m. We have then a quality of the concentric line of

$$Q = 1.628 \times 10^{-3} \times 3.02 \times 10^4 \sqrt{f} = 49.2 \sqrt{f^{(Mc)}} \tag{742}$$

We see from this result that for the ratio $q = D/d = 3.6$, which holds for the smallest attenuation (highest Q value), and a No. 14 inside copper wire at 100 Mc/sec, we have $Q = 49.2 \times 10 = 492$ while at 200 Mc/sec we have only 1.41 times this value, which is $Q = 732$ in the case of the principal wave.

From the discussion of this section, we note that magnetic H fields due to electric currents can be derived from a magnetic vector potential A volt · seconds per meter,

which, for a current parallel to the Z axis, depends on the current function $\psi = \Lambda_z/\mu$, where ψ amperes is a scalar as defined by (716). We have then

$$H_x = \frac{\partial\psi}{\partial y}; \quad H_y = -\frac{\partial\psi}{\partial x}; \quad H_r = \frac{\partial\psi}{r\,\partial\varphi}; \quad H_\varphi = -\frac{\partial\psi}{\partial r} \quad \text{amp/m} \tag{743}$$

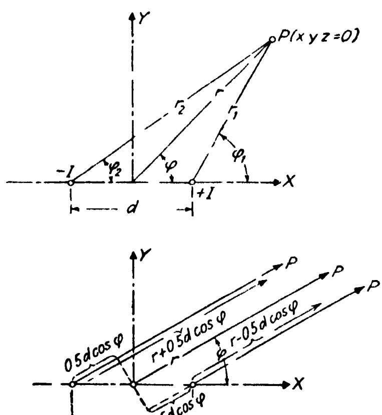
where ψ, in regions free of sources, satisfies the two-dimensional Laplace relations

$$\underbrace{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}}_{\text{Cartesian coordinates}} = 0; \qquad \underbrace{r\frac{\partial\left(r\dfrac{\partial V}{\partial r}\right)}{\partial r} + \frac{\partial^2 V}{\partial\varphi^2}}_{\text{cylinder coordinates}} = 0 \tag{744}$$

since $\nabla^2 V = 0$ for such conditions and $\mathcal{E} = -\text{grad } V$. Hence,

$$\underbrace{\mathcal{E}_x = -\frac{\partial V}{\partial x}; \quad \mathcal{E}_y = -\frac{\partial V}{\partial y}}_{\text{for Cartesian coordinates}} \quad \text{and} \quad \underbrace{\mathcal{E}_r = -\frac{\partial V}{\partial r}; \quad \mathcal{E}_\varphi = -\frac{\partial V}{r\,\partial\varphi}}_{\text{for cylinder coordinates}} \tag{745}$$

which would apply also to a steady current flow. On pages 301, 471, we have already seen the usefulness of the concept of the current function ψ; and in this section, we



Case where P is so close that r_1, r, and r_2 are converging to field point P

Case where P is very far away compared with spacing d and r_1, r, and r_2 are practically parallel
$\varphi = \varphi_1 = \varphi_2$
$P(x, y, z = 0)$ is distant field point

FIG. 94. Infinitely long parallel-wire system along the Z direction.

have already made use of the relation $H_\varphi = I/(2\pi r)$, where H_φ amperes per meter is the fixed tangential magnetic field vector along the circular path $2\pi r$ meters in the *immediate* neighborhood of a thin electric-current filament of I amperes, since $H_r = 0$. Inasmuch as for a steady field there can be no displacement current parallel to the current filament, $I = 2\pi r H_\varphi$ amperes applies for any r value from the I filament besides in the immediate neighborhood of I.

Introducing the relation $H_\varphi = I/(2\pi r)$ and $H_r = 0$ into (745), we find

$$\frac{\partial\psi}{\partial r} = -H_\varphi = -\frac{I}{2\pi r}; \qquad \psi = -\frac{I}{2\pi}\int\frac{dr}{r} = -\frac{I}{2\pi}[\log_\epsilon(r) + K]$$
$$= -\frac{I}{2\pi}[2.303\log_{10}(r) + K] \qquad (746)$$

for the integration constant K. Therefore, when we have, as in Fig. 94, an infinitely long parallel-wire line spaced a distance d meters between centers, we have, for a field point $P(x, y)$ in the XY plane a distance r_1 from the $+I$ and r_2 from the $-I$ conductor, only to add up the scalar current function contributions ψ_1 and ψ_2 due to the $+I$ and to the $-I$ effect. We find then for the current functions ψ_1 and ψ_2 due to $+I$ and $-I$ conductors

$$\psi = \psi_1 + \psi_2 = \overbrace{-\frac{I}{2\pi}[\log_\epsilon(r_1) + K]}^{\text{due to } +I} + \overbrace{\frac{I}{2\pi}[\log_\epsilon(r_2) + K]}^{\text{due to } -I}$$
$$= \frac{I}{2\pi}[\log_\epsilon(r_2) - \log_\epsilon(r_1)] = \frac{I}{6.28}\log_\epsilon\left(\frac{r_2}{r_1}\right) \qquad \text{amp} \qquad (747)$$

If we take field points in the XY plane or planes parallel thereto so that the distances r_1 and r_2 are large in comparison with the spacing d, then we have the following details:

For such a condition r_1, r, and r_2 are practically parallel and $r_1 = r - \frac{1}{2}d\cos\varphi$ and $r_2 = r + \frac{1}{2}d\cos\varphi$. Therefore,

$$\psi = \frac{I}{6.28}\log_\epsilon\left(\frac{r_2}{r_1}\right) = \frac{I}{6.28}\log_\epsilon\left(\frac{r + 0.5d\cos\varphi}{r - 0.5d\cos\varphi}\right) = \frac{I}{6.28}\log_\epsilon\left[\frac{1 + (0.5d\cos\varphi)/r}{1 - (0.5d\cos\varphi)r}\right]$$
$$= \frac{I}{6.28}\log_\epsilon\left(\frac{1+a}{1-a}\right) \qquad \text{amp}$$

Since $a = (0.5d\cos\varphi)/r$ is very small compared with unity, we can use the approximation formula $\log_\epsilon(1 \pm a) = \pm a - \frac{1}{2}a^2$ and find

$$\psi = \frac{I}{6.28}\log_\epsilon\left(\frac{1+a}{1-a}\right) = \frac{I}{6.28}[\log_\epsilon(1+a) - \log_\epsilon(1-a)]$$
$$= \frac{I}{6.28}\left[a - \frac{a^2}{2} - \left(-a - \frac{a^2}{2}\right)\right] = \frac{I}{2\pi}2a = \frac{Id\cos\varphi}{6.28r} \qquad \text{amp} \qquad (748)$$

which in (743) yields

$$H_r = \frac{1}{r}\frac{\partial\psi}{\partial\varphi} = -\frac{Id\sin\varphi}{6.28r^2}; \qquad H_\varphi = -\frac{\partial\psi}{\partial r} = +\frac{Id\cos\varphi}{6.28r^2} \qquad \text{amp/m} \qquad (749)$$

The field can also be found by means of the magnetic field relation $H_\varphi = I/(2\pi r)$, but vectorial addition of the H contributions must be used, since H is a vector and not a scalar like ψ. It is understood that for Fig. 94, we deal with sustained parallel currents $\pm I$ amperes. Also the two-dimensional Poisson equation for the total

current density J amperes per square meter applies, from which the magnetic field energy P_m per unit length along the Z axis can be found. We have then

$$\left.\begin{aligned}
\nabla^2\psi &= \underbrace{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}}_{\text{Cartesian coordinates}} = \underbrace{r\frac{\partial\left(r\frac{\partial\psi}{\partial r}\right)}{\partial r} + \frac{\partial^2\psi}{\partial\varphi^2}}_{\text{cylinder coordinates}} = -J \qquad \text{amp/m}^2 \\
P_m &= 0.5\mu \int\!\!\int_{xy\ \text{area}} (H_x^2 + H_y^2)\,dx\,dy = 0.5\mu \int\!\!\int_S \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] dS \\
&\qquad \text{Green theorem for } l \text{ as boundary of area } S \text{ normal to } Z \text{ axis } n \text{ is normal} \\
&= 0.5\mu\left[\int_l \psi \frac{\partial\psi}{\partial n}\,dl - \int\!\!\int_S \psi\,\nabla^2\psi\,dS\right] \qquad \text{joules/m along } Z \text{ axis}
\end{aligned}\right\} \tag{750}$$

where the integration normally extends over an area S filled with current.

Now the basic solution (747) for the parallel-wire line shown in Fig. 91 denotes the current function ψ amperes for sustained currents $\pm I$ amperes in respective parallel current filaments. However, the scalar potential and current functions V volts and ψ amperes solve the two-dimensional Laplace equation $\nabla^2 V = 0 = \nabla^2\psi$. Therefore, such solutions relate also to TEM waves, since we have for the general wave medium μ, κ, and σ

$$V = -\frac{\partial\psi/\partial z}{\sigma + j\omega\kappa} \quad \text{volts}; \qquad \psi = +j\frac{\partial V/\partial z}{\omega\mu} \quad \text{amp} \tag{751}$$

For a pure dielectric medium, we have $\mu, \kappa, \sigma = 0$ and, according to (717),

$$\kappa\,\partial V/\partial t = -\partial\psi/\partial z; \qquad \mu\,\partial\psi/\partial t = -\partial V/\partial z; \qquad \text{or}$$
$$j\omega\kappa V = -\partial\psi/\partial z; \qquad j\omega\mu\psi = -\partial V/\partial z$$

and

$$V = -\frac{\partial\psi/\partial z}{j\omega\kappa}; \qquad \psi = -\frac{\partial V/\partial z}{j\omega\mu} = +j\frac{\partial V/\partial z}{\omega\mu}$$

because for the general medium we have $\sigma + j\omega\kappa$ in place of only $j\omega\kappa$. This is evident when we compare the corresponding field relations

$$\mathcal{E}_x = -\frac{1}{\sigma + j\omega\kappa}\frac{\partial H_y}{\partial z}; \qquad \mathcal{E}_y = +\frac{1}{\sigma + j\omega\kappa}\frac{\partial H_x}{\partial z}; \qquad H_x = \frac{1}{j\omega\mu}\frac{\partial \mathcal{E}_y}{\partial z};$$
$$H_y = -\frac{1}{j\omega\mu}\frac{\partial \mathcal{E}_x}{\partial z} \quad \text{for medium } \mu, \kappa, \text{ and } \sigma$$

$$\mathcal{E}_x = -\frac{1}{j\omega\kappa}\frac{\partial H_y}{\partial z}; \qquad \mathcal{E}_y = +\frac{1}{j\omega\kappa}\frac{\partial H_x}{\partial z}; \qquad H_x = \frac{1}{j\omega\mu}\frac{\partial \mathcal{E}_y}{\partial z};$$
$$H_y = -\frac{1}{j\omega\mu}\frac{\partial \mathcal{E}_x}{\partial z} \quad \text{for dielectric medium } \mu, \kappa, \sigma = 0$$

for TEM waves with $\mathcal{E}_z = 0 = H_z$. Applying relations (747) to (751) in connection with Fig. 94, we find

$$\frac{\partial V}{\partial z} = -j\omega\mu\psi = -j6.28 \times 10^6 f\mu I \log_\epsilon \left(\frac{r_2}{r_1}\right);$$

$$V = -\frac{\partial\psi/\partial z}{\sigma + j10^6\omega\kappa} = -\frac{\log_\epsilon (r_2/r_1)}{6.28(\sigma + j10^6\omega\kappa)}\frac{\partial I}{\partial z} \quad \text{volts} \quad (752)$$

for $\omega/6.28 = f$ in megacycles per second.

94. Notes on Radiation from Transmission Lines. This has reference to at least two conductors (go and return), the case of a single aperiodic feeder being discussed in Sec. 90. Customary transmission lines, such as the parallel-wire system and the coaxial lines, driven by an applied HF voltage cause TEM waves down the line with no restrictions as to frequency. The spacing between the go and the return conductor is then, as a rule, small compared with the operating wavelength. On account of the spacing, we always shall have a small electromagnetic field left, since the fields due to each conductor do not completely neutralize each other. At high frequencies, there will be no external field for a concentric line closed at either end with a metal disk. Such a case can exist for a parallel-wire system only when the go and the return lines coincide, and the line can no longer function as a means of power transfer by means of a go and a return. It may be said that customary concentric lines which are not closed by good conducting disks have a smaller radiation effect than a parallel-wire system. The disk at the far end of the line closes the current path by means of radial currents flowing in the disk. Hence, for an open-ended concentric line as well as a line in which the inside conductor produces out of the outside cylinder, the fields at the open end bulge out and cause radiation effects as well as some current to spill around the open end. This will also happen to a smaller extent with a matched load unless a quarter-wavelength sleeve is used to prevent such action.

Transmission lines are used with a load that closes the line. The power radiated depends on the distribution of the current and the charge along every portion of the line, and it is the combined effect in proper phase addition of all portions at a certain field point that counts

Customary transmission-line theory cannot account for radiation effects, since the relations of the voltage E and current I represent simple exponentially propagated waves, which depend only on the propagation constant $\Gamma = \sqrt{YZ}$ and the characteristic impedance $Z_0 = \sqrt{Z/Y}$, for $Z = R + j\omega L$ and $Y = G + j\omega C$. The quantities L and R denote the self-inductance and resistance per unit length of the line (go and return), and G and C the conductivity and capacitance per unit length across the line. The current is not attenuated by radiation, since the energy losses due to radiation appear as terminal corrections (reflection effects). The generator delivers,

therefore, the power transferred to the load and dissipated by the line, which is accounted for by customary transmission-line relations. Since the theory is not a complete solution, it ignores the radiation energy that comes likewise from the generator connected to the line. This can be realized from the following: When a parallel-wire system of a definite length feeds a load, the power transfer does not depend on the geographical direction of the different portions of the feeder as long as the line preserves uniformity as defined by the characteristic impedance Z_0 and the propagation constant Γ. However, with respect to radiation effects, a linear feeder behaves differently from the same feeder when of L shape (two-linear perpendicular-line sections). Generally, if at any point of a feeder, the spacing between the go and return conductors is changed, the radiation effect is also modified thereby, just as the geographical orientation of the line has a modifying effect.

Deriving the line equations by means of the electric and magnetic field vectors (Maxwell's equations) requires for boundary conditions that $\mathcal{E}$ and H be everywhere continuous throughout boundary planes. The solution of the wave equation requires, therefore, that the boundary conditions at the surfaces of the conductors be satisfied. This requires the continuity of the tangential components of $\mathcal{E}$ and H at said surfaces.[1] Comparing customary line theory with the Maxwellian theory, we have for the wave equation

$$\left.\begin{aligned}
&\left.\begin{aligned}
CL\frac{\partial^2 F}{\partial t^2} + (CR + LG)\frac{\partial F}{\partial t} + GRF &= \frac{\partial^2 F}{\partial z^2};\\
F = E \text{ volts} &= I \text{ amp}
\end{aligned}\right\}\ \text{transmission-line equation}\\
&\left.\begin{aligned}
&\kappa\mu\frac{\partial^2 F}{\partial t^2} + \mu\sigma\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\\
&F = \mathcal{E} \text{ volts/m} = H \text{ amp/m}\\
&\qquad = D \text{ coulombs/m}^2 = B \text{ weber/m}^2\\
&\quad = \Lambda_m \text{ volt}\cdot\text{sec/m} = \Lambda_e \text{ amp}\cdot\text{sec/m}\\
&\qquad\qquad = \prod \text{m}\cdot\text{volt}
\end{aligned}\right\}\ \text{electromagnetic field equation}
\end{aligned}\right\}\quad (753)$$

where the medium constants are μ henrys per meter, κ farads per meter, and σ mhos per meter and apply to the conductor as well as the medium about the conductor with magnitudes depending upon the medium. For air about the conductors, we have practically $\mu = \mu_0$, $\kappa = \kappa_0$, and $\sigma = 0$; and for copper conductors, the electromagnetic properties are $\mu = \mu_0$, $\kappa = \kappa_0$, and $\sigma = 5.8 \times 10^7$ mhos/m. It is seen that the trans-

[1] J. R. Carson (*Bell Sys. Tech. Jour.*, **7**, 268–280, April, 1928) has solved the wave equation in the general form for N parallel conductors. He found that in addition to N principal modes of propagation, an N-fold infinity of other modes of propagation exists. The latter modes are complementary and differ from the principal waves, since they are not quasi-plane and are rapidly attenuated. As regards currents and charges on the line as well as the near fields, the effect of the complementary waves can play an important part only in the vicinity of physical terminals. The *complementary waves must be used*, however, *to account for radiation.*

mission-line equation satisfies the *voltage* as well as the *current* at any point *on* the *line*. For the electromagnetic field equation, the relation is satisfied by the electric field, magnetic field, electric flux density, magnetic vector potential, and the Hertz[1] vector.

95. Review of the Carson Transmission-line Theory Using the Sommerfeld Integral. If, in the electromagnetic field equation of (753), we use a harmonic field excitation, we have the form $F = F_0\epsilon^{j\omega t}$ and find for the derivatives $\partial F/\partial t = j\omega F$ and $\partial^2 F/\partial t^2 = -\omega^2 F$. The wave equation then reads

$$\underbrace{(j\omega\mu\sigma - \omega^2\mu\kappa)}_{\Gamma^2}F = \underbrace{\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}}_{\nabla^2 F} \tag{754}$$

Since, for the general medium μ, κ, and σ effective, we have the field oscillation constant $\mu\kappa_e$, and

$$\Gamma = j\omega\sqrt{\mu\kappa_e} = j\omega\sqrt{\mu\left(\kappa - j\frac{\sigma}{\omega}\right)} \quad j\omega\sqrt{\mu\kappa - j\frac{\mu\sigma}{\omega}} = \sqrt{-\omega^2\mu\kappa + j\omega\mu\sigma}$$

The wave equation then reads $\nabla^2 F - F\Gamma^2 = 0$. For an exponential propagation along the Z axis of a uniform system of N parallel conductors, we have

$$F = F_0(x, y)\epsilon^{j(\omega t - \gamma z)} \tag{755}$$

for the two-dimensional wave function $F_0(x, y)$, solving the two-dimensional wave equation

$$\frac{\partial^2 F_0}{\partial x^2} + \frac{\partial^2 F_0}{\partial y^2} - (\Gamma^2 - \gamma^2)F_0 = 0 \tag{756}$$

The N conductors along the positive Z axis may have any cross-sectional shape, but the conductors cannot vary physically (geometrically) along the Z axis, except at the terminations (points of discontinuity). The classic transmission-line theory, based on voltage and current distributions, is correct only for $\Gamma^2 - \gamma^2 = 0$ in the medium about the conductors and $\Gamma^2 = \infty$ in the space occupied by the conductors and the

[1] Radiation from lines were discussed already by H. Hertz, *Wied. Ann.*, **36,** 1 (1889), and G. Mie, *Ann. Physik.*, **2,** 201 (1900). C. Mannebeck [*Jour. AIEE*, **42,** 289 (1923)] cleared up the situation of traveling-wave radiation (rectangular current wave) and gave a clear exposition of the two-wire line with stationary waves. J. R. Carson called attention in the following periodicals to the fact that transmission-line theory is not complete to explain radiation phenomena: *Jour. AIEE*, October, 1921; *Bell Sys. Tech. Jour.*, October, 1927. See also A. Sommerfeld, *Ann. Physik*, **67,** 233 (1899); M. Abraham, *Encykl. Math. Wiss.* Vol. II, Heft 3, p. 526; E. J. Sterba and C. B. Feldmann, *Proc. IRE*, **20,** 1199, July, 1932.

propagation constant $\gamma = j\omega\mu\kappa = j\omega\mu_0\kappa_0 = j\omega/c$. This means that for unattenuated propagation along the line, the velocity $c = 3 \times 10^8$ m/sec. This is a most ideal case, and we have a plane TEM wave guided by the line along the Z axis such as an $\mathcal{E}_y$, H_x TEM wave governed by the $\epsilon^{j\omega\left(t-\frac{z}{c}\right)}$ function. We can then derive the electric $\mathcal{E}$ and magnetic H fields from two scalar wave functions as is also shown in Sec. 92. One wave function is linear with the conductor charges, and the other linear with the conductor currents.

There are always losses in conductors as well as in the dielectric. However, for air, the dielectric losses are of no concern and for well-designed HF lines, the self-attenuation loss may be small enough so that the guided wave along the transmission line approximates the TEM wave mentioned above. We have then essentially $\Gamma^2 - \gamma^2 = 0$ in the neighborhood of the conductors and $\Gamma^2 - \gamma^2 \cong \Gamma^2$ in the conductor, since σ is so high. As is shown in Sec. 93, the solution of the two-dimensional wave equation leads to the same result as the classic transmission-line theory, since we obtain an Nth order equation for the possible principal waves (quasi-plane) for which $\mathcal{E}_z = 0$ and $H_z = 0$.

Now, if we assume conditions for which we do not expect $\gamma = j\omega/c$ to hold, we have a more difficult problem, since we deal with the general wave equation. We have then for N parallel conductors, in addition to N principal modes (*i.e.*, plane transverse $\mathcal{E}$, H waves), also an N-fold infinity of other modes (complementary waves) that are not quasi-plane and highly attenuated, with fields of sufficient magnitudes only in the neighborhood of physical terminals. Even at a small distance from terminations, the near field is then essentially due to the principal wave if the conductors use small cross sections. Hence, for large cross sections of the conductors or a wire with a ground return, the complementary waves have to be taken into account, as also when we deal with the radiated power that is caused by modified terminal conditions. In the classic theory, where we have only a continuity of currents and potentials at the boundary, we cannot expect a complete solution; while for the electromagnetic theory, the $\mathcal{E}_x$, H_x, $\mathcal{E}_y$, and H_y fields meet the boundary conditions everywhere in the boundary plane $z = 0$. Since the complementary waves have appreciable magnitudes in the immediate vicinity of terminals, we must expect that the *current and charge* distributions *near terminals* must be *affected by the complementary waves.*

When the fields of the current and charge distributions along the line are computed by means of the *retarded* magnetic vector potential Λ in volt · seconds per meter, we obtain two distinct parts. One part represents exactly the near field as due to the principal mode (propagated in

the same fashion as the charges and currents along the line). The other part is a field entirely different as regards propagation, since for a line system of small line attenuation (small line losses) it is more attenuated in the vicinity of the conductors. Hence, we have a way to account for the reactive field and the radiation field by means of magnetic retarded vector potentials based on the principal line current and line charge distribution. The complementary wave effect near the physical terminals then causes a small additional current and charge distribution that affects somewhat the corresponding distributions due to the principal current and charge wave near physical terminals. The additional distributions are however, relatively small and *next to the physical terminals* and can, as such, affect the field little as calculated from the vector potential.

N conductors parallel to the Z axis extending from $z = z_1$ to $z = z_2$ and terminated into any load show that in the classic transmission-line theory, the boundary conditions at the terminals are the continuity of N currents and N potentials with N direct and N reflected waves. However, in the electromagnetic field treatment, the $\mathcal{E}$ and H fields are everywhere continuous throughout the boundary planes $z = z_1$ and $z = z_2$ (this is not satisfied by the line theory). Therefore, the complete wave in the region $z_1 \leqq z \leqq z_2$ is represented by an infinite number of component waves, which satisfies Maxwell's equations as well as the geometry of the transmission system.

Suppose we have a conductor of radius $0.5d$, expressed in meters, along the Z axis, surrounded by a cylindrical return conductor of inside radius $0.5D$. Let us further assume that perfect conductivity prevails for the concentric line, while the dielectric is perfect in the space between $0.5d \leqq r \leqq 0.5D$. The principal wave (TEM wave) is then propagated along the Z axis with the velocity $c = 3 \times 10^8$ m/sec, and the propagation constant $\gamma = \alpha + j\beta = j\omega/c = j\beta_0$. Let us assume also that the concentric line is very long so that only an outgoing traveling wave exists along the $+Z$ axis. We have then for the principal current and charge waves

$$I = I_0\epsilon^{-j\beta_0 z}\epsilon^{j\omega t} \qquad \text{and} \qquad Q = Q_0 \epsilon^{-j\beta_0 z},$$

as well as for the conservation of electric charge (conservation of electricity)

$$-\frac{\partial I}{\partial z} = \frac{\partial Q}{\partial t} = j\omega Q$$

The retarded scalar and vector potentials V and Λ are the integrations *over all* elementary volumes of respective conductors. They refer to retarded charge and current densities

$$[\rho] = \rho\epsilon^{j\omega(t-r/c)} \text{ coulombs/m}^3; \qquad [J] = J\epsilon^{j\omega(t-r/c)} \text{ amp/m}^2$$

and r is the distance between the conductor element $d\tau$ and the point where the scalar potential V volts and the vector potential Λ volt · seconds per meter is calculated. The integration is extended over the entire conductor system of currents and charges. Then at a space point a distance r from the volume element $d\tau$, we have

$$V = \iiint_\tau \frac{[\rho]\,d\tau}{4\pi\kappa r} \text{ volts}; \qquad \Lambda = \iiint_\tau \frac{\mu[J]\,d\tau}{4\pi r} \text{ volt} \cdot \text{sec/m} \qquad (757)$$

The magnetic and electric field intensities are then found from

$$H = \frac{1}{\mu} \text{curl } \Lambda \text{ amp/m}; \qquad \mathcal{E} = -\text{grad }(V) - \frac{\partial \Lambda}{\partial t} = -\text{grad }(V) - j\omega\Lambda \text{ volts/m} \qquad (758)$$

In both relations of (757), besides the time function $\epsilon^{j\omega t}$, we must deal with a function $(\epsilon^{-j\beta_0 r})/r$ leading to the Sommerfeld integral, which, for $\beta_0 = j\omega/c$, yields

$$\frac{\epsilon^{-j\beta_0 r}}{r} = \int_0^\infty J_0(\rho\lambda)\epsilon^{-|z-z'|\sqrt{\lambda^2-\beta_0^2}} \frac{\lambda}{\sqrt{\lambda^2 - \beta_0^2}}\,d\lambda \qquad (759)$$

where $\rho = \sqrt{x^2 + y^2}$ and J_0 is the Bessel function of the zeroth order. When this system of currents and charges is expressed in terms of the Sommerfeld integral for the surface currents and charges at $\rho = 0.5d$ and $\rho = 0.5D$, we obtain for $z \geqq 0$

$$V = Q_0 \int_0^\infty J_0(\rho\lambda)\left[J_0\left(\frac{d}{2}\lambda\right) - J_0\left(\frac{D}{2}\lambda\right)\right]\epsilon^{-z\sqrt{\lambda^2-\beta_0^2}} \frac{\lambda}{\sqrt{\lambda^2 - \beta_0^2}}\,d\lambda \times \int_0^z \epsilon^{z'[\sqrt{\lambda^2-\beta_0^2}-j\beta_0]}\,dz' + Q_0 \int_0^\infty J_0(\rho\lambda)\left[J_0\left(\frac{d}{2}\lambda\right) - J_0\left(\frac{D}{2}\lambda\right)\right]\epsilon^{z\sqrt{\lambda^2-\beta_0^2}} \frac{\lambda}{\sqrt{\lambda^2 - \beta_0^2}}\,d\lambda \cdot \int_z^\infty \epsilon^{-z'[\sqrt{\lambda^2-\beta_0^2}+j\beta_0]}\,dz'$$

reducing to

$$V = \underbrace{2Q_0\epsilon^{-j\beta_0 z} \int_0^\infty J_0(\rho\lambda)\left[J_0\left(\frac{d}{2}\lambda\right) - J_0\left(\frac{D}{2}\lambda\right)\right]\frac{d\lambda}{\lambda}}_{\text{potential wave along the Z axis in exactly the same fashion as the current and the charge (homogeneous potential wave)}}$$

$$\underbrace{- Q_0 \int_0^\infty J_0(\rho\lambda)\left[J_0\left(\frac{d}{2}\lambda\right) - J_0\left(\frac{D}{2}\lambda\right)\right]\frac{\epsilon^{-z\sqrt{\lambda^2-\beta_0^2}}}{\sqrt{\lambda^2 - \beta_0^2} - j\beta_0}\frac{\lambda}{\sqrt{\lambda^2 - \beta_0^2}}\,d\lambda}_{\text{potential wave dying away for sufficiently large values of } z \text{ (heterogeneous wave). Propagation different from that of current and charge}} \qquad (760)$$

which is the Carson solution for $d\lambda$ and dz' as differentials, but $d/2$ and $D/2$ as radii.

The above solution is an extension of the original Sommerfeld solution, since we deal here with two cylindrical current surfaces of diameters d and D instead of one cylindrical wave function. In the section on the general wave equation $\nabla^2 w \pm w\Gamma^2 = 0$, we found[1] that in cylinder coordinates ρ, φ, z for $\rho = \sqrt{x^2 + y^2}$ and $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$, the wave equation reads, according to (391a) on page 300 or (429) on page 322, for $r = \rho$,

$$\underbrace{\frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}}_{\nabla^2 w} - w\Gamma^2 = 0 \tag{761}$$

In that section, it is shown that by means of elementary functions, we can separate equations. The functions are $w = \Phi_1(\rho)\Phi_2(\varphi)\Phi_3(z)$, yielding the Bessel equation (437) on page 325, which reads for $r = \rho$

$$\rho^2 \frac{d^2\Phi_1}{d\rho^2} + \rho \frac{d\Phi_1}{d\rho} + [(\beta^2 - \beta_z^2)\rho^2 - \beta_\varphi{}^2]\Phi_1 = 0; \qquad x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

standard form, with solution $y = J_n(x)$

We have, therefore, a solution

$$\Phi_1 = J_{\beta_\varphi}(\rho\sqrt{\beta^2 - \beta_z^2}); \qquad \beta_\varphi = \frac{\omega}{c} = \beta_0$$

where β_φ gives the order of the Bessel function and $\rho\sqrt{\beta^2 - \beta_z^2}$ its argument. Therefore, the general solution is based on cylinder functions of the form

$$\Phi_1 = Z_{\beta_\varphi}(\rho\sqrt{\beta^2 - \beta_z^2}) = Z_n(\rho\sqrt{\beta^2 - \beta_z^2})$$

The $J_n(x)$ solution is then applicable and reserved for the particular solution that is finite on the cylinder axis ($\rho = 0$). The scalar wave equation, where w may be the potential V volts or another such quantity, then reads

$$\nabla^2 w - \mu\kappa \frac{\partial^2 w}{\partial t^2} - \mu\sigma \frac{\partial w}{\partial t} = 0; \qquad w_n = \epsilon^{jn\varphi} Z_n(\rho\sqrt{\beta^2 - \beta_z^2})\, \epsilon^{j(\omega t \pm \beta_z z)} \tag{762}$$

where the movement constant β in the general case may be complex. The field along the Z axis is, therefore, in the general case, not always periodic along the axis of propagation. For a homogeneous isotropic region, we have generally a linear combination of the alternate solutions

$$\left.\begin{aligned} w_n &= \epsilon^{jn\varphi} J_n(\rho\sqrt{\beta^2 - \beta_z^2})\, \epsilon^{j(\omega t \pm j\beta_z z)} && \text{is solution for finite regions including } \rho = 0 \\ w_n &= \epsilon^{jn\varphi} H_n^{(1)}(\rho\sqrt{\beta^2 - \beta_z^2})\, \epsilon^{j(\omega t \pm j\beta_z z)} && \text{applies at great distances since } H_n^{(1)}(x) = J_n(x) + jN_n(x);\ \text{for } x \gg 1 \text{ an outward (radial) wave} \end{aligned}\right\} \tag{763}$$

[1] The expression ρ is used in this section instead of r, as in the section on the wave equation, since r in (759) stands for any distance in any direction from a volume element $d\tau$ of a current-carrying conductor to a field point.

The wave equation (761) possesses then the elementary solutions just given. Since the fields are symmetrical, we have also $n = 0$ for an isotropic medium. In these relations, β_z and n are real or complex constants. Using the substitution, $\lambda^2 = \beta^2 - \beta_z^2$, where generally $\Gamma = j\omega\sqrt{\mu\kappa_e} = \alpha + j\beta$, *i.e.*, $\beta = \omega\sqrt{\mu\kappa_e}$. Therefore, the parameter λ is also a real or generally a complex constant. The first solution of (763) can be generalized into

$$w_n = \int_k^\infty J_n(\rho\lambda)\epsilon^{\pm j\beta_z z}\Phi(\beta_z)\beta_z\, d\beta_z$$

which, for $n = 0$ and $k = \beta$, gives

$$\frac{\epsilon^{j\beta\rho}}{\rho} = \int_0^\infty J_0(\rho\lambda)\Phi(\beta_z)\lambda\, d\lambda; \qquad \text{hence, } \Phi(\beta_z) = \int_0^\infty J_0(\rho\lambda)\epsilon^{j\beta\rho}\, d\rho = \frac{1}{\sqrt{\lambda^2 - \beta^2}}$$

for $\beta > 0$

We have then the Sommerfeld integral

$$\frac{\epsilon^{j\beta r}}{r} = \int_0^\infty \epsilon^{\pm z\sqrt{\lambda^2-\beta^2}}\, J_0(\rho\lambda)\frac{\lambda}{\sqrt{\lambda^2 - \beta^2}}\, d\lambda$$

where the (+) sign is for $z < 0$ and the (−) sign for $z > 0$. The parameter λ as defined above should not be confused with the wavelength. The symbol λ is used by many writers for expressing several things.

96. Radiation of a Parallel-wire Line with Stationary Wave Distribution. This derivation is based on the fact that each elementary length dl of a linear radiator with its instantaneous current i, due to the current moment $i\,dl$ in meter · amperes, causes, at a field point P a distance r meters from the mid-point of dl, a radiation effect of

$$d\mathcal{E}_\theta = \frac{188.35I\,dl}{r\lambda} A \sin\theta \quad \text{volts/m}; \qquad dH_\varphi = \frac{d\mathcal{E}_\theta}{Z_0} \quad \text{amp/m};$$

$$A = \epsilon^{j(\omega t - \beta r)}$$

$$Z_0 = \sqrt{\frac{\mu}{\kappa}} \cong 376.7 \text{ ohms for air} \tag{764}$$

where θ denotes the angle r makes with dl and r is a distance reaching into the *distant* radiation region. It is, therefore, necessary only to add up at a certain field point r, φ, θ all elementary radiation contributions of the entire transmission line (go and parallel return wires) with respect to *magnitude* and *phase*. Corresponding line elements of the go and the return wire act then with maximum moments $+I\,dl$ and $-I\,dl$, respectively. According to Fig. 95, we also have to take the angle φ

into consideration, because a parallel-wire radiating system has no longer a symmetrical circular radiation pattern (H_φ = constant) about the reference axis determining the dl directions. The average radiated power P_r can be found from the real component of the Poynting vector. According to Fig. 95, we have then in the distant radiation region field components F_θ and F_φ, which are normal to each other and normal to the direction r of wave propagation, satisfying the Poynting vector formula. Both vectors F_θ and F_φ are *tangential* components as far as the growing spherical surface of the sphere, expanding along the r directions with a

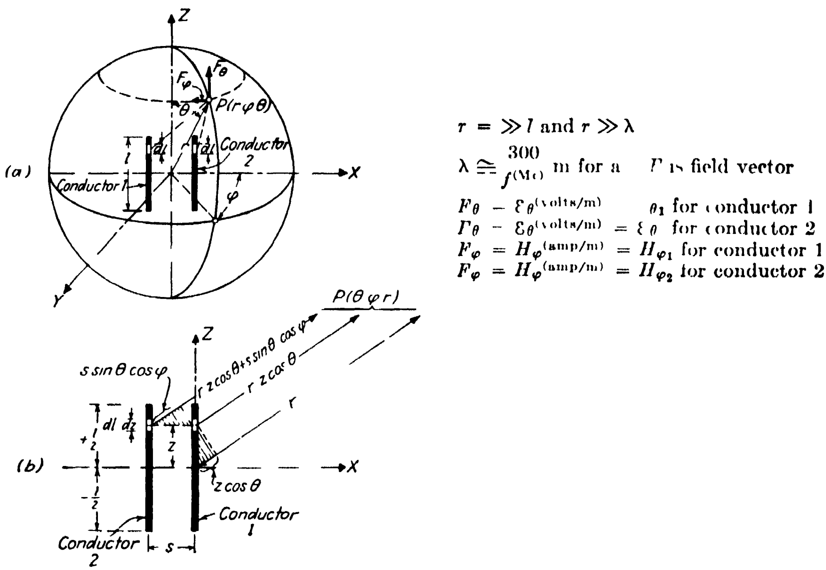


FIG. 95. Radiation from a Lecher wire feeder.

velocity $1/\sqrt{\mu\kappa}$ meters per second, is concerned (in an unobstructed dielectric medium). Taking the very general case where we have two conductors 1 and 2 with tangential meridian and azimuth components for each field vector, we have the maximum values

$$\mathcal{E}_{\theta_1},\ \mathcal{E}_{\varphi_1} \text{ volts/m and } H_{\varphi_1},\ H_{\theta_1} \text{ amp/m for conductor 1}$$
$$\mathcal{E}_{\theta_2},\ \mathcal{E}_{\varphi_2} \text{ volts/m and } H_{\varphi_2},\ H_{\theta_2} \text{ amp/m for conductor 2}$$

where the electric and magnetic field associations normal to each other form Poynting vector field components. We have then eight such possibilities and find the resultant radiated power P_r by employing the

conjugate H^* instead of the H values, from

$$P_r = \tfrac{1}{2}r^2 \operatorname{Re} \int (\mathcal{E}_\theta \cdot H^*_\varphi - \mathcal{E}_\varphi \cdot H^*_\theta) \underbrace{\sin\theta\, d\theta\, d\varphi}_{\substack{\text{elementary}\\ \text{solid angle}}} \qquad \text{watts/(surface } 4\pi r^2)$$

and

$$P_r = \tfrac{1}{2} \operatorname{Re} \int \Big[\underbrace{(\mathcal{E}_{\theta_1} \cdot H^*_{\varphi_1} - \mathcal{E}_{\varphi_1} \cdot H^*_{\theta_1})}_{\substack{\text{due to only conductor 1 (con-}\\ \text{ductor 2 not present)}}} + \underbrace{(\mathcal{E}_{\theta_2} \cdot H^*_{\varphi_2} - \mathcal{E}_{\varphi_2} \cdot H^*_{\theta_2})}_{\substack{\text{due to only conductor 2 (con-}\\ \text{ductor 1 not present)}}}$$
$$\left. + \underbrace{(\mathcal{E}_{\theta_2} \cdot H^*_{\varphi_1} + \mathcal{E}_{\theta_1} \cdot H^*_{\varphi_2} - \mathcal{E}_{\varphi_1} \cdot H^*_{\theta} - \mathcal{E}_{\varphi_2} \cdot H^*_{\theta_1})\Big]}_{\text{due to mutual interactions of conductors 1 and 2}} \right\} \quad \text{watts} \qquad (765)$$

This relation can be extended to any number of conductors. For a single conductor, we have only the product $\frac{1}{2}$ Re $\mathcal{E}_{\theta_1} \cdot H^*_{\varphi_1}$ watts if the linear conductor is oriented along the Z axis, *i.e.*, when θ is the angle that the direction to the space point makes with the length axis of the radiator. For two parallel conductors oriented in the Z directions, we have only $\mathcal{E}_{\theta_1}$, H_{φ_1} field associations for conductor 1 and $\mathcal{E}_\theta$, H_{φ_2} normal field associations for conductor 2. The values of $\mathcal{E}_\theta$ and H_φ for a particular conductor are found by integrating the $d\mathcal{E}_\theta$ relation over the total conductor length l, which will give the value $\mathcal{E}_\theta$ and H_φ from $\mathcal{E}_\theta/Z_0$ as is given by (764).

According to Fig. 95, we have, for a distant field point P for two parallel current filaments l meters in length, the radiation effects of all current elements $+I\,dl$ on line 1 and all current elements $-I\,dl$ on line 2. The minus sign occurs, since line 2 is taken as the return conductor; and if $I\,dl$ is *positive* at a certain instant on the go conductor 1, it must be *negative* at the same instant on the return conductor 2. The radiation center is as indicated in the representation (a) of Fig. 95. Because the distance r to a distant field point P is taken far enough to render parallel all rays from the above-mentioned radiation center as well as from the dl elements, which are a distance z above the XY plane, we can, as far as the inverse distance effect ($1/r$ effect) in (764) is concerned, take the distances from P to dl on lines 1 and 2 equal to $OP = r$ and shift the radiation center in representation (b) into the mid-point of conductor 1. Such a shift does not influence the relative phase effect, since we have for the line element on 1, the measure $r - z\cos\theta$ and, for the line element on 2, the measure $r - z\cos\theta + s\sin\theta\cos\varphi$, where the last term takes care of the separation of s meters between the go and the return conductor. The electric field contribution due to a pair of corresponding $+I\,dl$ and $-I\,dl$ elements is, according to (764),

$$d\mathcal{E}_\theta = \underbrace{\frac{188.35 I\,dl \sin\theta}{\underbrace{(r - z\cos\theta)}_{\cong r}\lambda}\epsilon^{j[\omega t - \beta(r - z\cos\theta)]}}_{\text{due to the 1, or go, conductor}}$$
$$- \underbrace{\frac{188.35 I\,dl \sin\theta}{\underbrace{(r - z\cos\theta + s\sin\theta\cos\varphi)}_{\cong r}\lambda}\epsilon^{j[\omega t - \beta(r - z\cos\theta + s\sin\theta\cos\varphi)]}}_{\text{due to the 2, or return, conductor}} \quad \text{volts/m} \quad (766)$$

The total electric field $\mathcal{E}_\theta$ at field point P is then the integral of this expression taken for $dl = dz$ over the limits $z = -\frac{1}{2}l$ to $z = +\frac{1}{2}l$ with the following details:

The term $(\sin\theta)/r$ represents only a *slowly varying function* and can be put before the integral sign. We have then

$$\left.\begin{aligned} \mathcal{E}_\theta^{(\text{volts/m})} &= \underbrace{K\int_{-l/2}^{+l/2} \epsilon^{j[\omega t - \beta(r - z\cos\theta)]}\, I\, dl}_{\text{due to go conductor}} \\ &\quad - \underbrace{K\int_{-l/2}^{+l/2} \epsilon^{j[\omega t - \beta(r - z\cos\theta + s\sin\theta\cos\varphi)]} I\, dl}_{\text{due to return conductor}} \\ K &= \frac{188.35\sin\theta}{r\lambda} \end{aligned}\right\} \quad (767)$$

We have still to define the current distribution I. Assuming a sinusoidal distribution along the line of length l meters, we have $I\sin\beta(l - d)$ for standing waves according to transmission-line theory. The quantity I then denotes the value at a current antinode (values at the loop). The input current to the line is $I_0 \sin\beta(l - d_0)$, where d_0 is the distance from the end of the double line from which d is measured to the point of the line where the input voltage is applied. To secure standing waves, we can have respective ends short-circuited or respective ends open. The voltage is then, in many cases, conveniently applied at the center of the double line, so that current and corresponding waves are propagated toward either open end and reflected back toward the source from either open end. Such a condition means that the current must be zero at either open end, *i.e.*, at $z = \pm l/2$ in Fig. 95. Such a condition is given by the current distribution

$$I_z = I\sin q\pi\left(\frac{z}{l} + \frac{1}{2}\right); \qquad \beta = \frac{q\pi}{l} \qquad (768)$$

where q is an integer and equal to the number of half wavelengths developed along the total length l and I is the value at a current antinode. If we apply this distribution in place of I to the go conductor (1) of (767),

$$\mathcal{E}_{\theta_1} = +j30I\left[\epsilon^{j\frac{q\pi}{2}\cos\theta} - (-1)^q\epsilon^{-j\frac{q\pi}{2}\cos\theta}\right]\frac{\epsilon^{j(\omega t - \beta r)}}{r\sin\theta} \qquad \text{volts/m} \qquad (769)$$

We have then for *odd* and *even* integer values of q, for $x = (q\pi/2)\cos\theta$,

$$\left.\begin{aligned} \epsilon^{jx} + \epsilon^{-jx} &= 2\cos x = 2\cos\left(\frac{q\pi}{2}\cos\theta\right) \qquad \text{for } q = 1, 3, 5, 7, \cdots \\ \epsilon^{jx} - \epsilon^{-jx} &= 2j\sin x = 2j\sin\left(\frac{q\pi}{2}\cos\theta\right) \qquad \text{for } q = 2, 4, 6, \cdots \end{aligned}\right\} \tag{770}$$

and

$$\left.\begin{aligned} &\mathcal{E}_{\theta_1}{}^{(\text{volts/m})} = j\frac{60I\epsilon^{j(\omega t-\beta r)}}{r}\underbrace{\left[\frac{\cos\left(\frac{q\pi}{2}\cos\theta\right)}{\sin\theta}\right]}_{D_{\theta_1}} \qquad \text{for } q = 1, 3, 5, 7, \cdots; \\ &D_{\theta_1} = \frac{\cos\left(\frac{q\pi}{2}\cos\theta\right)}{\sin\theta} \\ &\mathcal{E}_{\theta_1}{}^{(\text{volts/m})} = \frac{60I\epsilon^{j(\omega t-\beta r)}}{r}D_{\theta_1} \qquad \text{for } q = 2, 4, 6, 8, \cdots; \\ &D_{\theta_1} = \frac{\sin\left(\frac{q\pi}{2}\cos\theta\right)}{\sin\theta} \end{aligned}\right\} \tag{771}$$

where D_{θ_1} represents the respective direction factors for line 1. For the detailed solution in the case of the double line $l = \frac{1}{2}\lambda$, we have $\pm\frac{1}{2}l = \pm\frac{1}{4}\lambda$ in Fig. 95. Since the reference origin is at the mid-point of the line, the applied voltage of the double line would be at the origin and across the line with an input current of I amperes, where I is the loop current (represents the maximum time value at the center of each line). Hence, we deal with a cosine function, and

$$I_z = I\cos\beta z \tag{772}$$

and (767) reads, since $dl = dz$,

$$\mathcal{E}_\theta{}^{(\text{volts/m})} = \underbrace{KI\int_{-\lambda/4}^{+\lambda/4}\cos(\beta z)\epsilon^{j[\omega t-\beta(r-z\cos\theta)]}\,dz}_{\mathcal{E}_{\theta_1}} - \underbrace{KI\int_{-\lambda/4}^{+\lambda/4}\cos(\beta z)\epsilon^{j[\omega t-\beta(r-z\cos\theta+s\sin\theta\cos\varphi)]}\,dz}_{\mathcal{E}_{\theta_2}} \tag{773}$$

where $\mathcal{E}_{\theta_1}$ and $\mathcal{E}_{\theta_2}$ are the total field contributions at any distant field point r, φ, θ of the go and the return conductor, respectively. The respective phase factors can be written as

$$\left.\begin{aligned} &\underbrace{\epsilon^{j(\omega t-\beta r)}}_{\substack{\text{time}\\\text{function}}}\cdot\underbrace{\epsilon^{j\beta z\cos\theta}}_{\substack{\text{relative}\\\text{phase factor}}} \qquad \text{for go conductor} \\ &\underbrace{\epsilon^{j(\omega t-\beta r)}}_{\substack{\text{time}\\\text{function}}}\cdot\underbrace{\epsilon^{j\beta(z\cos\theta-s\sin\theta\cos\varphi)}}_{\text{relative phase factor}} \qquad \text{for return conductor} \end{aligned}\right\} \tag{774}$$

The relative phase is with respect to the condition at the radiation center (mid-point of conductor 1 in Fig. 95). For the field effect $\mathcal{E}_{\theta_1}$ of conductor 1, we can take the time factor before the integral sign and have

$$\mathcal{E}_{\theta_1} = KI\epsilon^{j(\omega t-\beta r)} \int_{z=-\lambda/4}^{z=+\lambda/4} \epsilon^{j\beta z \cos\theta} \cos(\beta z)\, dz \qquad \text{volts/m}$$

Putting $j\beta \cos\theta = m$ and $\beta = n$, we find for the integral

$$\int_{-\lambda/4}^{+\lambda/4} \epsilon^{mz} \cos(nz)\, dz \left| \frac{\epsilon^{mz}[m\cos(nz) + n\sin(nz)]}{m^2+n^2} \right|_{-\lambda/4}^{+\lambda/4} \qquad \text{a well-known solution (found in integral tables)}$$

Hence, for $\beta = 2\pi/\lambda$,

$$\int_{-\lambda/4}^{+\lambda/4} = \left| \frac{\epsilon^{j\beta z\cos\theta}[j\beta\cos\theta\cos(\beta z) + \beta\sin(\beta z)]}{\beta^2\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}} \right|_{z=-\lambda/4}^{z=+\lambda/4}$$

$$= \frac{\epsilon^{j90^\circ\cos\theta}[\overbrace{j\beta\cos\theta\cos 90^\circ}^{=0} + \overbrace{\beta\sin 90^\circ}^{=\beta}]}{\beta^2\sin^2\theta}$$

$$- \frac{\epsilon^{-j90^\circ\cos\theta}[\overbrace{j\beta\cos\theta\cos(-90^\circ)}^{=0} + \overbrace{\beta\sin(-90^\circ)}^{=-\beta}]}{\beta^2\sin^2\theta}$$

or

$$\int_{-\lambda/4}^{+\lambda/4} = \frac{\epsilon^{j90^\circ\cos\theta}}{\beta\sin^2\theta} + \frac{\epsilon^{-j90^\circ\cos\theta}}{\beta\sin^2\theta}$$

$$= \frac{\cos(90^\circ\cos\theta) + j\sin(90^\circ\cos\theta) + \cos(90^\circ\cos\theta) - j\sin(90^\circ\cos\theta)}{\beta\sin^2\theta}$$

$$= \frac{2\cos(90^\circ\cos\theta)}{\beta\sin^2\theta}$$

Therefore,

$$\mathcal{E}_{\theta_1}{}^{(\text{volts/m})} = KI\epsilon^{j(\omega t-\beta r)} \frac{2\cos(90^\circ\cos\theta)}{\frac{2\pi}{\lambda}\sin^2\theta} = \frac{188.35I\sin\theta\,\lambda}{r\lambda} \cdot \frac{\cos(90^\circ\cos\theta)}{\pi\sin^2\theta}\epsilon^{j(\omega t-\beta r)}$$

$$= \frac{60I\epsilon^{j(\omega t-\beta r)}}{r} \frac{\cos(90^\circ\cos\theta)}{\sin\theta} \qquad (775)$$

where r meters is the distance from the radiation center to field point r, φ, θ and I denotes the maximum time value of the line current in amperes at the origin (at mid-point of conductor 1). With respect to the field contribution $\mathcal{E}_{\theta_2}$, due to the return conductor (2 in Fig. 95), we can use the above derivation, since, according to (774), the relative phase of the return conductor can be written as $\epsilon^{-j\beta s\sin\theta\cos\varphi}\,\epsilon^{j\beta z\cos\theta}$. We find for conductor 2 (with respect to conductor 1)

$$\mathcal{E}_{\theta_2} = -KI\epsilon^{j(\omega t-\beta r)}\epsilon^{-j\beta s \sin\theta\cos\varphi}\underbrace{\int_{z=-\lambda/4}^{z=+\lambda/4}\epsilon^{j\beta z\cos\theta}\cos(\beta z)\,dz}_{\frac{2\cos(90^\circ\cos\theta)}{\beta\sin^2\theta}}$$

$$= -\frac{60I\epsilon^{j(\omega t-\beta r)}}{r}\underbrace{\underbrace{\left[\frac{\cos(90^\circ\cos\theta)}{\sin\theta}\right]}_{D_{\theta_1}}\epsilon^{-j\beta s\sin\theta\cos\varphi}}_{D_{\theta_2}} \quad \text{volts/m} \tag{776}$$

This result shows that the direction factor D_{θ_2} of the return conductor contains the direction factor D_{θ_1} of the go conductor as a factor.

By means of (775) and (776), we find the total electric field intensity $\mathcal{E}_\theta$ of a half-wavelength parallel-wire line open at either end as

$$\mathcal{E}_\theta = \mathcal{E}_{\theta_1} + \mathcal{E}_{\theta_2} = \frac{60I\epsilon^{j(\omega t-\beta r)}}{r}\frac{\cos(90^\circ\cos\theta)}{\sin\theta}(1 - \epsilon^{-j\beta s\sin\theta\cos\varphi}) \quad \text{volts/m} \tag{777}$$

showing that for a spacing $s = 0$ m, no radiation occurs (go and return currents cancel each other). The factor $\exp(-j\beta s\sin\theta\cos\varphi)$ is all the smaller the larger the spacing. Both the $\sin\theta$ and the $\cos\varphi$ functions fluctuate between $+1$, which means that they pass through zero value causing also zero radiation effects in certain space directions. When either $\sin\theta$ only or $\cos\varphi$ only has values between 0 and -1, the interference factor $\epsilon^{-j\delta}$ becomes $\epsilon^{+j\delta}$ for $\delta = \beta s\sin\theta\cos\varphi$ and

$$1 - \epsilon^{j\delta} = 1 - \cos\delta - j\sin\delta$$

We also note from (777) that the interference factor $\epsilon^{-j\delta}$ affects the magnitude of $\mathcal{E}_\theta$ as well as the shape of the resultant directivity pattern. The resultant pattern is then no longer symmetrical about the radiation center. Since the absolute amount of $1 - \epsilon^{-j\delta}$ becomes

$$1 - \epsilon^{-j\delta} = 1 - (\cos\delta - j\sin\delta) = 1 - \cos\delta + j\sin\delta$$

$$= \left|\sqrt{[\underbrace{(1-\cos\delta)}_{a} + \underbrace{(j\sin\delta)}_{b}][(1-\cos\delta) - (j\sin\delta)]}\right|$$

$$= \left|\sqrt{a^2+b^2}\right| = \sqrt{1 - 2\cos\delta + \underbrace{\cos^2\delta + \sin^2\delta}_{=1}}$$

$$= \left|\sqrt{2\underbrace{(1-\cos\delta)}_{=2\sin^2\frac{\delta}{2}}}\right| = \left|2\sin\frac{\delta}{2}\right|$$

we find the total resultant electric field intensity as

$$\mathcal{E}_\theta = \frac{120I\epsilon^{j(\omega t-\beta r)}}{r}\frac{\cos(90^\circ\cos\theta)}{\sin\theta}\sin\left(\frac{\beta s\sin\theta\cos\varphi}{2}\right) \quad \text{volts/m} \tag{777a}$$

According to (771), we find that for other stationary waves along the parallel-wire system, the above result likewise applies when the respective directivities D_{θ_1} and D_{θ_2} are used. Hence, we can say that for a parallel-wire line with both ends open,

we have a field

$$\mathcal{E}_\theta = \frac{K_1 I}{r} \epsilon^{j(\omega t - \beta r)} D_\theta (1 - \underbrace{\epsilon^{j\beta s \sin\theta \cos\varphi}}_{\substack{\text{interference}\\ \text{factor}}}) \qquad \text{volts/m} \tag{778}$$

if D_θ represents the directivity of each conductor by itself without the presence of the other conductor. On pages 502, 503 a formula is given for the directivity D_θ. This formula has reference to unobstructed radiation space, and I is the input current and K_1 a constant. The interference factor is mutual and does not belong to a particular conductor, since it is relative.

97. Remarks on the Radiated Power of a Parallel-wire System Excited with Standing Waves. The radiated power is found from the real component of the Poynting vector

$$p = \frac{1}{2} \text{Re}\, \mathcal{E}_\theta \cdot H_\varphi^* = \frac{1}{2} \text{Re} \sqrt{\frac{\kappa}{\mu}} |\mathcal{E}_\theta|^2 \qquad \text{watts/m}^2$$

since $H_\varphi = \sqrt{\kappa/\mu}\, \mathcal{E}_\theta$ amperes per meter. If we had only one conductor (go or return) without the presence of the other conductor and for a $\frac{1}{2}\lambda$ length, according to (775),

$$p = \frac{1}{2} \text{Re} \left[\sqrt{\frac{\kappa}{\mu}} |\mathcal{E}_{\theta_1}|^2 \right] = \frac{1}{2} \frac{1}{376.7} \left(\frac{60 I D_\theta}{r} \right)^2 = 4.78 \frac{I^2 D_\theta^2}{r^2} \qquad \text{watts/m}^2;$$

$$D_\theta = \frac{\cos(90^\circ \cos\theta)}{\sin\theta} \tag{779}$$

The total radiated power P_r in an unobstructed medium is found from the energy flux normal to the surface of a sphere of r meters, *i.e.*, from $4\pi r^2 p$ watts. Since P_r denotes the average power, we must also use the average value of D_θ^2 and find

$$P_r = 4\pi r^2 p = 4\pi r^2 4.78 \frac{I^2}{r^2} (D_\theta^2)_{\text{avg}} = 4 \times 4.78\pi I^2 \times 0.612 = 36.75 I^2 \text{ watts}$$

Since I represents the maximum time value of the current in amperes, the total radiated power is also equal to $\frac{1}{2} I^2 R_r$, where R_r is the radiation resistance in ohms. The radiation resistance then becomes $R_r = 2 \times 36.75 = 73.5$ ohms.

If we have any other half-wave distribution, the D_θ values of (771) hold and

$$P_r^{(\text{watts})} = \int_0^{2\pi} \int_0^{\pi} p r^2 \underbrace{\sin\theta\, d\theta\, d\varphi}_{\text{solid angle}} = 30 I^2 \int_0^{\pi} \frac{\cos^2\left(\frac{q\pi}{2}\cos\theta\right)}{\sin\theta} d\theta$$

for q an odd integer

$$P_r^{(\text{watts})} = \int_0^{2\pi}\int_0^{\pi} pr^2 \underbrace{\sin\theta\, d\theta\, d\varphi}_{\text{solid angle}} = 30I^2 \int_0^{\pi} \frac{\sin^2\left(\frac{q\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta$$

for q an even integer

where the solid angle $\sin\theta\, d\theta\, d\varphi$ represents the area that is intercepted by the angle on a unit sphere with the center at the apex of the solid angle. The integrals then yield

$$\int_0^{\pi} = \int_0^{\pi} \frac{\cos^2(\frac{1}{2}\pi q\cos\theta)}{\sin\theta}\, d\theta \qquad \text{for } q = 1, 3, 5, \cdots \left(l = \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \cdots\right)$$
$$= \tfrac{1}{2}[\log_\epsilon(2\pi qk) - \text{Ci}\,(2\pi q)]$$

$$\int_0^{\pi} = \int_0^{\pi} \frac{\sin^2(\frac{1}{2}\pi q\cos\theta)}{\sin\theta}\, d\theta \qquad \text{for } q = 2, 4, 6, \cdots (l = \lambda, 2\lambda, 3\lambda, \cdots)$$
$$= \tfrac{1}{2}[\log_\epsilon(2\pi qk) - \text{Ci}\,(2\pi q)]$$

where $k = 1.781$ and Ci is the cosine integral, which can be read off in tables or on curves. Since $\log_\epsilon(2 \times 1.781\pi q) = \log_\epsilon(11.2q) = \log_\epsilon(11.2) + \log_\epsilon(q) = 2.415 + 2.303 \log_{10}(q)$ and

$$S = \int_0^{\pi} = 1.2075 + 1.1513 \log_{10}(q) - \tfrac{1}{2}\,\text{Ci}\,(6.28q) \tag{780}$$

For $q = 1, 2, 3, 4, 5, \cdots$, we find

$$P_r = 30SI^2 = 30I^2[1.2075 + 1.1513 \log_{10}(q) - \tfrac{1}{2}\,\text{Ci}\,(6.28q)] = \tfrac{1}{2}R_rI^2 \quad \text{watts} \tag{781}$$

Hence, for the $\frac{1}{2}\lambda$ length of the conductor, we have $q = 1$, and

$$P_r = 30I^2[1.2075 + \underbrace{1.1513 \log_{10}(1)}_{=0,\text{ since } \log(1) = 0} - \underbrace{\tfrac{1}{2}\,\text{Ci}\,(6.28)}_{\text{about } 0}] \cong 30 \times 1.2075I^2 = 36.4I^2 \quad \text{watts}$$

which checks the result $36.65I^2$, as found before, very closely. That result is based on the average value 0.612, as found from the $D_\theta{}^2$ pattern. For a full-wavelength distribution, we have $q = 2$, and

$$P_r = 30I^2[1.2075 + 1.1513 \log_{10}(2) - \tfrac{1}{2}\,\text{Ci}\,(12.56)]$$
$$= 30I^2[1.2075 + 1.1513 \times 0.301 - \tfrac{1}{2}(-0.00708)] = 46.9I^2 \quad \text{watts}$$

as against $36.4I^2$ watts for half the length. The radiation resistance is then $2 \times 46.9 = 93.8$ ohms. All these computations refer to the parallel-wire system if only one conductor is present. For the last example, the voltage that causes the current flow can no longer be applied across the mid-points of respective parallel conductors, since for a full-wavelength distribution we have zero current (practically a pseudo node) at such points. Hence, the transmission line had to be excited at some points where a current flow is possible. Convenient points would be located a distance of $0.25l$ from one open end, since this would correspond to a very low resistance input value.

To find the actual radiation effect of the double line of Fig. 95, we have to use in the Poynting vector the $\mathcal{E}_\theta$ value, due to both conductors,

and find from (778)

$$\mathcal{E}_\theta^{(\text{volts/m})} = \frac{KID_\theta}{r}\epsilon^{j(\omega t-\beta r)}[\underbrace{1-\cos(\beta s\sin\theta\cos\varphi)}_{=A} + j\underbrace{\sin(\beta s\sin\theta\cos\varphi)}_{=B}]$$
$$= \frac{KID_\theta}{r}\epsilon^{j(\omega t-\beta r)}\sqrt{[1-\cos\underbrace{(\beta s\sin\theta\cos\varphi)}_{=\delta}]^2+\sin^2\delta}\;/\underline{\psi}$$
$$= \frac{KID_\theta}{r}\epsilon^{j(\omega t-\beta r)}\sqrt{1-2\cos\delta+\underbrace{\cos^2\delta+\sin^2\delta}_{=1}}\;/\underline{\psi}$$
$$= 1.41\frac{KID_\theta}{r}\epsilon^{j(\omega t-\beta r)}\sqrt{1-\cos\delta}\;/\psi; \qquad \psi = \tan^{-1}\left(\frac{B}{A}\right) \quad (782)$$

The Poynting vector for the go and return conductors then yields

$$p = \frac{1}{2}\mathcal{E}_\theta\cdot H_\varphi^* = \frac{1}{2}\underbrace{\frac{60ID_\theta(A-jB)}{r}}_{\mathcal{E}_\theta}\underbrace{\sqrt{\frac{\kappa}{\mu}}\frac{60ID_\theta(A+jB)}{r}}_{H_\varphi^*}$$
$$= \frac{1}{2}\frac{1}{376.7}\left(\frac{60ID_\theta}{r}\right)^2(A^2+B^2)$$
$$= 4.78\frac{I^2D_\theta^2}{r^2}(2-2\cos\delta) = 9.56\frac{I^2D_\theta^2}{r^2}(1-\cos\delta) \qquad \text{watts/m}^2 \quad (783)$$

since H_φ^* is conjugate to H_φ and H_φ has the factor $(A - jB)$ as has $\mathcal{E}_\theta$. This result shows that since $\delta = \beta s \sin\theta \cos\varphi$ and, at best, $\sin\theta$ and $\cos\varphi$ can attain values of unity, we have $\cos\delta = \cos(\beta s)$ or $\cos(-\beta s) = \cos(\beta s)$. Hence, it does not matter whether the argument $\delta = \beta s \sin\theta\cos\varphi$ comes out positive or negative, since in either case we have the factor $1 - \cos\delta$ in (783) and must expect a value of this factor between zero and unity. There is one great difference between the Poynting vector holding for the actual double line [Eq. (783)] and the case where we deal only with one conductor and the same wave distribution [Eq. (779)]. The difference lies in the fact that for the actual double line, p is a function of both the elevation $(90 - \theta)$ and the lateral bearing φ of the radiation field point in Fig. 95. Suppose the distant field point has an elevation of 45 deg and a bearing of 60 deg. Then we find $90 - \theta = 45$ deg or θ also equal to 45 deg; $\varphi = 60^2$, yielding $\sin\theta = 0.7071 = \cos\theta$, $\cos\varphi = 0.5$, and

$$D_\theta = \frac{\cos(90^\circ\cos\theta)}{\sin\theta} = \frac{\cos\overbrace{(0.7071\times 90^\circ)}^{63.5^\circ}}{0.7071} = \frac{0.4462}{0.7071} = 0.632$$

$$\text{for } l = \frac{\lambda}{2}, q = 1$$

$$D_\theta = \frac{\sin(0.5\pi q\cos\theta)}{\sin\theta} = \frac{\sin(\pi\cos\theta)}{\sin\theta} = \frac{\sin(180^\circ\cos\theta)}{\sin\theta}$$

$$= \frac{\sin\overbrace{(0.707\times 180^\circ)}^{127^\circ}}{0.7071} = \frac{\sin(90+37)^\circ}{0.7071} = \frac{\cos 37^\circ}{0.7071}$$

$$= \frac{0.7986}{0.7071} \cong 1.13 \text{ for } l = \lambda,\ q = 2$$

$$\delta = \beta s\sin\theta\cos\varphi = \frac{360^\circ}{\lambda} s\sin\theta\cos\varphi = 360^\circ \times 0.7071 \times 0.5\frac{s}{\lambda} = \frac{s}{\lambda}127^\circ$$

The Poynting vector in the θ, φ direction for $\theta = 45$ deg and $\varphi = 60$ deg, then becomes, for the double line $\frac{1}{2}\lambda$ and λ long, respectively,

$$\left.\begin{aligned}
p^{(\text{watts/m}^2)} &= 9.56\frac{I^2D_\theta^2}{r^2}(1-\cos\delta)\\
&= 9.56\frac{I^2}{r^2}0.632^2\left[1-\cos\left(127\frac{s}{\lambda}\right)^\circ\right]\\
&= 3.8\left[1-\cos\left(127\frac{s}{\lambda}\right)^\circ\right]\frac{I^2}{r^2} \qquad \text{for } l = \frac{\lambda}{2}\\
p^{(\text{watts/m}^2)} &= 9.56\frac{I^2D_\theta^2}{r^2}(1-\cos\delta)\\
&= 9.56\frac{I^2}{r^2}1.13^2\left[1-\cos\left(127\frac{s}{\lambda}\right)^\circ\right]\\
&= 12.2\left[1-\cos\left(127\frac{s}{\lambda}\right)^\circ\right]\frac{I^2}{r^2}\text{ for } l = \lambda
\end{aligned}\right\} \qquad (784)$$

$$\lambda^{(\text{m})} = \frac{300}{f^{(\text{Mc/sec})}}; \qquad 127^\circ\frac{s}{\lambda} = 0.423^\circ sf$$

It is, therefore, the product of the spacing s in meters between the go and return wires and the frequency f in megacycles per second or the ratio s/λ, where s and λ are in the same units (meters, for instance), that determines the effectiveness of the Poynting vector in a certain direction. Hence, when the product sf or the ratio s/λ has a fixed value, the magnitude of p is fixed. This means that a spacing $s = 0.1$ m and wavelength along the parallel wires of $\lambda = 3$ m have the same radiation effect as a spacing of 0.01 m and a wavelength $\lambda = 0.3$ m. It also means that a frequency $f = 100$ Mc/sec with a spacing $s = 0.1$ m gives the same effect as a frequency $f = 1{,}000$ Mc/sec with a spacing reduced to $s = 0.01$ m $=$ 1 cm, since in each case we preserve the magnitude $sf = 10$. Such an outcome is expected, since in wave propagation, we have to express physical dimensions of a radiating system in terms of the operating wavelength along the dimension. Suppose we deal with a parallel-wire

system open at either end and with a spacing of $s = 10$ cm ($= 10 \times 0.3937$ in. $= 3.937$ in.) between the centers of respective conductors. This assumes also that the radius of the conductors is small compared with the spacing. We have then the values given in Table 22 and note

TABLE 22

f, Mc/sec	λ, m	$sf = 0.1f$	$\delta° = 0.423sf$	$\cos \delta°$	$1 - \cos \delta°$
10	30	1	0.423	1.000	0.000
100	3	10	4.23	0.99725	0.00275
1000	0.3	100	42.3	0.7396	0 2604

Remarks. $\delta° = 0.423sf$ for $\theta = 45°$ and $\varphi = 60°$. Factor $1 - \cos \delta$ affects magnitude of $p_{\theta,\varphi}$ watts/m².

that at 10 Mc/sec, there is practically no radiation effect in the 45 deg elevation and 60 deg bearing direction in the distant field region. For the other frequencies, we find

$$\left.\begin{aligned} p^{(\text{watts/m}^2)} &= 3.8 \times 0.00275 \frac{I^2}{r^2} = 1.04 \times 10^{-2} \frac{I^2}{r^2} \quad \text{for } l = \frac{\lambda}{2} \\ p^{(\text{watts/m}^2)} &= 12.2 \times 0.00275 \frac{I^2}{r^2} = 3.35 \times 10^{-2} \frac{I^2}{r^2} \quad \text{for } l = \lambda \end{aligned}\right\} \text{for 100 Mc/sec}$$

$$\left.\begin{aligned} p &= 3.8 \times 0.2604 \frac{I^2}{r^2} = 0.99 \frac{I^2}{r^2} \quad \text{for } l = \frac{\lambda}{2} \\ p &= 12.2 \times 0.2604 \frac{I^2}{r^2} = 3.2 \frac{I^2}{r^2} \quad \text{for } l = \lambda \end{aligned}\right\} \text{for 1,000 Mc/sec}$$

while for the case when *only* the go or *only* the return conductor is *present*, we have in the same direction as much as

$$\left.\begin{aligned} p^{(\text{watts/m}^2)} &= 4.78 \times 0.399 \frac{I^2}{r^2} = 1.9 \frac{I^2}{r^2} \quad \text{for } l = \frac{\lambda}{2} \\ p^{(\text{watts/m}^2)} &= 4.78 \times 1.28 \frac{I^2}{r^2} = 6.1 \frac{I^2}{r^2} \quad \text{for } l = \lambda \end{aligned}\right\} \text{for either frequency}$$

since (779) applies for the $l = \lambda/2$ length and, for the $l = \lambda$ length, we use $D_\theta^2 = [\sin^2 (180° \cos \theta)]/\sin^2 \theta = 1.13^2 = 1.28$ in place of

$$D_\theta^2 = \frac{\cos^2 (90° \cos \theta)}{\sin^2 \theta} = 0.622^2 = 0.399$$

In the numerical evaluations for p, it is assumed that the distance r from the radiation center is so far away that a plane wave front of 1 sq

m holds essentially for the direction r, $\theta = 45$ deg, $\varphi = 60$ deg. The results for the double line are what count and also the frequency effect for a given spacing $s = 0.1$ m between the centers of the wires.

To find the radiation resistance of the double line, we must obtain a solution for the total radiated power in unobstructed radiation space. We have then

$$P_r = \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} pr^2 \underbrace{\sin\theta\, d\theta\, d\varphi}_{\text{solid angle}} \quad \text{watts}; \qquad p = \frac{1}{2} \mathcal{E}_\theta \cdot H_\varphi^* = \frac{1}{2} \operatorname{Re}\left(\sqrt{\frac{\kappa}{\mu}}\, |\mathcal{E}_\theta|^2\right)$$

watts/m²

$$\mathcal{E}_\theta = \frac{60I}{r} \epsilon^{j(\omega t - \beta r)} D_\theta (1 - \epsilon^{-j\delta}) \qquad \text{volts/m}$$

$$H_\varphi^* = \text{conjugate } (H_\varphi); \qquad \delta = \beta s \sin\theta \cos\varphi; \qquad \sqrt{\frac{\kappa}{\mu}} \cong \sqrt{\frac{\kappa_0}{\mu_0}} = \frac{1}{376.7} \qquad \text{mhos}$$

$$p = \frac{1}{2} \frac{60^2 I^2}{r^2} D_\theta^2 \frac{1}{376.7} (1 - \epsilon^{-j\delta})(1 - \epsilon^{+j\delta})$$

$\frac{1}{2}(60^2/376.7) = 4.78$; $(1 - \epsilon^{+j\delta})$ is conjugate to $(1 - \epsilon^{-j\delta})$;

$$(1 - \epsilon^{-j\delta})(1 - \epsilon^{+j\delta}) = 1 - \epsilon^{j\delta} - \epsilon^{-j\delta} + \underbrace{\epsilon^{-j\delta}\epsilon^{+j\delta}}_{=1} = 2 - (\epsilon^{j\delta} + \epsilon^{-j\delta})$$

Hence, $(1 - \epsilon^{-j\delta})(1 - \epsilon^{+j\delta}) = 2 - 2\cos\delta$, since $\epsilon^{j\delta} + \epsilon^{-j\delta} = \cos\delta + j\sin\delta + \cos\delta$

$$-j\sin\delta = 2\cos\delta; \qquad p = 9.56 \frac{I^2 D_\theta^2}{r^2} (1 - \cos\delta) \qquad \text{watts/m}^2$$

Therefore,

$$P_r = \int\int 9.56 \frac{I^2 D_\theta^2}{r^2} (1 - \cos\delta) r^2 \sin\theta\, d\theta\, d\varphi$$

$$= 9.56 I^2 \int\int (1 - \cos\delta) D_\theta^2 \sin\theta\, d\theta\, d\varphi$$

$$= 9.56 I^2 \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=\theta}^{\theta=\pi} D_\theta^2 [1 - \cos(\beta s \sin\theta \cos\varphi)] \sin\theta\, d\theta\, d\varphi \qquad \text{watts} \qquad (785)$$

Since the radiation resistance R_r multiplied by the effective current in square represents the radiated power and $I = \sqrt{2}\, I_{\text{eff}}$, we have

$$P_r = R_r \frac{I^2}{2} \qquad \text{watts}$$

$$R_r = \frac{2P_r}{I^2} = 19.12 \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} D_\theta^2 [1 - \cos(\beta s \sin\theta \cos\varphi)] \sin\theta\, d\theta\, d\varphi$$

ohms (786)

Relations (785) and (786) give then a means of computing the radiated power P_r and the radiation resistance of a parallel-wire transmission line

with stationary waves. The directivity D_θ denotes not the directivity for the resultant effect but the self-directivity of either the go or the return conductor when the other conductor is not present. The angle θ is with respect to the direction of the double line. The resultant directivity depends also on the interference factor $(1 - \cos \delta)$ or strictly $2(1 - \cos \delta)$. Since $2D_\theta(1 - \cos \delta)$ depends on both the θ and the φ variations, an unsymmetrical pattern about the direction of the parallel-wire line exists. With respect to the evaluation of the expressions for the radiated power P_r and the radiation resistance R_r, we have the following procedure:

In either case, we deal with the double integral

$$\int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} D_\theta^2(1 - \cos \delta) \sin \theta \, d\theta \, d\varphi = \underbrace{\iint \overbrace{D_\theta^2 \sin \theta}^{M_\theta} \, d\theta \, d\varphi}_{A} - \underbrace{\iint M_\theta \cos \delta \, d\theta \, d\varphi}_{B}$$

and

$$R_r = 19.12(A - B) \qquad \text{ohms} \tag{787}$$

The A term is the same as used for one conductor only, with a meridian directivity D_θ. The B term accounts for the interference of the mutual radiation effect of the go and the return conductors on each other. We have in either double integral

$$\left.\begin{aligned} M_\theta &= D_\theta^2 \sin \theta = \frac{\cos^2 (0.5\pi q \cos \theta)}{\sin \theta} \qquad \text{for } q = 1, 3, 5, 7, \cdots \text{ or} \\ &\qquad\qquad l = \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, 7\frac{\lambda}{2}, \cdots \\ M_\theta &= \frac{\sin^2 (0.5\pi q \cos \theta)}{\sin \theta} \qquad \text{for } q = 2, 4, 6, \text{ or } l = \lambda, 2\lambda, 3\lambda \end{aligned}\right\} \tag{788}$$

We find then, for $l = 0.5\lambda, 1.5\lambda, 2.5\lambda, 3.5\lambda, \cdots$, *i.e.*, q an *odd* integer, and $u = \cos \theta$, resulting in $du/d\theta = -\sin \theta$, $d\theta = du/(-\sin \theta)$,

$$\begin{aligned} A &= \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_{\theta=0}^{\theta=\pi} M_\theta \, d\theta = 2\pi \int_0^\pi M_\theta \, d\theta = 2\pi \int_0^\pi \frac{\cos^2 (0.5\pi q \cos \theta)}{\sin \theta} d\theta \\ &= 2\pi \int_{u=-1}^{u=+1} \frac{\cos^2 (0.5\pi q u)}{1 - u^2} du = \pi \int_{-1}^{+1} \frac{1 + \cos (\pi q u)}{1 - u^2} du \\ &= \frac{\pi}{2}\left[\int_{-1}^{+1} \frac{1 + \cos (\pi q u)}{1 + u} du + \int_{-1}^{+1} \frac{1 + \cos (\pi q u)}{1 - u} du\right] \\ &= 3.14 \int_{-1}^{+1} \frac{1 + \cos (\pi q u)}{1 + u} du \end{aligned}$$

For the substitution $v = \pi q(1 + u)$, we find

$$u = \frac{v}{\pi q} - 1; \qquad \frac{du}{dv} = \frac{1}{\pi q}; \qquad du = \frac{dv}{\pi q};$$

$$A = 3.14 \int_{v=0}^{v=2\pi q} \frac{1 + \cos (v - \pi q)}{v/(\pi q)} \frac{dv}{\pi q} = 3.14 \int_0^{2\pi q} \frac{1 - \cos v}{v} dv$$

But

$$\int_0 \frac{1 - \cos v}{v}\, dv = \int_0 \frac{1 - \left(1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \cdots\right)}{v}\, dv$$

$$= \int_{v=0}^{v=2\pi q} \left(\frac{v}{2} - \frac{v^3}{24} + \frac{v^5}{720} - \cdots\right) dv$$

$$\int_0^v \frac{1 - \cos v}{v}\, dv = 0.577 + 2.3026 \log_{10}(6.28q) - \mathrm{Ci}\,(6.28q); \qquad 0.577 \text{ is Euler constant}$$

$$\log\,(6.28q) = \log\,(6.28) + \log\,(q)$$

The second solution is *complete*, while the first solution is convenient if only a few terms give a satisfactory approximation. We find then

$$A = 3.14\left(\left|\frac{1}{2}\,\frac{v^2}{2}\right| - \left|\frac{1}{24}\,\frac{v^4}{4}\right| + \left|\frac{1}{720}\,\frac{v^6}{6}\right| - \cdots\right)_{v=0}^{v=6.28q}$$

$$= 3.14\left[\frac{1}{4}(6.28q)^2 - \frac{(6.28q)^4}{24 \times 4} + \frac{(6.28q)^6}{720 \times 6} - \cdots\right]$$

$$= 3.14(9.87q^2 - 16.2q^4 + 14.2q^6 - \cdots)$$

This result shows that many terms must be used, since $q = 1, 3, 5, \cdots$. It is, therefore, the second solution that must be employed, and we find

$$A = 3.14[\underbrace{0.577 + 2.303 \times 0.798}_{2.415} + 2.303 \log_{10}(q) - \mathrm{Ci}\,(6.28q)]$$

$$= 7.6 + 7.22 \log_{10}(q) - 3.14\,\mathrm{Ci}\,(6.28q) = 6.28 \int_{\theta=0°}^{\theta=90°} M_\theta\, d\theta \qquad (789)$$

Exactly the same result is obtained for $q = 2, 4, 6, \cdots$, and it does not matter which expression for $D_\theta^2 \sin\theta$ in (788) is integrated. We have then the formula

$$A = 7.6 + 7.22 \log_{10}(q) - 3.14\,\mathrm{Ci}\,(6.28q); \qquad q = 1, 2, 3, 4, \cdots \quad \text{or}$$
$$l = 0.5\lambda;\ \lambda;\ 1.5\lambda;\ 2\lambda;\ 2.5\lambda \text{ etc.} \cdots \qquad (789a)$$

For a half-wavelength distribution ($l = 0.5\lambda$), we should have $A \cong 7.6$, since $q = 1$, $\log(q) = 0$, and Ci (6.28) is also about zero. Since the integral cosine function Ci x starts with minus infinity value[1] and oscillates about the zero-reference axis with increasing x values toward Ci $x = 0$ with practical zero value from about $x = 12$ on, we can say that for $q \geqq 2$ or $l \geqq \lambda$ (and still higher distributions such as $l = 1.5\lambda, 2\lambda, \cdots$), the cosine integral term is negligible. It happens to be nearly zero value also for $q = 1$ as mentioned above.

As regards the second term B in (787), we have, for the same expressions for $D_\theta^2 \sin\theta$ as given in (788),

$$B = \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \underbrace{D_\theta^2 \sin\theta}_{M_\theta} \cos\delta\, d\theta\, d\varphi = \int_0^{2\pi} \int_0^{\pi} M_\theta \cos(\beta s \sin\theta \cos\varphi)\, d\theta\, d\varphi \qquad (790)$$

[1] See Fig. 147. See also A. HUND "Frequency Modulation," Fig. 90, p. 282, McGraw-Hill Book Company, Inc., New York, 1942.

with a somewhat more involved solution, because we can *no longer* separate the θ and the φ functions, as was done for the A term where we could write

$$A = \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_{\theta=0}^{\theta=\pi} M_\theta \, d\theta$$

and treat each integral like a factor of a product. The solution of (790) requires *partial integration* as well as *integration by parts*. Both partial integration and integration by parts can be understood if we quickly review partial differentiation and differentiation of products of functions.

Suppose u is a function of x and y and, for instance, as in $u = 10x^3 + 5y^3 + ax^2y + bxy^2$; then

$$\frac{\partial u}{\partial x} = 30x^2 + 0 + 2axy + by^2; \qquad \frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial\left(\dfrac{\partial u}{\partial x}\right)}{\partial y} = 0 + 0 + 2ax + 2by$$

Now $\partial u/\partial x$ is the partial differentiation of the function u with respect to x, and the opposite operation (partial integration) leads to

$$\partial u/\partial x = 30x^2 + 0 + 2axy + by^2$$

$$u = \int_x (30x^2 + 0 + 2axy + by^2) \, \partial x = \frac{30x^3}{3} + f(y) + \frac{2ax^2y}{2} + bxy^2$$
$$= 10x^3 + ax^2y + bxy^2 + f(y)$$

where $f(y)$ accounts for the integration constant. In exactly the same way, we can reverse the operation from

$$\frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial\left(\dfrac{\partial u}{\partial x}\right)}{\partial y} = 0 + 0 + 2ax + 2by$$

to

$$\frac{\partial u}{\partial x} = \int_y (0 + 0 + 2ax + 2by) \, \partial y = 0 + f(x) + 2axy + \frac{2by^2}{2}$$

$$= 0 + \underbrace{f(x)}_{\text{integration constant}} + 2axy + by^2$$

to

$$u = \int_x [0 + f(x) + 2axy + by^2] \, \partial x = f(y) + \int f(x) \, dx + \frac{2ax^2y}{2} + bxy^2$$

$$= \underbrace{f(y)}_{\substack{\text{must be} \\ 5y^3}} + \underbrace{\int f(x) \, dx}_{\substack{\text{must be} \\ 10x^3}} + ax^2y + bxy^2$$

For the differentiation of products, we have, if u and v are functions of z,

$$\frac{d(u \cdot v)}{dz} = u\frac{dv}{dz} + v\frac{du}{dz}; \quad \text{and} \quad u \cdot v = \int u \frac{dv}{dz} dz + \int v \frac{du}{dz} dz = \int u \, dv + \int v \, du;$$

or

$$\int_a^b u\,dv = \Big|u \cdot v\Big|_a^b - \int_a^b v\,du \cdots (791); \quad \text{or}$$

$$\int_a^b u\frac{dv}{dz}dz = \Big|u \cdot v\Big|_a^b - \int_a^b v\frac{du}{dz}dz \cdots \quad (792)$$

Applying partial operation of this type to relation (790), we can write (790) as

$$\frac{\partial^2 B}{\partial\theta\,\partial\varphi} = M_\theta \cos(\beta s \sin\theta\cos\varphi) = \frac{\partial(\partial B/\partial\theta)}{\partial\varphi}$$

with a partial integration

$$\frac{\partial B}{\partial\theta} = \int_{\varphi=0}^{\varphi=2\pi} M_\theta \cos(\underbrace{\beta s \sin\theta}_{a}\cos\varphi)\,\partial\varphi = M_\theta \int_{\varphi=0}^{\varphi=2\pi}\cos(a\cos\varphi)\,d\varphi = M_\theta F$$

since neither M_θ nor $a = \beta s \sin\theta$ is a function of φ. The factor F represents the solution of the integral taken over the limits $\varphi = 0$ to 2π. This factor is a function of θ only, since it contains a term $a = \beta s \sin\theta$. The value of B is then found upon integration with respect to θ, yielding

$$B = \int_{\theta=0}^{\theta=\pi} F M_\theta\,d\theta$$

This relation is then solved by integration in parts, by putting $F = u$ and $M_\theta = dv/d\theta$, yielding

$$B = \int_{\theta=0}^{\theta=\pi} u\frac{dv}{d\theta}d\theta = \Big|uv\Big|_{\theta=0}^{\theta=\pi} - \int_{\theta=0}^{\theta=\pi} v\frac{du}{d\theta}d\theta$$

where

$$M_\theta = D^2_\theta \sin\theta = \frac{\cos^2(0.5\pi q\cos\theta)}{\sin\theta} \quad \text{for} \quad q = 1, 3, 5, \cdots \quad \text{or} \quad l = 0.5\lambda, 1.5\lambda, 2.5\lambda$$

$$M_\theta = \frac{\sin^2(0.5\pi q\cos\theta)}{\sin\theta} \quad \text{for} \quad q = 2, 4, 6, \cdots \quad \text{or} \quad l = \lambda, 2\lambda, 3\lambda, \cdots$$

$$v = \int_{\theta=0}^{\theta=\pi} M_\theta\,d\theta = \underbrace{1.2075 + 1.1513\log_{10}(q) - 0.5\,\mathrm{Ci}(6.28q)}_{\text{pages 512, 643–648}} \quad \text{for}$$

$$q = 1, 2, 3, 4, \cdots \quad \text{or} \quad l = 0.5\lambda, \lambda, 1.5\lambda, 2\lambda, \cdots$$

$$\frac{du}{d\theta} = \frac{dF}{d\theta}; \quad F = \int_{\varphi=0}^{\varphi=2\pi}\cos(a\cos\varphi)\,d\varphi; \quad a = \beta s\sin\theta; \quad \beta = \frac{2\pi}{\lambda}$$

We find then

$$B = \underbrace{[1.2075 + 1.1513\log_{10}(q) - 0.5\,\mathrm{Ci}(6.28q)]}_{v} \cdot \Big|F\Big|_{\theta=0}^{\theta=\pi} - v\int_{\theta=0}^{\theta=\pi}\frac{dF}{d\theta}d\theta \quad (793)$$

where the integration for F with respect to φ is done by integration by substitution, similar to the method employed for $\int M_\theta\,d\theta$.

A much simpler solution for the radiation resistance is described on page 191, applying for a stationary-wave distribution along the double line, for a length l up to a half wavelength. The method can also be used for longer lines if the concept of the effective length l_e is not used and the elementary current element effects on the go and the return conductors in the radiation field are added up with respect to phase and amplitude.

The method on page 235 is not based on the Poynting vector concept which gives a means for computing the radiation resistance from the outward electromagnetic field energy flow at a great distance from the radiation system. The method is based on the fact that the product of the current density and the inphase induced electric field $\mathcal{E}_t$ when integrated over the conductor surface will give the radiated energy coming from the conductor. Also this procedure requires the knowledge of the current distribution on the conductor system and is based on the tangential compounds H_t and $\mathcal{E}_t$, since at the surface of a conductor, the tangential magnetic field H_t amperes per meter is normal to the current density J amperes per square meter. The product of the tangential inphase induced electric component $\mathcal{E}_t$ volts per meter and J is numerically equal to the magnitude of H_t(even though $\mathcal{E}_t J$ is in watts per cubic meter). Hence, when $\mathcal{E}_t J$ is integrated over the conductor surface, which is in square meters, we obtain the total radiated power P_r in watts. The quantity $\mathcal{E}_t \cdot H_t$ in watts per square meter represents then the real part of the Poynting vector. The same result is obvious from the fact that the radiated power which is lost from a conductor system must be coming from the source that causes the conductor current. Now for a perfect conductor ($\sigma = \infty$ mhos per meter), there can be no tangential component $\mathcal{E}_t$ along the conductor surface. We then have to assume an additional external field $\mathcal{E}'_t$ which just compensates the $\mathcal{E}_t$ field. This compensating field then delivers to the conductor system just as much power as is being radiated from it. For a thin single conductor, this would give, on account of $\mathcal{E}'_t = -\mathcal{E}_t$,

$$P_r = \tfrac{1}{2}\,\mathrm{Re}\int_{z=0}^{z=l} \mathcal{E}'_t \cdot I_z^*\,dz = 0.5R_rI^2 \qquad \text{watts}$$

where $\mathcal{E}'_t$ is along the conductor and I_z^* denotes the conjugate current distribution along the conductor direction Z for the actual distribution I_z. Since $\mathcal{E}'_t$ is in volts per meter, the corresponding voltage $\mathcal{E}'_t\,dz$ volts along the conductor length *dz must be* 180 *deg out of phase* with the conductor current. We can, therefore, compute the total radiated power (average value) from a system of currents from

$$P_\tau^{(\text{watts})} = -\tfrac{1}{2}\,\text{Re}\iiint_\tau \mathcal{E}_t \cdot J^*\,d\tau \qquad \text{integration of power per unit volume over volume } \tau \text{ of current distribution refers only to space } \tau \text{ occupied by actual radiators}$$

$$P_\tau^{(\text{watts})} = \iint_S p\,dS \qquad \text{integration of normal component of Poynting vector } p \text{ in watts/m}^2 \text{ over a closed surface of } S \text{ sq m enveloping completely all radiating sources and reaching into the radiation region} \tag{794}$$

On page 163, we found that the electric field intensity, at any distance from a Hertzian doublet radiator, is

$$\mathcal{E}_\theta^{(\text{volts/m})} = \frac{\beta^3 m \sin\theta}{4\pi\kappa}\left[-\frac{1}{\beta r} + \frac{j}{(\beta r)^2} + \frac{1}{(\beta r)^3}\right]$$

$$m = q\,dl = \int m_e\,dt = \int I\,dl\epsilon^{j(\omega t-\beta r)}\,dt = \frac{m_e}{j\omega} = \frac{I\,dl\epsilon^{j(\omega t-\beta r)}}{j\omega}$$

where it does not matter whether we deal with a doublet radiator of charge moment $m = q\,dl$ meter · coulombs or an elementary current radiator of electric current moment $m_e = i\,dl$ meter · amperes, since we have the interrelation $m = m_e/(j\omega)$. Since $\omega = c\beta = \beta/\sqrt{\mu\kappa}$ and

$$\frac{\beta^3 m}{4\pi\kappa} = \frac{\beta^3 m_e}{j\omega 4\pi\kappa} = \frac{\beta^3 m_e}{j\beta 4\pi\kappa}\sqrt{\mu\kappa} = \frac{\beta^2 m_e}{j4\pi}\sqrt{\frac{\mu}{\kappa}} = -j\frac{m_e\beta^2 Z_0}{4\pi}$$

we find

$$\mathcal{E}_\theta = \frac{Z_0 m_e \sin\theta}{4\pi}\left(\frac{j\beta}{r} + \frac{1}{r^2} - \frac{j}{\beta r^3}\right) = \frac{Z_0 I\,dl\epsilon^{j\omega t}}{4\pi}\left(\frac{j\beta}{r} + \frac{1}{r^2} - \frac{j}{\beta r^3}\right)\epsilon^{-j\beta r}$$

But

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots; \qquad 3! = 1 \times 2 \times 3; \qquad \text{or}$$

$$\epsilon^{-j\beta r} = 1 - j\beta r - \frac{\beta^2 r^2}{2} + j\frac{\beta^3 r^3}{6}\cdots$$

Therefore,

$$\mathcal{E}_\theta^{(\text{volts/m})} = \frac{Z_0 I\,dl\epsilon^{j\omega t}\sin\theta}{4\pi}\Big(-\frac{2}{3}\beta^2 \underbrace{- j\frac{\beta}{r} - j\frac{2}{\beta r^3} + \cdots}_{\text{reactive terms}}\Big)$$

Only the real term

$$-(\beta^2 Z_0 I\,dl\epsilon^{j\omega t}\sin\theta)/(6\pi)$$

can contribute to the radiated field power, according to the first of Eqs. (794). The induced voltage E in the length dl of the elementary $i\,dl$ conductor is the tangential

component of $\mathcal{E}_\theta$ along the conductor times the length dl, or

$$E^{(\text{volts})} = -\frac{\beta^2 Z_0 I\, dl \sin 90^\circ}{6\pi}\, dl = -\frac{\left(\frac{2\pi}{\lambda}\right)^2 Z_0 I\, (dl)^2}{6\pi} = -\frac{2\pi Z_0 I}{3}\left(\frac{dl}{\lambda}\right)^2$$

and this voltage, multiplied by the current I, gives the power that is radiated, where E and I are the maximum time values. Therefore,

$$P_r = EI = \frac{2\pi Z_0 I^2}{3}\left(\frac{dl}{\lambda}\right)^2 \quad \text{watts} = I_{\text{eff}}^2 R_r = \frac{I^2 R_r}{2}$$

$$R_r = \frac{E_{\text{eff}}}{I_{\text{eff}}} = \frac{E}{I} = \frac{2\pi}{3} Z_0 \left(\frac{dl}{\lambda}\right)^2 = \frac{2\pi}{3} 376.7 \left(\frac{dl}{\lambda}\right)^2 = 789.5\left(\frac{dl}{\lambda}\right)^2 \quad \text{ohms} \tag{795}$$

It is this expression which is used in Sec. 37 for computing the radiation resistance of Lecher feeders excited in the quarter- and half-wavelength distribution.

98. Radiation Resistance of an Open-ended Double Line of Short Electrical Length. This again has reference to a parallel-wire line and the case of unobstructed radiation space. We have then conditions as shown in Fig. 96 and essentially a linear current decrease from the place where the driving voltage E is applied toward the open end. Figure 96 also shows the case of the symmetrical or center feed. Also in this case, it is convenient to express the radiation resistance with respect to the current flow I of the double line at the input points. The effective length of the double line is then $l_e = 0.5l$, and the electric current moment along the Z direction is $m_e = l_e I \epsilon^{j(\omega t - \beta r)} = \frac{1}{2} l I \epsilon^{j(\omega t - \beta r)}$ meter · amperes. Applying (795), we find, for the self-radiation resistance R_1 of the go conductor and R_2 of the return conductor,

$$R_1 = R_2 = 789.5\left(\frac{l_e}{\lambda}\right)^2 = \frac{789.5}{4}\left(\frac{l}{\lambda}\right)^2 = 197.25\left(\frac{l}{\lambda}\right)^2 \quad \text{ohms}$$

This means that the total self-radiation resistance is twice this value, or $R_s = 394.5(l/\lambda)^2$ ohms. The total mutual radiation resistance R_m is, according to formula (187), on page 190,

$$R_m = 9.56\frac{l_e^2 \lambda}{s^3}\left\{6.28\frac{s}{\lambda}\cos\left(\frac{s}{\lambda}360^\circ\right) - \left[1 - \left(6.28\frac{s}{\lambda}\right)^2\right]\sin\left(\frac{s}{\lambda}360^\circ\right)\right\} \quad \text{ohms} \tag{796}$$

if s meters is the spacing between the centers of the go and the return conductors of the parallel-wire transmission line and l_e is again equal to half the actual line length l. For a spacing of $s = 0.1$ m, *i.e.*, $s = 39.37 \times 0.1 = 3.937$ in., we find for a 100-Mc current, the free-space wavelength $\lambda_0 = 300/100 = 3$ m and the wavelength along the parallel-wire

line about $\lambda = 0.95 \times 3 = 2.85$ m. We have then

$$9.56\,\frac{l_e^2\lambda}{s^3} = 9.56\,\frac{0.25l^2\lambda}{s^3} = \frac{9.56 \times 2.85}{4 \times 0.1^3}\,l^2 = 6.78 \times 10^3 l^2$$

$$6.28\,\frac{s}{\lambda}\cos\left(\frac{s}{\lambda}\,360°\right) - \left[1 - \left(6.28\,\frac{s}{\lambda}\right)^2\right]\sin\left(\frac{s}{\lambda}\,360°\right) = 6.5 \times 10^{-3}$$

details on page 190

$$R_m = 6.78 \times 10^3 \times 6.5 \times 10^{-3} l^2 = 44l^2 \quad \text{ohms}$$

showing that we have a mutual radiation resistance that increases with the square of the actual length. Inasmuch as the value $l_e = \frac{1}{2}l$ is

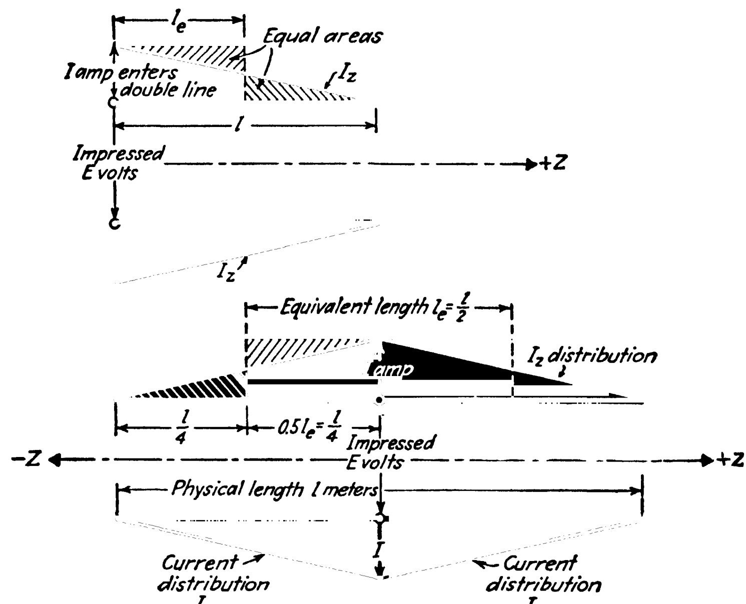


FIG. 96. Current distributions along short electrical lengths of a double-line feeder and equivalent length l_e of such a feeder system. Equivalent length $l_e = 0.5l$ m is found by means of equal shaded areas. A linear current decrease occurs for l in short with respect to the operating wavelength λ along the conductor if a sinusoidal I_z distribution is assumed.

based on a linear current decrease from the source toward the open end of the double line, we have a limit beyond which l cannot be made larger. For a sinusoidal current distribution, linearity exists when in sin x the argument x is equal to sin x. A trigonometric table shows that x radians is equal to sin x up to arguments of about $x \leqq 11$ deg, since even for 11.46 deg, we have $x = 0.2$ radians and sin x = sin 11.46 deg = 0.1987, which is almost 0.2. Hence, when the double line does not have an electrical length βl of more than 11 deg, we may expect a linear current decrease. But $\beta l = 2\pi l/\lambda = 6.28l/\lambda$, or $360°l/\lambda \leqq 11$ deg. This means

that $l \leqq 3.05 \times 10^{-2}\lambda$ is the range of l in terms of the wavelength, for which we have a linear current decrease. Hence, we cannot exceed about 3.05 per cent of the wavelength developed along the double line. Since $\lambda = 2.85$ m, we have for a 100-Mc/sec current $l \leqq 3.05 \times 2.85 \times 10^{-2} \leqq 8.7 \times 10^{-2}$ m $\leqq 39.37 \times 8.7 \times 10^{-2}$ in. $\leqq 3.42$ in.

According to Sec. 37 the total radiation resistance is

$$R_r = R_s - R_m = 394.5\left(\frac{l}{\lambda}\right)^2 - 44l^2 \qquad \text{ohms}$$

for the above-mentioned frequency and spacing. Taking the longest permissible length $l = 3.42$ in. $= 8.7 \times 10^{-2}$ m, we find

$$R_r = 394.5\left(\frac{8.7 \times 10^{-2}}{2.85}\right)^2 - 44(8.7 \times 10^{-2})^2$$
$$= 0.366 - 0.333 = 0.033 \text{ ohm}$$

0.366 ohm is the total self-effect of both conductors; 0.333 ohms is total mutual effect on both conductors. The resultant radiation resistance R_r is then quite small, which would be expected from a short electrical double line.

99. Remarks on the Radiation Resistance and Radiated Power of Customary Transmission Lines for Electric Power. If we think in terms of commercial power frequencies such as 50 or 60 cycles/sec, we can, as a good approximation, just use the actual length l instead of the effective length. For the self-radiation resistance of the go and return conductor, we have then

$$R_s = R_1 + R_2 = 2R_1 = 1{,}579\left(\frac{l}{\lambda}\right)^2 \qquad \text{ohms}$$

and the mutual radiation resistance as given in (796), by using l in place of l_e. Since the spacing s is very much smaller than the wavelength λ developed along the transmission line, we can, for $x = 2\pi s/\lambda$, use the solution

$$R_m = 9.56\,\frac{l^2\lambda}{s^3}\left\{x\cos x - (1 - x^2)\sin x\right\}$$

$$= x\Bigl(\underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!}}_{\cos x}\ \underbrace{- \cdots}_{\text{neglected}}\Bigr) - (1 - x^2)\Bigl(\underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!}}_{\sin x}\ \underbrace{- \cdots}_{\text{neglected}}\Bigr)$$

$$\cong x\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) - (1 - x^2)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

$$= x - \frac{x^3}{2} + \frac{x^5}{24} - x + \frac{x^3}{6} - \frac{x^5}{120} + x^3 - \frac{x^5}{6} + \frac{x^7}{120}$$

$$= \frac{2}{3}x^3 - \frac{2}{15}x^5 + \frac{1}{120}x^7 = \left(\frac{2}{3}\right)\frac{8\pi^3}{\lambda^3} - \left(\frac{2}{15}\right)\frac{32\pi^5}{\lambda^5} + \left(\frac{1}{120}\right)\frac{128\lambda^7}{\lambda^7}$$

$$9.56\left\{\quad\right\} = 1{,}579\left(\frac{s}{\lambda}\right)^3 - 12{,}400\left(\frac{s}{\lambda}\right)^5 + 30{,}500\left(\frac{s}{\lambda}\right)^7$$

The mutual radiation resistance for a commercial alternating current along the parallel-wire transmission line in unobstructed space (image effect of the ground neglected) then becomes, for $p = l/\lambda$ and $q = s/\lambda$,

$$R_m = \frac{l^2\lambda}{s^3}\left[1{,}579\left(\frac{s}{\lambda}\right)^3 - 12{,}400\left(\frac{s}{\lambda}\right)^5 + 30{,}500\left(\frac{s}{\lambda}\right)^7\right]$$
$$= 1{,}579p^2 - 12{,}400p^2q^2 + 30{,}500p^2q^4 \qquad \text{ohms} \qquad (797)$$

The total radiation resistance of the transmission line then is

$$R_r = R_s - R_m = 1{,}579p^2 - R_m = p^2q^2(12{,}400 - 30{,}500q^2) \qquad \text{ohms} \qquad (797a)$$

where p is the length of the double line in terms of the wavelength λ along the line and q is the spacing between the parallel conductors in terms of λ.

An application of this result is then as follows: Suppose we deal with a commercial frequency of 60 cycles/sec. Then we have a free-space wavelength $\lambda_0 = 3 \times 10^8/60 = 5 \times 10^6$ m/sec. The phase velocity of a power line is smaller than the free-space velocity 3×10^8 m/sec. It can be computed from $c' = \omega/\beta$, where, for a power double line,

$$\beta = \sqrt{\tfrac{1}{2}[\sqrt{\omega^4(CL)^2 - \omega^2(RC)^2} - \omega^2 CL]} = \sqrt{\tfrac{1}{2}\omega C[\sqrt{(\omega L)^2 - R^2} - \omega L]}$$
circular radians/m

where $\omega = 6.28 \times 60 = 377$ radians/sec, C is the capacitance in farads across each meter length of the double line, L is the self-inductance of 1 m length (go and return combined) in henrys, and R is the corresponding resistance in ohms per meter. For power overhead lines, air is the dielectric between the conductors, and the leakance G mhos per meter across the line vanishes. We may then have order of magnitudes like $R =$ 0.16 ohm/mile, $L = 0.002$ henry/mile, $C = 0.015 \times 10^{-6}$ farad/mile, depending upon the size and the material of the conductor and the spacing s between the go and the return conductors. Suppose the phase velocity $c' = \omega/\beta$ comes out such that we have the wavelength λ along the double line 10 per cent shorter than the free-space wavelength λ_0; then $\lambda = 0.9\lambda_0 = 0.9 \times 5 \times 10^6 = 4.5 \times 10^6$ m. Suppose we have a spacing of $s = 3$ m and a line length of 10 km; then $l = 10^4$ m. Therefore

$$p = \frac{l}{\lambda} = \frac{10^4}{4.5 \times 10^6} = 2.222 \times 10^{-3}; \qquad p^2 = 4.95 \times 10^{-6}$$

$$q = \frac{s}{\lambda} = \frac{3}{4.5 \times 10^6} = 6.67 \times 10^{-7}; \qquad q^2 = 4.45 \times 10^{-13}$$

The total radiation resistance of the parallel-wire line excited at the commercial frequency of 60 cycles/sec, according to (797), becomes

$$R_r = \underbrace{4.95 \times 10^{-6} \times 4.45 \times 10^{-13}}_{2.2 \times 10^{-18}}(12{,}400 - \underbrace{30{,}500 \times 4.45 \times 10^{-13}}_{1.36 \times 10^{-8} \text{ (negligible compared with 12,400)}})$$

$$= 2.2 \times 10^{-18} \times 1.24 \times 10^4 = 2.74 \times 10^{-14} \text{ ohm}$$

This is, indeed, a very small radiation resistance for such a long double line ($l = 10$ km $= 0.6214 \times 10 = 6.214$ statute miles). Even if we had a circulating alternating current of 1,000 amp, we should have a radiated power of only

$$P_r = 2.74 \times 10^{-14} \times 10^6 = 2.74 \times 10^{-8} \text{ watt}$$

i.e., as little as 0.0274 μw.

100. Notes on Radiation Effects of a Parallel-wire System When Carrying Traveling Waves. This has reference to a system where a Lecher wire feeder is matched at the far end to the characteristic impedance of the feeder. We know from the discussion in Sec. 90 that a matched feeder *will radiate*. A matched feeder shows essentially a fixed-current distribution when the self-attenuation (attenuation of the double line) is negligible. For the self-effect of the radiation field, we have for each line a value for $\mathcal{E}_\theta$ in volts per meter as given by (700) and (701) on page 463; and for the self-radiation resistance for each line without the presence of the other line, formulas (705*a*) show what happens in the case of a matched feeder (traveling waves) and of standing waves. By means of the retarded vector potential method, we can then also find the field for a double line and compute the radiated power from the complex Poynting vector by means of its real part $p = \frac{1}{2}\mathcal{E}_\theta H_\varphi^*$ watts per square meter and the integration over a spherical surface reaching into the distant radiation region. This is expressed by the second of Eqs. (795). The first formula shows a method whereby the radiated power is computed from the work done in driving the electric current against the electric force caused by it. Since in the case of parallel-wire feeders, the diameter of the wire is very small compared with the wavelength, the volume element $d\tau$ simplifies to a line element dl, and the complex power becomes

$$P = -\frac{1}{2} \int_{-0.5l}^{0.5l} \mathcal{E}_t \cdot I^* \, dz \qquad \text{watts}$$

TABLE 23

Frequency f, Mc/sec	Wavelength λ, m	$\beta l = \frac{38.3}{\lambda}$	$2\beta l = \frac{4\pi}{\lambda} l$	$\log_{10} (2\beta l)$	$2.303 \log_{10} (2, l)$	Ci $(2\beta l)$
1	300	0 12767	0 25534	−0 59295	−1 365	−0 8
10	30	1 2767	2 5534	0 40705	0 94	+0 268
100	3	12 767	25 534	1 40705	3 25	−0 0084
1,000	0 3	127 67	255 34	2 40705	5 57	−0 0038

f, Mc/sec	λ, m	$\beta s = \frac{2\pi}{\lambda} s = \frac{1\ 92}{\lambda}$	Ci $\left(\frac{1.92}{\lambda}\right)$	2 Ci $\left(\frac{1.92}{\lambda}\right)$	$(\beta s)^\circ = \frac{108^\circ}{\lambda}$, deg	sin $(\beta s)^\circ$	$\frac{\sin (\beta s)^\circ}{\beta s}$
1	300	0 0064	−4 5	−9 0	0 36	0 0064	1 000
10	30	0 064	−2 2	−4 4	3 6	0 0628	0 98
100	3	0 64	+0 03	+0 06	36	0 5878	0 915
1,000	0 3	6 4	−0 0156	−0 0312	360	0 0000	0 000

f, Mc/sec	λ, m	$2\beta l = \frac{4{,}400}{\lambda}$ deg	sin $\left(\frac{4{,}400}{\lambda}\right)^\circ$	$\frac{\sin (2\beta l)}{2\beta l}$	$\delta^\circ = \frac{38.353}{\lambda}$ numeric	2δ numeric	$\delta^\circ = \frac{2200}{\lambda} \cong (\beta l)^\circ$	$\varphi^\circ = \delta^\circ - \beta l^\circ \cong \delta^\circ - \delta^\circ$
1	300	14 67	0 2538	0 995	0 1278	0 2556	7 335	= 0
10	30	146 7	0 519	0 215	1 278	2 556	73 35	
100	3	1,467(= 27°)	0 454	0 0178	12 78	25 56	733 5	
1,000	0 3	14,670(= 270°)	−1 000	0 00392	127 8	255 6	7335	

f, Mc/sec	$\psi^2 = \delta^\circ + (\beta l)^\circ \cong 2\delta^\circ \cong 2(\beta l)^\circ$	$\sin \varphi + \sin \psi \cong \sin \psi$	$\frac{\sin \varphi + \sin \psi}{2\delta}$	$\varphi = \frac{0\ 053}{\lambda}$ numeric	Ci $\left(\frac{0.053}{\lambda}\right)$	$\psi = \frac{76\ 653}{\lambda}$ numeric	Ci $\left(\frac{76.653}{\lambda}\right)$
1	14 67	0 2538	0.99	0 000176	−8 217	0 2555	−0 8
10	146 7	0 549	0 214	0 00176	−5 9	2 555	+0 268
100	1,467	0 6018	0 0235	0 0176	−3 5	25 55	−0 0084
1,000	14,670	−1 000	0 00391	0 176	−1 175	255 5	−0 0038

Remarks: l is the length of the balanced parallel-wire aperiodic feeder and s is the spacing between centers of the parallel wires. The cosine integral functions (Ci) are taken from tables employing graphical interpolation. 1,467 deg = 4 × 360 deg + 27 deg, and we operate with the argument of 27 deg. Strictly, the wavelength distribution along the line is about 5 per cent shorter than in empty space.

if we had only one wire (either a go or return wire,) of length l meters. Applying this to the double line and taking the real part of P yield also the radiated power P_r watts, from which, as is shown in the preceding sections, the total radiation resistance R_r can be found. Doing this, we arrive, for negligible attenuation along the aperiodic double line, at the expression

$$\left.\begin{aligned} R_r = 120\Big[&-0.4228 + 2.303 \log_{10} (2\beta l) - \mathrm{Ci}\,(2\beta l) + \frac{\sin\,(2\beta l)}{2\beta l} \\ &- 2\,\mathrm{Ci}\,(\beta s) + \frac{\sin\,(\beta s)}{\beta s} - \frac{\sin\varphi + \sin\psi}{2\delta} \\ &\qquad + \mathrm{Ci}\,(\varphi) + \mathrm{Ci}\,(\psi)\Big] \quad \text{ohms} \\ \beta = \frac{2\pi}{\lambda};\quad \varphi &= \delta - \beta l;\quad \psi = \delta + \beta l;\quad \delta = \beta\sqrt{l^2 + s^2} \end{aligned}\right\} \tag{798}$$

where l is the length of double line and s is the spacing between wires in meters.

This expression is not difficult to apply. Suppose we have an aperiodic Lecher wire feeder, with a spacing $s = 1$ ft $= 1/3.281$ m $= 0.3048$ m, and a length $l = 20$ ft $= 6.096$ m. We find then

$$\beta l = \frac{2\pi}{\lambda}\,6.096 = \frac{38.3}{\lambda};\quad 2\beta l = \frac{76.6}{\lambda};\quad (\beta l)^2 = \frac{1{,}467}{\lambda^2};\quad \beta s = \frac{2\pi}{\lambda}\,0.3048 = \frac{1.92}{\lambda}$$

$$(\beta s)^2 = \frac{3.69}{\lambda^2};\qquad \delta = \beta\sqrt{l^2 + s^2} = \sqrt{(\beta l)^2 + (\beta s)^2}$$

$$= \frac{\sqrt{1{,}467 + 3.69}}{\lambda} = \frac{\sqrt{1{,}470.69}}{\lambda} = \frac{38.353}{\lambda}$$

$$\varphi = \delta - \beta l = \frac{38.353 - 38.3}{\lambda} = \frac{0.053}{\lambda};\quad \psi = \delta + \beta l = \frac{38.353 + 38.3}{\lambda} = \frac{76.653}{\lambda}$$

$$(2\beta l)\ \text{deg} = \frac{6.096 \times 720^\circ}{\lambda} = \frac{4{,}100}{\lambda}\ \text{deg};\qquad (\beta s)\ \text{deg} = \frac{0.3048 \times 360^\circ}{\lambda} = \frac{108}{\lambda}\ \text{deg}$$

$$\delta\ \text{deg} = \frac{360^\circ}{\lambda}\sqrt{l^2 + s^2} = \frac{360^\circ}{\lambda}\sqrt{\underbrace{6.096^2}_{37.21} + \underbrace{0.3048^2}_{0.093}} \cong \frac{6.096 \times 360^\circ}{\lambda} = \frac{2{,}200}{\lambda}\ \text{deg}$$

leading to the component values of Table 23. The Ci functions were taken from the cosine integral tables.[1] The values Ci x, for very small arguments x, were obtained by plotting a curve and extending this curve toward minus infinity, since this is the limit for $x = 0$. Great accuracy cannot, therefore, be claimed for very small arguments of the Ci functions in this table. From Table 23 and formula (798), we find for $f = 1{,}000$ Mc/sec.

[1] *Ibid.*

$$R_r = 120[-0.4228 + 5.57 - (-0.0038) + 0.00392 - (-0.0312) + 0.0 - 0.00391 + (-1.175) + (-0.0038)]$$
$$= 120(-0.4228 + 5.57 + 0.0038 + 0.00392 + 0.0312 - 0.00391 - 1.175 - 0.0038)$$
$$= 120(\underbrace{5.60892 - 1.60551}_{4.00341}) \cong 480 \text{ ohms}$$

TABLE 24

f, Mc/sec	$R_r = 120\left[-2.303 \log_{10}(2\beta l) - \text{Ci}(2\beta l) + \frac{\sin(2\beta l)}{2\beta l} - 2\,\text{Ci}(\beta s) + \frac{\sin(\beta s)}{\beta s} - \frac{\sin\varphi + \sin\psi}{2\delta} + \text{Ci}\,\varphi + \text{Ci}\,\psi\right]$ ohms
10	$120(-0.4228 + 0.94 - 0.268 + 0.215 + 4.4 + 0.98 - 0.214 - 5.9 + 0.268) = 120(6.803 - 6.8048) = 0.0018 \times 120 = 0.216$ ohms (the polarity is of no concern here, since only the absolute amount counts in the final result)
100	$120(-0.4228 + 3.25 + 0.0084 + 0.0178 - 0.086 + 0.915 - 0.0235 - 3.5 - 0.0084) = 120(4.1912 - 4.0407) = 0.1505 \times 120 = 23.7$ ohms
1,000	$120(5.6092 - 1.60551) = 4.00341 \times 120 \cong 480$ ohms
1	$120(-0.4228 - 1.365 + 0.8 + 0.995 + 9.0 + 1.0 - 0.99 - 8.217 - 0.8) = 120(11.795 -$ about $11.7948)$ which is about $0.0002 \times 120 \cong 0.024$ ohms This particular evaluation shows only that we deal with a very small radiation resistance, which is probably still smaller than found. The value of -8.217 for $\text{Ci}\,\varphi = \text{Ci}\,0.000176$ is only an approximation, besides the fact that all other values in the parenthesis for 120 $(\cdot\cdot\cdot)$ are not carried to enough decimals to claim too much accuracy for the 1-Mc/sec example

This is quite a high radiation resistance for unobstructed radiation space. Table 24 gives the evaluations for all four frequencies used in Table 23. The solution for the total radiation resistance degenerates to an expression

$$R_r = 1{,}579\left(\frac{s}{\lambda}\right)^2 \quad \text{ohms} \tag{799}$$

if the ratio s/λ is very small and $l \gg s$. This expression is similar to the self-radiation resistance of an elementary current radiator $I\,dl$, for which we have

$$R_r = 1{,}579\left(\frac{dl}{\lambda}\right)^2 \quad \text{ohms}$$

in unobstructed radiation space. Hence, for such a condition, the long aperiodic Lecher wire feeder behaves like a Hertzian doublet with oscillating charges $\pm q$, but

separated by the spacing s between the centers of the go and the return wires. The length of the feeder then no longer appears in formula (799). We have then, for a 1-ft spacing (= 0.3048 m) and a 1-Mc current in the aperiodic Lecher feeder,

$$R_r = 1{,}579\left(\frac{0.3048}{300}\right)^2 = 1{,}579\,\underbrace{(1.016 \times 10^{-3})^2}_{1.0323\,\times\,10^{-6}} = 0.00163 \text{ ohm}$$

From this outcome, we note that the power radiated from a long double line is *independent of* the *length of* the *line*, since $P_r = R_r I_{\text{eff}}^2$ watts.

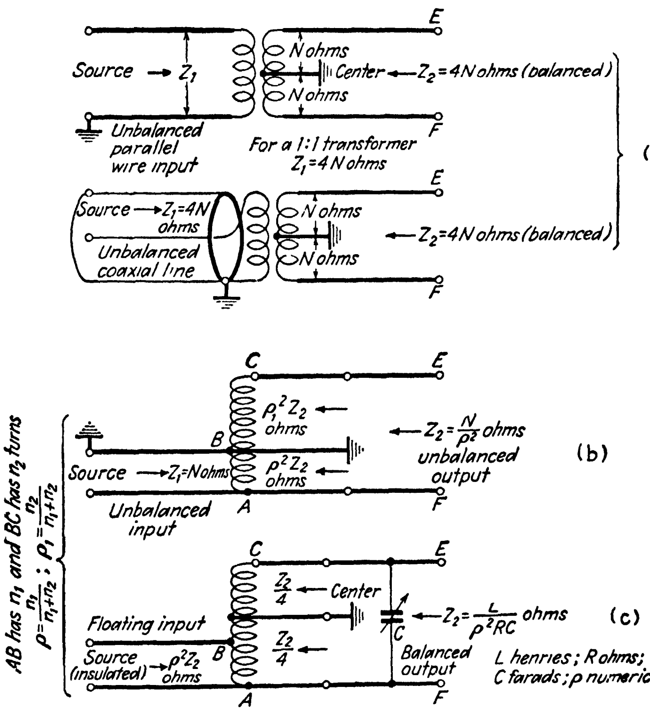


FIG. 97. Systems for converting to balanced output feeder supply. All turns ($n_1 + n_2$) have the inductance L henrys and resistance R ohms, and C farads is set for parallel resonance.

101. Balanced and Unbalanced Feeders. For currents whose frequency is not in the short-wave and microwave range, transformers, as in Fig. 97, can be used for converting impedances and transforming an unbalanced input into a balanced output feeder. Figure 97*b* gives the general case of impedance transformation, for which the ground is not symmetrically disposed of with respect to the output terminals E and F. The input turns AB are then taken as n_1, and the output turns as $n_1 + n_2$.

Hence, looking into the autotransformer, we have $Z_1 = N$ ohms for the unbalanced input impedance, while the total impedance is $Z_2 = N/\rho^2$ ohms. With respect to the branch output impedances, we have $\rho^2 Z_2$ and $\rho_1^2 Z_2$ ohms, respectively. Applying these relations to a 1:1 autotransformer, or a customary transformer as in Fig. 97*a*, we have $\rho = \rho_1 = \frac{1}{2}$, and therefore the total output impedance Z_2 equals $4N$ ohms, when N is the output impedance of the respective half secondaries. This must also be the primary impedance on account of the 1:1 transformation. Figure 97*c* gives the case for a tuned secondary with a center tap to ground. For UHF's, a half-wavelength line as in Fig. 98 acts as a 1:1 transformer. Hence, when a 75-ohm unbalanced concentric line feeds the 1:1 impedance line transformer, we obtain a balanced $4 \times 75 = 300$-ohm Lecher wire output. A quarter-wave line can also be used for a balanced output, not by virtue of no impedance transformation (1:1) but by virtue of a transformation from a short circuit on one side of the $\frac{1}{4}\lambda$-line section to an infinite impedance (practically a very high impedance) on the other side. Hence, the quarter wavelength insulates the apparently grounded output terminals of the feeder from ground (dynamically). Figure 99 gives a practical application. The dimension of the "bazooka" coaxial sleeve is then chosen so that the inside diameter D of the sleeve to the outside diameter d of the concentric-line feeder has the same ratio as the inside diameter d_i of the outside cylinder of the concentric feeder to the outside diameter d_0 of the axial conductor. The balanced output can then be terminated into a load requiring a balanced output. If a half-wave dipole is connected to A and B of Fig. 99*b*, a 75-ohm feeder will match the radiation resistance but still leave the inductive reactance of 42 ohms uncompensated. This reactance can be canceled by a stub. Figure 100 shows a balanced dipole feeder and a feeder for a single pole, where in each case the length l has to be somewhat shorter than $\lambda/4$, in order to compensate the reactance effect of the input impedance. For the true $\lambda/2$ dipole, the impedance would be $74 + j42$ ohms and, for a $\lambda/4$, pole $37 + j21$ ohms. Figure 101 shows a balanced line output, by means of an "apparent" ground return 5,2 to an unbalanced input. It is assumed that the $Z_0 =$

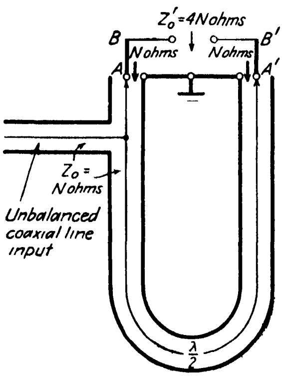
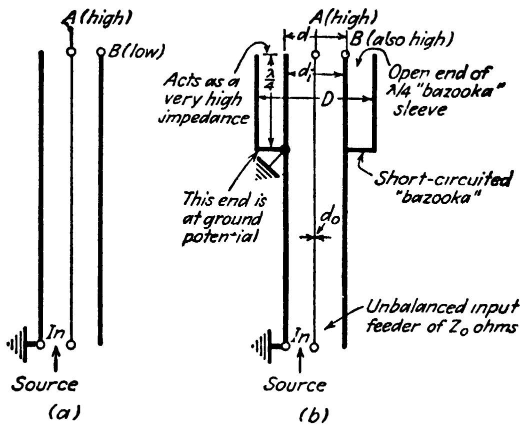


FIG. 98. A conversion of an unbalanced N-ohm coaxial line input by means of a half-wave line transformer into a balanced $4N$-ohm parallel-wire output. AB and $A'B'$ are a balanced Lecher output. If $N = 75$ ohms, a 75-ohm concentric feeder which is unbalanced is converted into a parallel-line balanced output of $4 \times 75 = 300$ ohms.



(a) Coaxial feeder is unbalanced at either end, since outside conductor is at ground potential

(b) Balanced output with both A and B terminals insulated from ground

FIG. 99. Quarter-wavelength sleeve open at the output end of a coaxial feeder leaves both ends of the coaxial line insulated at the output side.

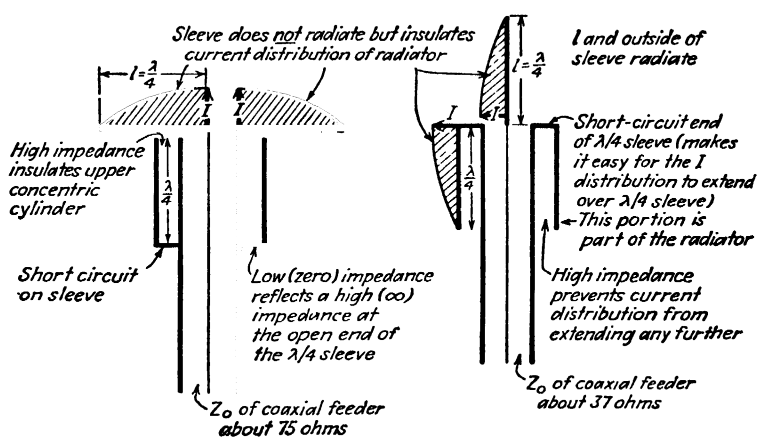


FIG. 100. Quarter-wavelength sleeve about a concentric feeder acts either as an insulator or as a radiator depending on which side the short-circuited end of the sleeve is located.

75-ohm twisted line is matched so that the input impedance $Z_i = 75$ ohms. The spacing d between the go and the return, respectively with respect to the ground wire 2,5 is chosen such that the characteristic impedance Z_0' is high, say $Z_0' = 1{,}000$ ohms. Representations (b) and (c) act then as 1,000-ohm lines. Representation (b) is then a $\frac{1}{4}\lambda$ double line with the end 2,2 short-circuited; and looking into the 3,5 terminals, it

gives an infinite impedance. Hence, as far as the load terminal 3 is concerned, the connection from 5 to 2 is as though dynamically absent, and input terminal 2 is dynamically connected to terminal 3 only by means of the go lead of the twisted line. Representation (*c*) is a ¼λ double line, which looks to the left of terminals 1,2 into an impedance

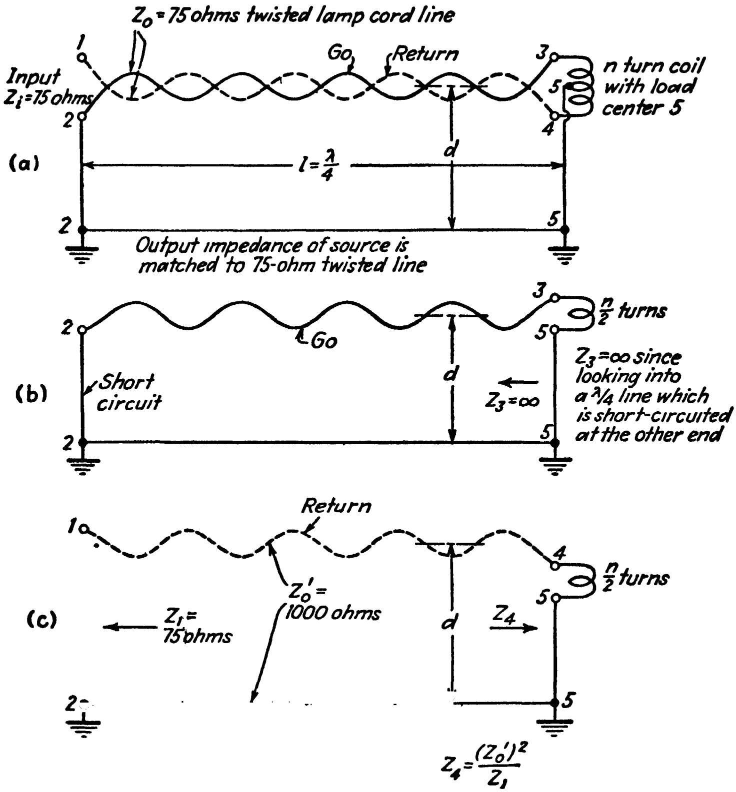


FIG. 101. Balanced output by means of a third quarter-wavelength line.

$Z_1 = 75$ ohms and to the right into an impedance $Z_4 = 1{,}000^2/75 = 13{,}333$ ohms. This is a very large value compared with the impedance of $n/2$ turns of the coil, since the impedance can never exceed $75/2 = 37.5$ ohms. Hence, also as far as load terminal 4 is concerned, the ground return 5,2 cannot be practically effective. Therefore, the short-circuiting effect of the 5,2 line appears, in effect, absent.

CHAPTER VI

UNOBSTRUCTED SPACE RADIATION

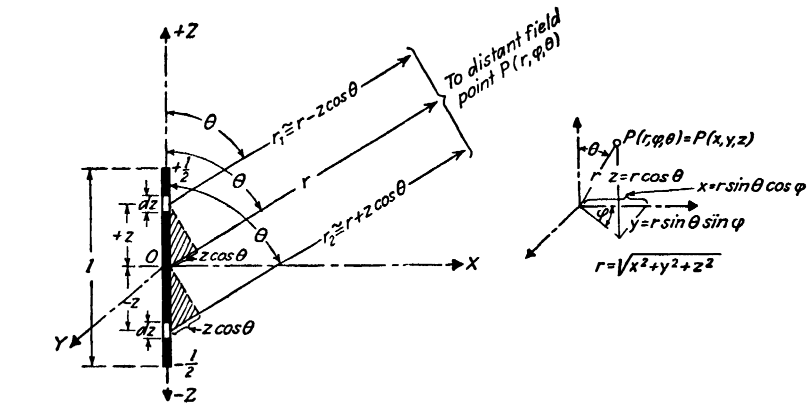
Such electromagnetic radiation has reference to *free*-field intensities, since it is assumed that any obstructing mediums are so far away from the transmitter antenna and the receiver antenna that essentially only the *direct* electromagnetic ray from the transmitter antenna is effective at a space point where the electromagnetic field is received. It does not matter what the polarization of the transmitted wave is, since in the true radiation field, we always orient the receiving antenna in such a way as to have maximum pickup. This happens when a linear antenna is parallel with the direction of the electric lines of force that arrive with the wave front. For a loop receiver, the plane of the loop points, for maximum reception, toward the radiation center. Unobstructed radiation patterns are basic as far as antenna design is concerned, and unobstructed or free-field intensities are likewise a measure for the radiation effect of a certain radiation system. Commercial antennas have not the ideal field pattern as a point source radiator in an unobstructed field region. We may have regions of no fields as well as regions with fields. Linear radiators longer than a half wavelength, unless all current distributions are equally phased, will give rise to destructive wave interference, so that zones of reception and zones of no reception will be produced in the radiation field. All fields and directivity patterns described in this chapter refer to the true radiation field, *i.e.*, to the case of transverse electromagnetic space waves.

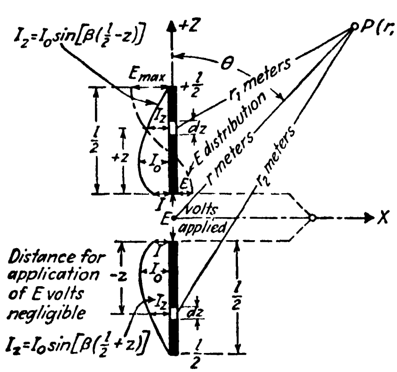
102. Radiation from a Linear Antenna. This has reference to a linear radiator that can be driven at its mid-point as long as this point does not represent a current node, since this would mean theoretically an input impedance that is infinitely high and practically an impedance that is very high. Figure 102 shows the general case where the diameter of the linear radiator is assumed small compared with the length of the radiator and the length of the radiator is considered very small compared with the distance to a field point in the far radiation region. This means that the distance from any radiation element dz along the linear radiator to the field point $P(r, \varphi, \theta)$ is fixed as far as the inverse distance effect in the elementary radiation formula [Eq. (161) on page 165]

$$d\mathcal{E}_\theta = \frac{188.35 I\,dl}{r\lambda} P \sin\theta = \frac{0.628 f I\,dl}{r} P \sin\theta \quad \text{volts/m};$$

$$dH_\varphi = d\mathcal{E}_\theta / 376.7 \quad \text{amp/m}$$

for an elementary electric current element $I\,dl$ is concerned. In our case, $dl = dz$, since the isolated linear radiator is along the Z axis. The field point $P(r, \varphi, \theta)$ is in spherical coordinates, with φ around the radiator





For distant radiation region r_1, r, and r_2 are essentially parallel and $\theta_1 \cong \theta \cong \theta_2$.

$r_1 = \sqrt{r^2 + z^2 - 2rz\cos\theta} \cong r - z\cos\theta$

$r_2 = \sqrt{r^2 + z^2 - 2rz\underbrace{\cos(180° - \theta)}_{= -\cos\theta}} \cong r - z\cos\theta$

$E\epsilon^{j\omega t}$ is input voltage. $I\epsilon^{j\omega t}$ is input current for resonance. Sinusoidal stationary current distribution is in time phase along the entire length of l m.

FIG. 102. Analysis for a thin linear dipole of total length l m with a sinusoidal current distribution.

and θ the angle that the distance r makes with the linear radiator. Inasmuch as we deal with an unobstructed radiation space, we have the radiator as symmetry axis for the radiation pattern, and we have to solve only for the radiation effects in the XZ plane of Fig. 102. The method is then based on dividing the entire length of the radiator into current elements ($I_z\,dz$ elements) and summing up at the field point $P(r, \varphi, \theta)$ all the incremental effects $d\mathcal{E}_\theta$ with respect to amplitude and phase.

To do this, we have to assume a current distribution along the radiator. If the radiating conductor has a very small cross-sectional diameter in comparison with the length l of the conductor, a sinusoidal current distribution may be assumed. In the expression for the elementary linear radiator (may also be called "differential" linear radiator), we have $d\mathcal{E}_\theta$ volts per meter divided by dH_φ amperes per meter equal to the three-dimensional unbounded medium impedance $Z_0 = \sqrt{\mu/\kappa}$ ohms for a pure dielectric wave medium and essentially equal to 376.7 ohms for air as a medium. We can then use either the differential field contribution $d\mathcal{E}_\theta$ or its associated magnetic differential contribution dH_φ. Using the former, we have

$$d\mathcal{E}_\theta = KI_z\,dz\,\epsilon^{j(\omega t-\beta r)} \quad \text{volts/m}; \qquad K = \frac{188.35 \sin\theta}{r\lambda} \tag{800}$$

Since in Fig. 102, the radiating conductor is drawn symmetrically with respect to the origin of the indicated coordinate systems, it is necessary only to group the differential line elements of infinitesimal length dz with respect to the origin and add up their field contributions at a space point. Doing this, we find the total field

$$\mathcal{E}_\theta = -\frac{60 I_0 D_\theta}{r}\,\epsilon^{j(\omega t-\beta r)} \quad \text{volts/m};$$

$$\text{directivity } D_\theta = \frac{\cos(0.5\beta l\cos\theta) - \cos(0.5\beta l)}{\sin\theta} \tag{801}$$

with the following details:

$$\mathcal{E}_\theta = \underbrace{K\int_0^{l/2} \epsilon^{j(\omega t-\beta r_1)} I_0 \sin\left[\beta\left(\frac{l}{2}-z\right)\right]dz}_{\text{due to } I_z\,dz \text{ current element a distance } +z \text{ m from origin}} + \underbrace{K\int_0^{l/2} \epsilon^{j(\omega t-\beta r_2)} I_0 \sin\left[\beta\left(\frac{l}{2}-z\right)\right]dz}_{\text{due to } I_z\,dz \text{ current element a distance } -z \text{ m}}$$

$$I_z = I_0 \sin\left[\beta\left(\frac{l}{2}-z\right)\right]$$

$$\mathcal{E}_\theta = KI_0\epsilon^{j\omega t}\left\{\int_0^{l/2} \epsilon^{-j\beta(r-z\cos\theta)} \sin\left[\beta\left(\frac{l}{2}-z\right)\right]dz + \int_0^{l/2} \epsilon^{-j\beta(r+z\cos\theta)} \sin\left[\beta\left(\frac{l}{2}-z\right)\right]dz\right\}$$

$$\mathcal{E}_\theta = K_1\left\{\int_0^{l/2} \epsilon^{j\beta z\cos\theta} \sin\left[\beta\left(\frac{l}{2}-z\right)\right]dz + \int_0^{l/2} \epsilon^{-j\beta z\cos\theta} \sin\left[\beta\left(\frac{l}{2}-z\right)\right]dz\right\} \quad \text{volts/m}$$

$$K_1 = KI_0\epsilon^{j(\omega t-\beta r)}$$

These are well-known integrals of the form and solution

$$\int \epsilon^{az} \sin (pz + q)\, dz = \frac{\epsilon^{az}}{a^2 + p^2} [a \sin (pz + q) - p \cos (pz + q)]$$

For the first integral, we have $a = j\beta \cos \theta$; and for the second integral, we have $a = -j\beta \cos \theta$. If we choose the integration limits in either case between 0 and $l/2$, we have for both integrals $p = -\beta$, $q = \beta l/2$, $a^2 = -\beta^2 \cos^2 \theta$, and $p^2 = \beta^2$. Hence, for the first integral, we have

$$\int_0^{l/2} \epsilon^{az} \sin (pz + q)\, dz = \left| \frac{\epsilon^{az}}{a^2 + p^2} [a \sin (pz + q) - p \cos (pz + q)] \right|_{z=0}^{z=l/2}$$

$$= \frac{\epsilon^{al/2}}{a^2 + p^2} \left[a \sin \left(p\frac{l}{2} + q \right) - p \cos \left(p\frac{l}{2} + q \right) \right] - \frac{1}{a^2 + p^2} (a \sin q - p \cos q)$$

$$= \frac{\epsilon^{j(\beta l/2) \cos \theta}}{a^2 + p^2} \Bigg[\underbrace{j\beta \cos \theta \overbrace{\sin \left(-\frac{\beta l}{2} + \frac{\beta l}{2} \right)}^{=0}}_{\text{hence, } =0} + \underbrace{\beta \overbrace{\cos 0}^{=1}}_{=\beta} \Bigg]$$

$$- \frac{1}{a^2 + p^2} \left[j\beta \cos \theta \sin \left(\frac{\beta l}{2} \right) + \beta \cos \left(\frac{\beta l}{2} \right) \right]$$

$$= \frac{\beta}{a^2 + p^2} \epsilon^{j(\beta l/2) \cos \theta} - \frac{\beta}{a^2 + p^2} \left[j \cos \theta \sin \left(\frac{\beta l}{2} \right) + \cos \left(\frac{\beta l}{2} \right) \right]$$

For the second integral, we have

$$\int_0^{l/2} \epsilon^{az} \sin (pz + q)\, dz = \frac{\epsilon^{j(\beta l/2) \cos \theta}}{a^2 + p^2} [\underbrace{-j\beta \cos \theta \overbrace{\sin \theta}^{=0}}_{\text{hence, } =0} + \underbrace{\beta \overbrace{\cos \theta}^{=1}}_{=\beta}]$$

$$- \frac{1}{a^2 + p^2} \left[-j\beta \cos \theta \sin \left(\frac{\beta l}{2} \right) + \beta \cos \left(\frac{\beta l}{2} \right) \right]$$

$$= \frac{\beta \epsilon^{-j(\beta l/2) \cos \theta}}{a^2 + p^2} - \frac{\beta}{a^2 + p^2} \left[-j \cos \theta \sin \left(\frac{\beta l}{2} \right) + \cos \left(\frac{\beta l}{2} \right) \right]$$

Hence,

$$\underbrace{\text{First integral}}_{\text{for upper } l/2 \text{ of radiator}} + \underbrace{\text{second integral}}_{\text{for lower } l/2 \text{ of radiator}} = \frac{\beta}{a^2 + p^2} \left\{ \overbrace{[\epsilon^{j(\beta l/2) \cos \theta} + \epsilon^{-j(\beta l/2) \cos \theta}]}^{2 \cos \left(\frac{\beta l}{2} \cos \theta \right)} - 2 \cos \left(\frac{\beta l}{2} \right) \right\}$$

$$= \frac{2\beta}{a^2 + p^2} \left[\cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \left(\frac{\beta l}{2} \right) \right] = \frac{2\beta}{\beta^2 - \beta^2 \cos^2 \theta} \left[\cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \left(\frac{\beta l}{2} \right) \right] = \frac{2 \left[\cos \left(\frac{\beta l}{2} \cos \theta \right) - \cos \left(\frac{\beta l}{2} \right) \right]}{\beta \sin^2 \theta}$$

and

$$\mathcal{E}_\theta = \frac{2K_1}{\beta}\overbrace{\left[\frac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\left(\frac{\beta l}{2}\right)}{\sin^2\theta}\right]}^{=A} = \frac{2}{\beta}KI_0A\epsilon^{j(\omega t-\beta r)}$$

$$= \frac{376.7\sin\theta}{\beta r\lambda}I_0A\epsilon^{j(\omega t-\beta r)}; \qquad \beta = 2\pi/\lambda$$

$$\mathcal{E}_\theta = \frac{60\sin\theta}{r}I_0A\epsilon^{j(\omega t-\beta r)} = \frac{60I}{r}\left[\frac{\cos\left(\frac{\beta l}{2}\cos\theta\right) - \cos\left(\frac{\beta l}{2}\right)}{\sin\theta}\right]\epsilon^{j(\omega t-\beta)} \quad \text{volts/m}$$

Equation (801) gives the instantaneous electric field intensity of a linear antenna, where I_0 denotes the maximum time value at a current loop of the linear antenna of Fig. 102 in amperes and the field point is a distance of r meters away from the radiation center. The angle θ is then the angle of the direction r to the field point with the l direction of the isolated radiator in unobstructed space. The expression D_θ represents the directivity of a linear radiator For the effective field intensity and effective loop current we have then the simple formula

$$\mathcal{E}_e^{(\text{volts/m})} = \frac{60D_\theta I_e^{(\text{amp})}}{r^{(\text{m})}} \tag{801a}$$

We note that the directivity for a linear radiator in unobstructed space depends only on the angle θ, since a circular pattern prevails about and around such a radiator. This is by no means the case if several linear radiators are used.

103. Field Intensity of a Half-wave Radiator and Its Directivity Compared with an Elementary Linear and Elementary Point Radiator. An application of formula (801) is also a simple matter now, since in short-wave work, we use only efficient structures, because the physical antennas have small dimensions. For a half-wave antenna, we have then in Fig. 102 a length l in meters equal to the half-wavelength distribution along the radiator. We again must have zero current at the ends of the radiator, since there is no conductor that can support a conduction current beyond the ends. Hence, we have a half-period sinusoidal distribution along the radiator with a current antinode at the mid-point of l. Since we have $l = \lambda/2$, we have $\beta l = 2\pi l/\lambda$, or $\beta l = 2\pi l/(2l) = \pi$ radians, or 180 electrical degrees along the radiator. Therefore, $\beta l/2 = 90$ deg and

$$D_\theta = \frac{\cos(90°\cos\theta) - \cos(90°)}{\sin\theta} = \frac{\cos(90°\cos\theta)}{\sin\theta}$$

which is to be used in (801*a*), in the case of a *half-wave dipole* (which is, in effect, a half-wave linear radiator). Now a *point source* radiator, in

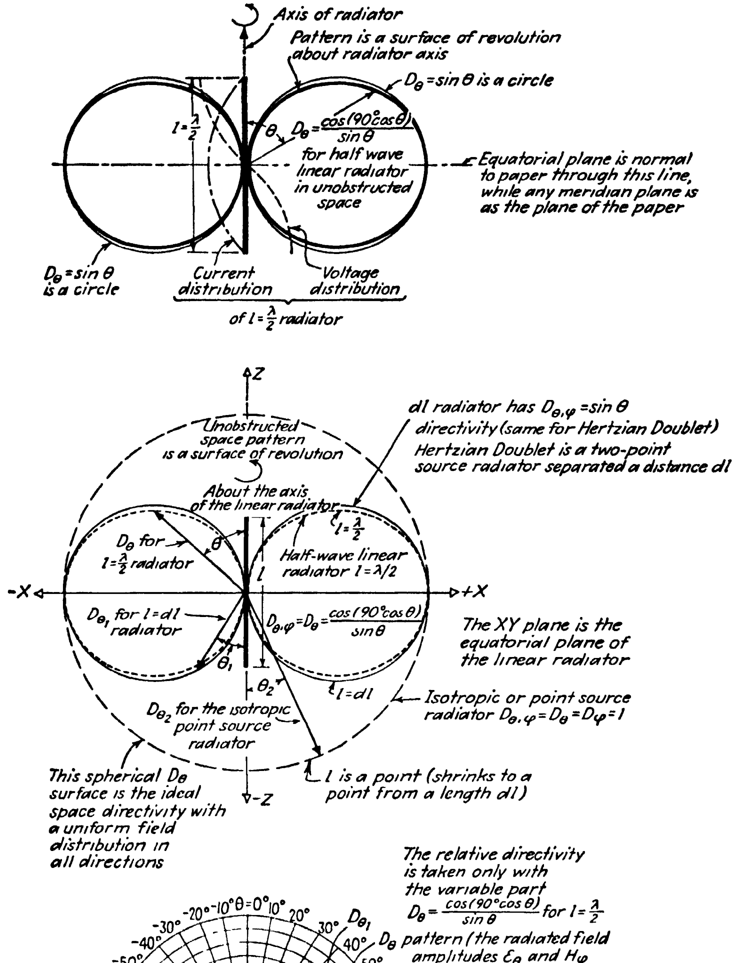


FIG. 103. Directivity patterns for linear radiators assuming a sinusoidal current distribution. These patterns give the *relative* directivities, but *not* the absolute magnitude of the field. *Note:* For the dl radiator (infinitesimal length dl) the tangent circles are exceedingly small if representing field values instead of mere directivities D_θ. The same is true for the isotropic radiator, for which the circle is exceedingly small.

unobstructed radiation space, must radiate equally well in all directions, *i.e.*, have a directivity pattern $D_\theta = D_\varphi = D_{\theta,\varphi} = 1$, while for the half-wave radiator only $D_\varphi = 1$, but $D_\theta = D_\theta D_\varphi = D_{\theta,\varphi}$ is as was just found and has already been used in this text in many cases. A two-point

TABLE 25. HALF-WAVE LINEAR RADIATOR

θ, deg	sin θ	cos θ	90° cos θ, deg	cos (90° cos θ)	D_θ	D_θ^2
0	0.0000	1.0000	90	0.0000	0.0000	0 000
5	0.0872	0.9962	89.6	0.0070	0.0803	0 0064
10	0.1736	0.9848	88.6	0.0244	0.142	0.0206
15	0.2588	0.9659	87	0.0523	0.202	0 0408
20	0.3420	0.9387	84.5	0.0951	0.28	0.0784
25	0.4226	0.9063	81.3	0.1513	0.357	0.1274
30	0.5000	0.8660	78	0.2079	0.417	0.1739
35	0.5736	0.8192	73 9	0.2773	0.481	0 2314
40	0.6428	0.7660	69	0.3584	0 556	0 3091
45	0.7071	0.7071	63.4	0.4478	0.634	0.4020
50	0.7660	0.6428	58	0.5299	0.693	0.4802
55	0.8192	0.5736	51 8	0.6184	0.756	0.5715
60	0.8660	0.5000	45	0.7071	0.816	0.6659
65	0.9063	0.4226	38	0.7880	0.870	0 7569
70	0.9397	0.3420	30.8	0 859	0.912	0 8317
75	0.9659	0.2588	23 3	0.9184	0.950	0 9025
80	0.9848	0.1736	5.6	0.965	0.98	0.9604
85	0.9962	0.0872	0.79	0.999	0.999	0 9980
90	1.0000	0.0000	0.000	1.0000	1.0000	1.000

Remarks: $D_\theta = \dfrac{\cos(90° \cos\theta)}{\sin\theta}$; $(D_\theta^2)_{avg} = 0.612$

The average value is most accurately obtained by plotting the D_θ^2 polar pattern, finding its area by means of a planimeter, and changing the area found to a circular area whose radius is $(D_\theta^2)_{avg}$ (see Fig. 137).

source (elementary) is a Hertzian doublet radiator, since the two points are infinitesimally close to each other and infinitesimally small. We know that the Hertzian doublet has, in effect, the same radiation pattern as an elementary linear radiator [formula (161) on page 165]. The only factor that has anything to do with the directivity θ, φ in space is the sin θ term. Hence, sin θ must be equal to $D_{\theta,\varphi}$ and equal to D_θ, because D_φ is again equal to unity. We have then the comparison formulas for the respective directivity patterns

$$\left.\begin{array}{ll} D_\theta = D_{\theta,\varphi} = 1 & \text{for isotropic point source radiator} \\ D_\theta = D_{\theta,\varphi} = \sin\theta & \text{for elementary linear radiator and doublet} \\ D_\theta = D_{\theta,\varphi} = \dfrac{\cos(90° \cos\theta)}{\sin\theta} & \text{for linear } \dfrac{\lambda}{2} \text{ radiator} \end{array}\right\} \quad (802)$$

Table 25 and Fig. 103 give the results. It is then necessary to compute the directivity for only one quadrant, since for the remaining three quadrants we repeat symmetrically the same directivity radius vectors. From (801*a*) and the method described on page 465, we note that the radiation resistance is proportional to the *average* value of D_θ^2 since the radiated power is proportional either to $\mathcal{E}_\theta^2$ or to H_φ^2. By means of (801), we can readily compute the D_θ values for different electrical lengths βl and find the polar D_θ^2 pattern. The average value of D_θ^2 can also be found by integration, as is shown on page 538, but may in some cases be more laborious. It must be realized that in Fig. 103, the space radiation patterns are not drawn to the same scale, but only with respect to *shape*. The pattern for the half-wave radiator is the only one that has a finite size. It will be noted that over quite a range the pattern for the half-wave antenna coincides with that for the infinitesimal linear radiator. This is also borne out in Table 25, since the sin θ column gives the D_θ values for the *elementary* linear radiator. Hence, for almost ± 10 deg above and below the equatorial plane of the radiator, [cos (90° cos θ)]/sin θ has the same shape as the sin θ pattern.

104. Radiated Power and Radiation Resistance of a Linear Radiator with a Sinusoidal Current Distribution. In the distant radiation field, we have, for an arriving electromagnetic wave, the respective field associations $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter, and $\mathcal{E}_\theta/H_\varphi = \sqrt{\mu/\kappa} \cong 376.7$ ohms for propagation in customary atmosphere. The quantity $\sqrt{\mu/\kappa}$ is known as the three-dimensional space impedance Z_0 holding in the wave front of an advancing electromagnetic wave. We have then for the Poynting vector

$$p = \mathcal{E}_\theta \cdot H_\varphi = \frac{\mathcal{E}_\theta \cdot \mathcal{E}_\theta}{Z_0} = \frac{\mathcal{E}_\theta^2}{Z_0} \qquad \text{watts/m}^2$$

The quantities $\mathcal{E}_\theta$ and H_φ are taken here as the effective values $\mathcal{E}_e$ and H_e; in (801), the phase factor $\epsilon^{j(\omega t - \beta r)} = \cos(\omega t - \beta r) + j \sin^{(\omega t - \beta r)}$ gives a cosine term for the *real value*, and

$$p = \frac{\mathcal{E}_\theta^2}{Z_0} = \frac{60^2 D_\theta^2 I_0^2 \cos^2(\omega t - \beta r)}{r^2 Z_0}; \qquad \text{average } [\cos^2(\omega t - \beta r)] = \frac{1}{2}$$

$$p = \frac{1{,}800 D_\theta^2 I_0^2}{376.7 r^2} = 4.76\,\frac{D_\theta^2 I_0^2}{r^2} \qquad \text{watts/m}^2 \tag{803}$$

where p denotes the *average* power flow and I_0 is the *maximum* time value that occurs at the current antinode along the linear radiator. If I_e is the effective current value in amperes measured at a current antinode of a linear radiator of electrical length βl, we have $I_0 = \sqrt{2}\, I_e$, $I_0^2 = 2I_e^2$, and

$$p = 9.52 \frac{D_\theta^2 I_e^2}{r^2} \quad \text{watts/m}^2;$$

$$D_\theta = \frac{\cos(0.5\beta l \cos\theta) - \cos(0.5\beta l)}{\sin\theta} \tag{803a}$$

This relation is easy to apply. Let us take a half-wave radiator, which is a half-wave dipole. If the current flowing to the dipole gap is 5 amp effective, we find at a field point 1 mile away under a direction of 60 deg with the linear radiator, the following computations: 1 mile = 1,609 m = r; r^2 = 1,609² = 2.592 × 10⁶ sq m. From Table 25, we find

$$D_\theta = D_{60} = \frac{\cos(90° \cos 60°)}{\sin 60°} = 0.816; \qquad D_\theta^2 = 0.6659$$

$$p = 9.52 \frac{0.6659 \times 25}{2.592 \times 10^6} = 6.12 \times 10^{-5} \text{ watt/m}^2 \doteq 61.2 \ \mu\text{v./m}^2$$

The total power in unobstructed radiation space then becomes

$$P = \int\int_S p\, dS = \int_{\theta=0}^{\theta=\pi} p \underbrace{2\pi r^2 \sin\theta\, d\theta}_{dS}$$

$$= \int_0^\pi 2\pi 9.52 D_\theta^2 I_e^2 \sin\theta\, d\theta = 60 I_e^2 \int_0^\pi D_\theta^2 \sin\theta\, d\theta = R_r I_e^2 \qquad \text{watts}$$

where R_r denotes the radiation resistance in ohms. But

$$D_\theta^2 \sin\theta = \frac{[\cos(0.5\beta l \cos\theta) - \cos(0.5\beta l)]^2}{\sin\theta}$$

and

$$\left.\begin{aligned} P &= 60 I_e^2 \int_0^\pi \frac{[\cos(0.5\beta l \cos\theta) - \cos(0.5\beta l)]^2}{\sin\theta}\, d\theta \qquad \text{watts} \\ R_r &= 60 \int_0^\pi \frac{[\cos(0.5\beta l \cos\theta) - \cos(0.5\beta l)]^2}{\sin\theta}\, d\theta \qquad \text{ohms} \end{aligned}\right\} \tag{804}$$

These expressions are also not difficult to apply when graphical integration is used or the process as described in detail on page 535, since $P = R_r I_e^2$ watts and

$$\begin{aligned} R_r^{(\text{ohms})} = 34.632 + 138.18 \log_{10}(\beta l) - 60 \text{ Ci}(\beta l) \\ + 30 \sin(\beta l)[\text{Si}(2\beta l) - 2 \text{ Si}(\beta l)] \\ + 30 \cos(\beta l)\left[0.5772 + 2.303 \log_{10}\left(\frac{\beta l}{2}\right) + \text{Ci}(2\beta l) - \text{Ci}(\beta l)\right] \end{aligned} \tag{804a}$$

for the sine integral (Si) and the cosine integral (Ci) functions, with the following details:

By means of the substitution $x = \cos\theta$, given on page 511, we find

$$\int_0^\pi D_\theta^2 \sin\theta\, d\theta = \int_0^\pi \frac{[\cos(0.5\beta l\cos\theta) - \cos(0.5\beta l)]^2}{\sin\theta}\, d\theta$$

$$= \int \frac{\left[\cos\left(\frac{\beta l}{2}x\right) - \cos\left(\frac{\beta l}{2}\right)\right]^2}{x^2 - 1}\, dx$$

$$= 0.5772 + \log_\epsilon(\beta l) - \text{Ci}(\beta l) + \frac{1}{2}\sin(\beta l)[\text{Si}(2\beta l) - 2\,\text{Si}(\beta l)]$$

$$+ \frac{1}{2}\cos(\beta l)\left[0.5772 + \log_\epsilon\left(\frac{\beta l}{2}\right) + \text{Ci}(2\beta l) - 2\,\text{Ci}(\beta l)\right]$$

When we use graphical integration by evaluating, from the D_θ^2 pattern, the $(D_\theta^2)_{avg}$ value as on page 535, we have $\int_0^\pi 2\pi r^2 \sin\theta\, d\theta = 4\pi r^2$, which is the area of the spherical surface of radius r about the radiation center, but passing entirely through the distant radiation region. Therefore,

$$P = \int\int_S p\, dS = 4\pi r^2 (p)_{avg} = 4\pi r^2 \frac{9.52 I_e^2}{r^2}(D_\theta^2)_{avg} = 120(D_\theta^2)_{avg} I_e^2;$$

$$R_r = 120(D_\theta^2)_{avg} \quad \text{ohms}$$

The radiation resistance is then 120 times the *average* value of D_θ^2. For the half-wave linear radiator, Table 25 gives $(D_\theta^2)_{avg} = 0.612$, and $120 \times 0.612 = 73.4$ ohms represents the radiation resistance with respect to the current flow *at the center* of the half-wave radiator.

105. Radiation Fields and Directivity Patterns for Higher Harmonic Current Distributions along a Linear Radiator. The direction factor D_θ for any actual length l and the corresponding electrical length $\Omega = \beta l = 2\pi l/\lambda$, for an isolated linear radiator in unobstructed radiation space, can be readily computed from the formula

$$D_\theta = \frac{\cos\Omega\cos(\Omega\cos\theta) + \cos\theta\sin\Omega\sin(\Omega\cos\theta) - 1 + j[\cos\Omega\sin(\Omega\cos\theta) - \cos\theta\sin\Omega\cos(\Omega\cos\theta)]}{\sin\theta} \tag{805}$$

Applying this relation to a half-wave radiator, we have $l = \lambda/2$, $\Omega = \pi$; and for $\pi\cos\theta = \delta$ and Fig. 104*a*,

$$D_\theta = \frac{\cos\pi\cos\delta + \cos\theta\sin\pi\sin\delta - 1 + j(\cos\pi\sin\delta - \cos\theta\sin\pi\cos\delta)}{\sin\theta}$$

$$\cos\pi = \cos 180^\circ = -1; \qquad \sin\pi = \sin 180^\circ = 0$$

$$D_\theta = \frac{-\cos\delta - 1 - j\sin\delta}{\sin\theta} = \underbrace{\left|\frac{\sqrt{(-1-\cos\delta)^2 + (-\sin\delta)^2}}{\sin\theta}\right|}_{\text{relative } magnitude \text{ of } D_\theta \text{ (counts only)}}$$

$$= \frac{\sqrt{2(1+\cos\delta)}}{\sin\theta} = \frac{2\sqrt{\tfrac{1}{2}(1+\cos\delta)}}{\sin\theta} = \frac{2\cos\delta/2}{\sin\theta} = 2\frac{\cos(90^\circ\cos\theta)}{\sin\theta}$$

which confirms the result found in Sec. 103. The factor 2 has nothing to do with the relative directivity and belongs properly to the corresponding field intensity formula, which contains D_θ as a factor. For a full-

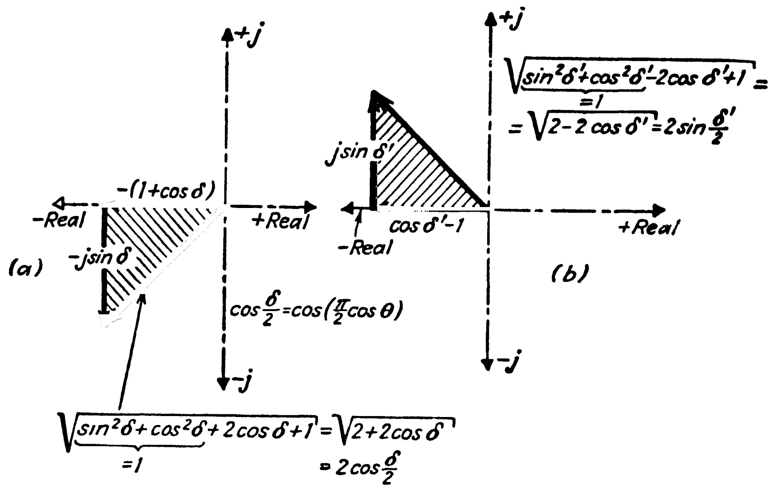


FIG. 104. Vector diagrams for directivity for a half-wave linear radiator (for which $\delta = \pi \cos \theta$ radians).

wave current distribution $l = \lambda$, we have $\Omega = 2\pi l/\lambda = 2\pi$. For $2\pi \cos \theta = \delta'$ and Fig. 104*b*, we have

$$D_\theta = \frac{\cos (2\pi) \cos \delta' + \cos \theta \sin (2\pi) \sin \delta' - 1 + j[\cos (2\pi) \sin \delta' - \cos \theta \sin (2\pi) \cos \delta']}{\sin \theta}$$

$$= \frac{\cos \delta' - 1 + j \sin \delta'}{\sin \theta} = \left| \frac{\sqrt{(\cos \delta' - 1)^2 + \sin^2 \delta'}}{\sin \theta} \right|$$

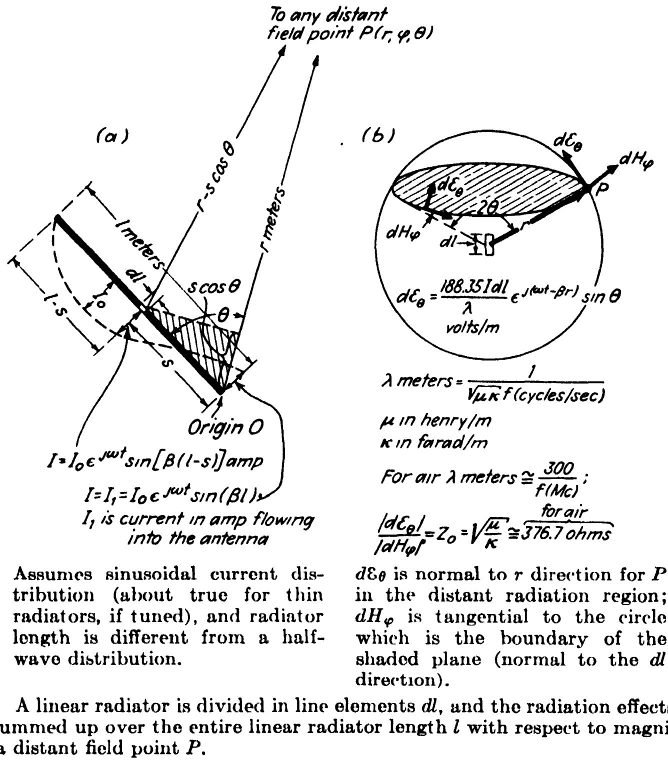
$$= \frac{\sqrt{2(1 - \cos \delta')}}{\sin \theta} = \frac{2\sqrt{0.5(1 - \cos \delta')}}{\sin \theta} = \frac{2 \sin \delta'/2}{\sin \theta}$$

$$= 2 \frac{\sin (180^\circ \cos \theta)}{\sin \theta}$$

where again the factor 2 belongs to the corresponding field intensity expression and strictly not to the D_θ expression.

Formula (805), even though lengthy, is, therefore, not difficult to apply to practical UHF cases, where we deal mostly with fundamental and higher modes of current distributions, since the physical dimensions are not prohibitive as would be the case in the standard broadcast range of frequencies, for instance. Formula (105) is based on the following details:

We have a linear radiator (assuming a sinusoidal current distribution) that can be arbitrarily oriented in space as in Fig. 105. As we have unobstructed radiation space, we are, in the case of *one* linear radiator, *interested only in the θ coordinate* of a spherical coordinate system, besides the r coordinate, which takes care of the distance to the field point in the radiation region. Figure 105*b* gives the elementary field contribution at any field point P a distance of r meters from the radiation center. By radiation center is meant the symmetry center of a radiator. Hence, for an elementary $I\,dl$ radiator, it would be at the mid-point of the infinitesimal length dl.



Assumes sinusoidal current distribution (about true for thin radiators, if tuned), and radiator length is different from a half-wave distribution.

$d\mathcal{E}_\theta$ is normal to r direction for P in the distant radiation region; dH_φ is tangential to the circle which is the boundary of the shaded plane (normal to the dl direction).

FIG. 105. A linear radiator is divided in line elements dl, and the radiation effects $d\mathcal{E}_\theta$ and dH_φ are summed up over the entire linear radiator length l with respect to magnitude and phase at a distant field point P.

For a finite length l of an actual radiator in most cases, we also use the radiation center (mid-point of length l) as the reference point. This was done, for instance, in Sec. 102 in connection with Fig. 102 and the result of Eq. (801). As far as the derivation in connection with Fig. 105*a* is concerned, we choose the origin at one end of the radiator at which a current $I_1 = I_0\epsilon^{j\omega t}\sin(\beta l)$ can be impressed on the isolated radiator of length l. At the far end, there can then be no current; while at a current element $dl = ds$ a distance s from the origin, we have a current $I_0\epsilon^{j\omega t}\sin[\beta(l - s)]$, where I_0 is the maximum time value at a current antinode and $(l - s)$ the distance from the far end of the radiator to the current element $I\,ds$. We have then for such a current element

$$d\mathcal{E}_\theta = \frac{188.35 I\,ds\sin\theta}{r\lambda}\epsilon^{j[\omega t - \beta(r - s\cos\theta)]} \quad \text{volts/m}; \qquad dH_\varphi = \frac{d\mathcal{E}_\theta}{376.7} \quad \text{amp/m}$$

and for the total field intensity, $\mathcal{E}_\theta = \int_0^l d\mathcal{E}_\theta$, $H_\varphi = \mathcal{E}_\theta/376.7$, $I = I_0$, or

$$\mathcal{E}_\theta^{(\text{volts/m})} = \frac{188.35 I_0}{r\lambda} \int_{s=0}^{s=l} \sin\theta \sin[\beta(l-s)] \epsilon^{j(\omega t - \beta r + \beta s \cos\theta)}\, ds$$

$$= \underbrace{\frac{188.35 I_0}{r\lambda} \epsilon^{j(\omega t - \beta r)}}_{k} \int_0^l \sin\theta \sin[\beta(l-s)] \underbrace{\epsilon^{j\overbrace{\beta s \cos\theta}^{=\delta}}}_{\cos\delta + j\sin\delta} ds$$

$$= k \int_0^l \sin\theta \sin \overbrace{(\beta l - \beta s)}^{-q} (\cos\delta + j\sin\delta)\, ds = A + jB$$

$$A = k \int_0^l \underbrace{\sin\theta \sin q}_{D} \cos\delta\, ds; \qquad B = k \int_0^l D \sin\delta\, ds$$

showing that we have a *complex* value for the *total field.* Inasmuch as we assume the distant radiation region, for which all distances for the successive $I\,dl$ elements to the field point P are practically parallel, we can take the $\sin\theta$ term out of the integral and find for $k \sin\theta = k_1$

$$\mathcal{E}_\theta = k_1 \int_0^l \sin q \cos\delta\, ds + jk_1 \int_0^l \sin q \sin\delta\, ds \qquad \text{volts/m}$$

But

$$\sin q \cos\delta = \tfrac{1}{2}[\sin(q+\delta) + \sin(q-\delta)];$$
$$\sin q \sin\delta = \tfrac{1}{2}[\cos(q-\delta) - \cos(q+\delta)]$$

and

$$A = k_1 \int_0^l \sin q \cos\delta\, ds = \tfrac{1}{2}k_1 \left[\int_0^l \sin(q+\delta)\, ds + \int_0^l \sin(q-\delta)\, ds\right] \quad \text{volts/m}$$

$$B = k_1 \int_0^l \sin q \sin\delta\, ds = \tfrac{1}{2}k_1 \left[\int_0^l \cos(q-\delta)\, ds - \int_0^l \cos(q+\delta)\, ds\right] \quad \text{volts/m}$$

$$q + \delta = \beta l + \beta s \cos\theta = \beta[l + s\underbrace{(\cos\theta - 1)}_{a/\beta}]$$

$$q - \delta = \beta[l - s\underbrace{(\cos\theta + 1)}_{b/\beta}]$$

and for $\beta l = \Omega$, we find

$$A = \frac{k_1}{2}\left[\int_0^l \sin(\Omega + as)\, ds + \int_0^l \sin(\Omega - bs)\, ds\right] \qquad \text{volts/m}$$

$$B = \frac{k_1}{2}\left[\int_0^l \cos(\Omega - bs)\, ds - \int_0^l \cos(\Omega + as)\, ds\right] \qquad \text{volts/m}$$

But

$$\sin(\Omega + as) = \sin\Omega \cos(as) + \cos\Omega \sin(as)$$
$$\sin(\Omega - bs) = \sin\Omega \cos(bs) - \cos\Omega \sin(bs)$$
$$\cos(\Omega - bs) = \cos\Omega \cos(bs) + \sin\Omega \sin(bs)$$
$$\cos(\Omega + as) = \cos\Omega \cos(as) - \sin\Omega \sin(as)$$

and

$$\int_0^l \sin(\Omega + as)\,ds = \sin\Omega \int_0^l \cos(as)\,ds + \cos\Omega \int_0^l \sin(as)\,ds$$

$$= \Big|_0^l \sin\Omega \frac{\sin(as)}{a} - \frac{\cos(as)}{a}\cos\Omega\Big| = \Big| -\frac{\cos(\Omega + as)}{a}\Big|_{s=0}^{s=l}$$

$$= -\frac{\cos(\Omega + al) - \cos\Omega}{a} = \frac{\cos\Omega - \cos(\Omega + al)}{a}$$

$$\int_0^l \sin(\Omega - bs)\,ds = +\Big|\frac{\cos(\Omega - bs)}{b}\Big|_{s=0}^{s=l} = \frac{\cos(\Omega - bl) - \cos\Omega}{b}$$

$$\int_0^l \cos(\Omega + as)\,ds = \Big|\frac{\sin(\Omega + as)}{a}\Big|_{s=0}^{s=l} = \frac{\sin(\Omega + al) - \sin\Omega}{a}$$

$$\int_0^l \cos(\Omega - bs)\,ds = \Big| -\frac{\sin(\Omega - bs)}{b}\Big|_{s=0}^{s=l} = \frac{\sin\Omega - \sin(\Omega - bl)}{b}$$

$$a = \beta(\cos\theta - 1); \qquad b = \beta(\cos\theta + 1); \qquad ab = \beta^2(\cos^2\theta - 1)$$

We can find then the values for A and B since all integrals are solved. The process is based on expanding and collecting terms that belong together, as is indicated for the real A term:

$$A = 0.5k_1\left[\frac{\cos\Omega - \cos(\Omega + al)}{a} + \frac{\cos(\Omega - bl) - \cos\Omega}{b}\right]$$

$$= \frac{k_1}{2ab}[b\cos\Omega - b\cos(\Omega + al) + a\cos(\Omega - bl) - a\cos\Omega]$$

$$= \frac{k_1}{2ab}\{b\cos\Omega - b[\cos\Omega\cos(al) - \sin\Omega\sin(al)] + a[\cos\Omega\cos(bl) + \sin\Omega\sin(bl) - a\cos\Omega]\}$$

$$= \frac{k_1}{2ab}[b\cos\Omega - b\cos\Omega\cos(al) + b\sin\Omega\sin(al) + a\cos\Omega\cos(bl) + a\sin\Omega\sin(bl) - a\cos\Omega]$$

$$= \frac{k_1}{2ab}[\beta\cos\theta\cos\Omega + \beta\cos\Omega - \beta\cos\theta\cos\Omega\cos(al) - \beta\cos\Omega\cos(al) + \beta\cos\theta\sin\Omega\sin(al) + \beta\sin\Omega\sin(al) + \beta\cos\theta\cos\Omega\cos(bl) - \beta\cos\Omega\cos(bl) + \beta\cos\theta\sin\Omega\sin(bl) - \beta\sin\Omega\sin(bl) - \beta\cos\theta\cos\Omega + \beta\cos\Omega] = \cdots$$

Doing this also for the jB term, we find the expression for the total electric field intensity and the direction factor as given in formula (805).

Applying formula (805) to a linear radiator one-quarter of a wavelength long, we have $l = \lambda/4$, $\Omega = 2\pi l/\lambda = 2\pi/4 = \pi/2$; and for $\delta = (\pi/2)\cos\theta = \Omega\cos\theta$,

$$D_\theta = \frac{1}{\sin\theta}\left\{\overbrace{\cos\left(\frac{\pi}{2}\right)}^{=0}\cos\delta + \cos\theta\,\overbrace{\sin\left(\frac{\pi}{2}\right)}^{=1}\sin\delta - 1 + j\left[\overbrace{\cos\left(\frac{\pi}{2}\right)}^{=0}\sin\delta - \cos\theta\,\overbrace{\sin\left(\frac{\pi}{2}\right)}^{=1}\cos\delta\right]\right\}$$

$$= \frac{1}{\sin\theta}(\cos\theta\sin\delta - 1 - j\cos\theta\cos\delta)$$

$$= \underbrace{\left|\frac{1}{\sin\theta}\sqrt{(\cos\theta\sin\delta - 1)^2 + (-\cos\theta\cos\delta)^2}\right|}_{\text{magnitude}}$$

$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta\sin^2\delta - 2\cos\theta\sin\delta + 1 + \cos^2\theta\cos^2\delta}$$

$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta(\sin^2\delta + \cos^2\delta) - 2\cos\theta\sin\delta + 1}$$

$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta - 2\cos\theta\sin\delta + 1}$$

$$= \frac{\sqrt{\cos^2\theta - 2\cos\theta\sin(90^\circ\cos\theta) + 1}}{\sin\theta}$$

For an angle of 60 deg with the linear radiator, we have $\theta = 60$ deg, $\sin\theta = 0.866$, $\cos\theta = 0.5$, $\cos^2\theta = 0.25$, $\sin(90^\circ\cos\theta) = \sin(0.5 \times 90^\circ) = 0.7071$, and

$$D_{60^\circ} = \frac{\sqrt{0.25 - 2 \times 0.5 \times 0.7071 + 1}}{0.866} = \frac{\sqrt{0.5429}}{0.866} = \frac{0.74}{0.866} = 0.855$$

which is 85.5 per cent of the maximum directivity. This occurs for $\theta = 90$ deg with the linear radiator of Fig. 105 and is in a plane normal to the input end of the radiator and yields, of course, $D_{90^\circ} = 1$, because the square-root value becomes unity and $\sin\theta = \sin 90^\circ = 1$. For a three-quarter-wavelength distribution along the radiator, we find $l = 3\lambda/4$, $\Omega = \beta l = 2\pi l/\lambda = 3\pi/2$ or 270 electrical degrees; and for $\delta = 270^\circ\cos\theta$,

$$D_\theta = \frac{1}{\sin\theta}[\cos 270^\circ\cos\delta + \cos\theta\sin 270^\circ\sin\delta - 1 + j(\cos 270^\circ\sin\delta - \cos\theta\sin 270^\circ\cos\delta)]$$

$$\cos 270^\circ = 0; \qquad \sin 270^\circ = -1$$

$$D_\theta = \frac{1}{\sin\theta}(-\cos\theta\sin\delta - 1 - j\cos\delta\cos\theta)$$

$$= \underbrace{\left|\frac{1}{\sin\theta}\sqrt{[-(\cos\theta\sin\delta + 1)]^2 + (-\cos\theta\cos\delta)^2}\right|}_{\text{magnitude}}$$

$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta\sin^2\delta + 2\cos\theta\sin\delta + 1 + \cos^2\theta\cos^2\delta}$$
$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta(\sin^2\delta + \cos^2\delta) + 2\cos\theta\sin\delta + 1}$$
$$= \frac{1}{\sin\theta}\sqrt{\cos^2\theta + 2\cos\theta\sin(3\times 90^\circ\cos\theta) + 1}$$

for $\theta = 60$ deg we find, since $\sin 135^\circ = \cos 45^\circ = 0.7071$,

$$D_{60^\circ} = \frac{1}{0.866}\sqrt{0.25 + 2\times 0.5\sin(0.5\times 270^\circ) + 1}$$
$$= \frac{1}{0.866}\sqrt{0.25 + 0.7071 + 1} = \frac{\sqrt{1.9571}}{0.866} = 1.62$$

as against unity value for D_{90°, which is in the plane normal to the radiator at the input side. Such a field gain in the 60-deg direction is due to radiation interference, which happens to be geometrically additive. Radiation interference occurs because the radiator is now longer than a half wavelength, and there is a quarter-wavelength section along the radiator for which the antenna current flows in opposition to the remaining half-wavelength section. This will be followed up after discussing other directivity patterns.

When we apply formula (805) to radiator lengths $l = k\lambda/2$ for k an *odd* integer such as 1, 3, 5, . . . , we find directivities D_θ with a numerator $\cos(k\,90^\circ\cos\theta)$ and a denominator $\sin\theta$; but for *even* integers, such as $k = 2, 4, 6, \cdots$, the numerator reads $\sin(k\,90^\circ\cos\theta)$. We have then the directivity formulas

$$\left.\begin{aligned} &D_\varphi = 1; \qquad D_{\theta,\varphi} = D_\theta; \qquad D_\theta = \frac{\cos(k\,90^\circ\cos\theta)}{\sin\theta} \\ &\qquad\qquad\qquad\qquad \text{for } k = \frac{l}{0.5\lambda} = 1, 3, 5, 7, \cdots \\ &D_\theta = \frac{\sin(k\,90^\circ\cos\theta)}{\sin\theta} \qquad \text{for } k = \frac{l}{0.5\lambda} = 2, 4, 6, 8, \cdots \end{aligned}\right\} \tag{806}$$

with the directivity patterns shown in Fig. 106. The corresponding field intensity formulas are then

$$\left.\begin{aligned} &\mathcal{E}_\theta^{(\text{volts/m})} = j\frac{60I_0}{r}\frac{\cos(k\,90^\circ\cos\theta)}{\sin\theta}\epsilon^a \qquad \text{for } k = 1, 3, 5, 7, \cdots \\ &\mathcal{E}_\theta^{(\text{volts/m})} = \frac{60I_0}{r}\frac{\sin(k\,90^\circ\cos\theta)}{\sin\theta}\epsilon^a \qquad \text{for } k = 2, 4, 6, 8, \cdots \\ &H_\varphi = \frac{\mathcal{E}_\theta}{376.7}\ \text{amp/m}; \qquad a = j(\omega t - \beta r) \end{aligned}\right\} \tag{807}$$

where r is in meters, l and λ in same units (namely in meters), $\mathcal{E}_\theta$ and H_φ are instantaneous fields, and I_0 in amperes is maximum value. The indicated directivity patterns hold for *natural* phasing, for which a phase reversal occurs after each half period along the radiator, *i.e.*, after each half wavelength, if we measure from the open end of the radiator for which a current antinode exists. Since the magnetic flux lines are circles around the radiator, we have a directivity pattern that is a surface of revolution about the radiator. The expressions for maximum and

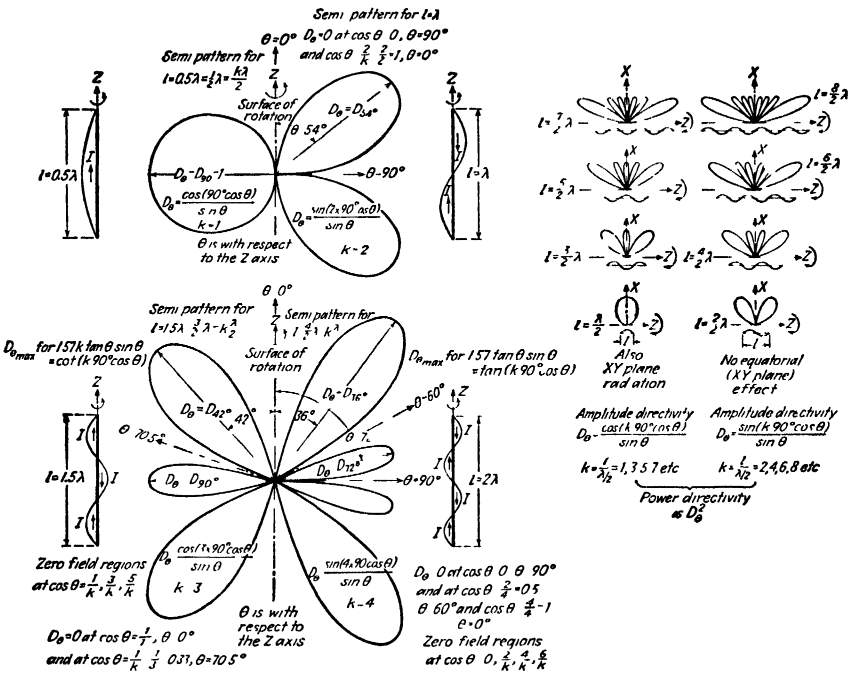


FIG. 106. Directivity patterns for the field amplitude of linear radiators which are an integer multiple of half a wavelength (for unobstructed radiation space).

minimum (zero) directivity are obtained by differentiating the directivity with respect to θ and putting $d(D_\theta)/d\theta$ equal to zero, provided $d^2(D_\theta)/d\theta^2$ is not zero. If the second derivative is positive for the value of θ, then the corresponding value of D_θ is a minimum; while for a negative value of the second derivative, the corresponding value of D_θ is a maximum. Formulas (807) show that the respective field associations $\mathcal{E}_\theta$ and H_φ are proportional to the directivity D_θ. Therefore, the patterns shown in Fig. 106 can be taken also as the field intensity patterns.

Inasmuch as the radiation power is $376.7H_\varphi^2$ or $\mathcal{E}_\theta^2/376.7$ watts per square meter in the distant radiation field, where H_φ is effective amperes

per meter and $\mathcal{E}_\theta$ effective volts per meter, at a distant field point for the radiated power received, we must also consider the D_θ^2 patterns, since they give a measure for the actual power directivity. The power lobes (or ears) will then be much more pronounced than the field lobes, while any field minimum effects are more of minimum effects in the case of field power patterns. This has reference to minimum values, which are smaller than unity, in the case of the respective field patterns. Figure

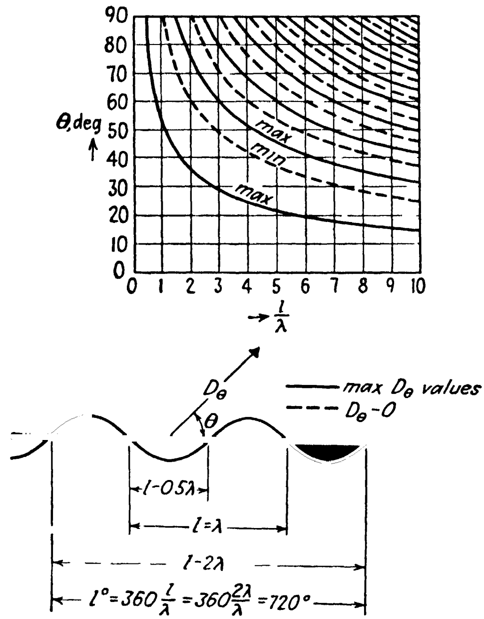


FIG 107. Directivity D_θ of a linear radiator in unobstructed radiation space

$$D_\theta = \frac{\sqrt{1 + \cos^2 l^\circ + \cos^2 \theta \sin^2 l^\circ - 2\cos(l^\circ \cos\theta)\cos l^\circ - 2\cos\theta \sin(l^\circ\cos\theta)\sin l^\circ}}{\sin\theta}$$

l is actual length of linear radiator in meters, l° is electrical length $l = 360^\circ \frac{l}{\lambda}$ deg

107 represents graphs from which we can find the maximum and minimum (zero) directivities of linear radiators with a sinusoidal current distribution. Thus for a radiator that is two operating wavelengths long, we have $l = 2\lambda$, where λ is the full period sinusoidal current distribution and $k = l(\frac{1}{2}\lambda) = 4$, yielding $l/\lambda = 2\lambda/\lambda = 2 = \frac{1}{2}k$. The $l/\lambda = 2$ ordinate in Fig. 107 cuts the solid curves at $\theta = 36$ and $\theta = 72$ deg, which checks the values of Fig. 106. The same ordinate cuts the broken curves at $\theta = 60$ and $\theta = 90$ deg, yielding the $D_{60} = 0$ and the $D_{90} = 0$ directivities in Fig. 106. For the $l = 5\lambda$ case of Fig. 106, we find, from Fig. 107, the main maximum at $\theta = 22.5$ deg, then one at

$\theta = 45$ deg, the next at about $\theta = 60.5$ deg, and the following at $\theta =$ 72.5 and 85 deg, respectively. The zero directivities, where the various lobes slide in tangentially, occur in succession at about $\theta = 38$, 53, 67, 79, and 90 deg, respectively.

106. Remarks on Natural and Equal Phasing along Linear Radiators. By *natural* phasing is meant the phasing that occurs automatically when an emf is impressed on a linear radiator at one end with respect to ground or another conductor. Therefore, the current distributions in Figs. 102, 103, 105, and 107 are examples of natural phasing. This phasing begins at the far end, since we deal with stationary or standing waves for which reflection occurs at the open end and causes a current minimum (a current zero) at the open end, since zero current is due to an outgoing and reflected current disturbance that cancels at this end.[1] Since the current distribution depends upon the phase velocity and the radian frequency of $2\pi f$ radians per second depends upon the frequency f in cycles per second, the phase distribution along a conductor must depend upon both the length of the radiator and the operating frequency. The wavelength λ along the radiator is then the distance moved by the current along the radiator during the time of one period of the exciting current, *i.e.*, during $1/f$ seconds. We note from the current distributions referred to that there are line sections which are in antiphase with other line sections if the line is longer than one-half of the operating wavelength. By operating wavelength is meant the effective wavelength along the radiator that is not necessarily equal to the free-space wavelength $\lambda_0 = 300/f$ in meters, where f is in megacycles per second. It is nearly equal to λ_0 for a good linear radiator. Now, for the case of natural phasing, as shown in Fig. 108, we obtain, according to the directivity patterns of Fig. 106, an acceptable radiation lobe with only zero effect along either direction of a linear radiator as long as the radiator does not exceed one-half wavelength. For more than one-half wavelength, we have inphase and antiphase current sections and, therefore, also such electromagnetic fields, since the fields depend upon the relative instantaneous polarities of all current values flowing in the linear radiator. For more than one-half-wavelength distribution and natural phasing, the patterns of Fig. 106 show that the single rotation surface (holding for $l \leqq \lambda/2$), breaks up into several directivity surfaces, producing additional regions of *no* electromagnetic field energy. We have, for instance, for a full-wave current distribution, two side lobes that produce space directivity surfaces. For a three-half-wavelength distribution, we have three side lobes; for a 2-wavelength antenna, we have four

[1] This is also due to the fact that beyond the open end there is no longer a conductor which can support a conduction current flow.

side lobes; etc. Hence, every additional half-wavelength section produces, as it were, another side lobe, making 10 such lobes altogether in the case of a 5-wavelength distribution, since we have 10 half-wave sections, *i.e.*, five inphase and five antiphase natural current flows, at any instant in one and the same current-carrying conductor. For most applications, this is an undesirable feature, since field energy is being

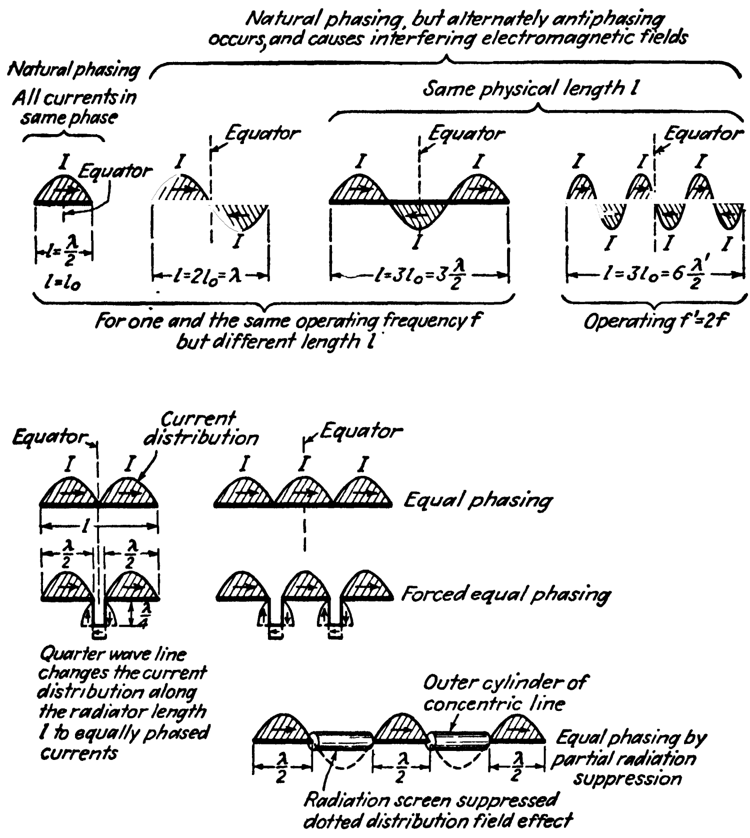


FIG. 108. Radiators with natural phasing and equal phasing (shaded current distributions cause appreciable radiation fields).

diverted in certain regions and suppressed in other regions. It is true that if we had an FM carrier of sufficient frequency swing, we could fill in the blind field zones, but this would require a wider carrier channel. It could also be done with small channel width if we transmitted with two frequencies at the same time, where one frequency fills in the gaps left field-free for the other frequency, Also a supersonic frequency switching (electronic) could be used as a remedy. Nevertheless, all

these ideas do not strike at the cause. The cause is the breaking up of one side lobe into several side lobes as the number of half-wave distributions is increased. Equal phasing will cure the interference and can be accomplished conveniently by quarter-wave lines in UHF work or by screening off the antiphase effects, as is shown in Fig. 108. Equal phasing has another advantage that natural phasing does not have. It takes the side lobes and, so to speak, "throws them into the main lobe," thus emphasizing the directivity of the main lobe and causing decided beaming effects. This is taken up in other sections.

107. Polarization and Wave Front Due to Linear Radiators. Inasmuch as an electromagnetic wave has an electric as well as a magnetic field vector, which at a certain space point varies with time, we can speak of the polarization of the electric vector and the polarization of the magnetic vector. It has become customary to refer the polarization to that of the electric vector, since in the distant radiation field both the $\mathcal{E}_\theta$ field and the H_φ field are normal to each other and in time phase and in the wave front. There are several kinds of polarizations that can be readily understood from the Lissajous figures[1] on the screen of a cathode-ray tube. If we have a linear polarization, the displacement has a constant direction at a point in space. If we apply an alternating voltage $E\epsilon^{j\omega t}$ across the deflection condenser (deflection plates) of a cathode-ray tube, we have a corresponding electric field $\mathcal{E} = (E/d)\epsilon^{j\omega t}$ between the two deflection plates which moves the cathode-ray beam in a plane normal to the plates, producing a linear fluorescent trace of the cathode-ray spot on the screen of the tube. This is what is known as "linear polarization," since only the magnitude of $\mathcal{E}$ volts per meter varies periodically, but always along the same direction. Hence, $(E/d)\cos\omega t$ represents a linear polarization, where d in meters is the distance by which the deflection plates are separated and E is in volts. Also $j(E/d)\sin\omega t$ represents a linear polarization that is normal to the former polarization. We deal, in either case, only with a change in magnitude, since the $\epsilon^{j\omega t} = \cos\omega t + j\sin\omega t$ does *not* denote a space plane in the real sense, *i.e.*, a circular polarization, but means that we deal with a complex time vector. However, if we apply to one deflection condenser an electromotive force $E\epsilon^{j\omega t}$ and to the other deflection condenser (which is in space quadrature with the first deflection condenser) a voltage $E\epsilon^{j\omega t}$ of the same magnitude and the same phase, the effect of one voltage alone would be a linear polarization due to $(E/d)\cos\omega t$ volts per meter normal to the deflection plates. The effect of the other voltage alone would be a linear polarization $(E/d)\cos\omega t$ normal to the deflection plates of the

[1] For details, consult A. HUND, "High-frequency Measurements," 2d ed., pp. 75–78, McGraw-Hill Book Company, Inc., New York, 1951.

second condenser. When, therefore, both fields act simultaneously, both fields of equal magnitude and phase add geometrically in space, causing a resultant linear polarization $\sqrt{E^2 \cos^2 \omega t + E^2 \cos^2 \omega t} = 1.41E \cos \omega t$ along a 45-deg direction between the component linear polarizations. If two inphase voltages $E_1 \cos \omega t$ and $E_2 \cos \omega t$ of unequal amplitudes E_1 and E_2 but in phase were applied to respective deflection condensers, we should have a resultant linear polarization (linear fluorescent trace) proportional to the magnitude $\sqrt{E_1^2 + E_2^2} \cos \omega t$, but under an angle $\psi = \tan^{-1} (E_1/E_2)$. Hence, when linear electric field polarizations $(E_1/d) \cos \omega t$ and $(E_2/d) \cos \omega t$ of unequal magnitude but equal phase add geometrically in space, we still have a resultant *linear* polarization $\sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2} \cos \omega t$, except along another *fixed* direction. This is, however, *not* the case when we add up geometrically linear polarizations that are *out of phase*, even though we have equal component amplitudes. This can also be realized on a cathode-ray screen. The details are then as follows:

Putting $\omega t = q$ and orienting the cathode-ray tube such that one deflection condenser causes a linear polarization along the $+X$ coordinate and the other deflection condenser causes a linear polarization along the $\pm Y$ coordinate of an XYZ system, then we find for a time phase difference φ, the linear component polarizations $x = \mathcal{E}_1 \cos q$ and $y = \mathcal{E}_2 \cos (q - \varphi)$ at any instant. Since

$$\cos q = \frac{x}{\mathcal{E}_1}; \qquad \sin q = \sqrt{1 - \cos^2 q} = \sqrt{1 - \frac{x^2}{\mathcal{E}_1^2}} = \frac{\sqrt{\mathcal{E}_1^2 - x^2}}{\mathcal{E}_1}$$

$$y = \mathcal{E}_2 \cos (q - \varphi) = \mathcal{E}_2(\cos q \cos \varphi + \sin q \sin \varphi) = \frac{\mathcal{E}_2(x \cos \varphi + \sqrt{\mathcal{E}_1^2 - x^2} \sin \varphi)}{\mathcal{E}_1}$$

This is the equation of a general ellipse. The out-of-phase geometrical addition of the two linear polarizations causes an elliptical[1] polarization. For the general ellipse, we have the following special cases:

$\varphi = 0$ time degrees $\cdots y = \frac{\mathcal{E}_2}{\mathcal{E}_1} x$ a line of inclination $\psi = \tan^{-1} \frac{\mathcal{E}_2}{\mathcal{E}_1}$

$\varphi = 90° \cdots y = \frac{\mathcal{E}_2}{\mathcal{E}_1} \sqrt{\mathcal{E}_1^2 - x^2}$ or $\left(\frac{x}{\mathcal{E}_1}\right)^2 + \left(\frac{y}{\mathcal{E}_2}\right)^2 = 1$ an ellipse

$\varphi = 90° \cdots \mathcal{E}_1 = \mathcal{E}_2$ yields a circle;

$\varphi = 180°; y = \frac{\mathcal{E}_2}{\mathcal{E}_1} x$ a conjugate inclined line

$\varphi = 270°$ as for $\varphi = 90°$; $\varphi = 360°$ as for $\varphi = 0°$

From this discussion, we note that even for linear polarization components, we generally arrive at an elliptical polarization. Since it is

[1] For other details, consult A. Hund, "Phenomena in High-frequency Systems," pp. 394–403, 499–517, McGraw-Hill Book Company, Inc., New York, 1936.

customary, in the case of electromagnetic waves, to take the direction of the electric displacement as the direction of polarization (direction of electric flux density $D = \kappa\mathcal{E}$), a vertically polarized wave is a linearly

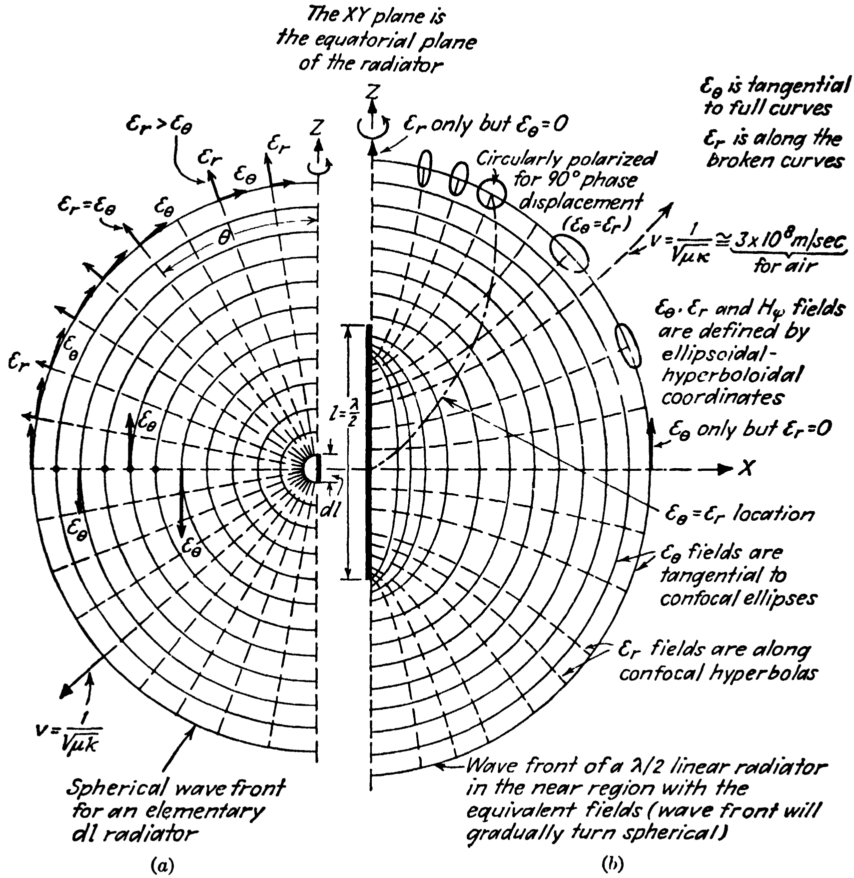


FIG. 109. Elliptical polarization of the electrical field in the near region of a linear radiator. For an elementary radiator we have spherical coordinates and a spherical wave front. For a finite linear radiator length, we have ellipsoidal-hyperboloidal coordinates, and the ellipsoidal wave front turns gradually into a spherical and then into a plane wave front as we approach the distant radiation region.

polarized wave whose polarization (whose $\mathcal{E}$ oscillations) is vertical. For a horizontally polarized wave, the $\mathcal{E}$ variations are horizontal.[1] For an elliptically polarized wave, the direction of the electric displacement $\kappa\mathcal{E}$ at a point rotates about the point in a plane and the magnitude of $\kappa\mathcal{E}$

[1] For other details reference is made to Table 44 where lengthy derivations are given for vertically and horizontally polarized waves. Generally, we must distinguish whether the electric vector is normal or parallel with the plane of wave incidence.

coulombs per square meter varies as the radius of the ellipse. For a circularly polarized electromagnetic wave, the direction of the electric displacement $\kappa\mathcal{E}$ at a point rotates with a constant angular velocity $\omega = 2\pi f$ radians per second about an axis in the direction of wave propagation, while the magnitude of the electric displacement of $\kappa\mathcal{E}$ coulombs per square meter is fixed (independent of its direction). In a right-hand elliptically polarized wave, the direction of the electric displacement is clockwise for an observer looking in the direction the wave is traveling. For a left-hand elliptically polarized wave, the electric displacement $\kappa\mathcal{E}$ is counterclockwise. These are present-day conventional definitions and in accord with the Standards on Radio Wave Propagation of the Institute of Radio Engineers.

We have already discussed what is meant by a wave front (pages 393, 365–368). We have found that it is an equiphase surface which is propagated with a finite velocity; and at the beginning of the wave, it is the surface over which the displacement from zero starts at the same moment. If we deal with a thin linear radiator that is at resonance with the exciting current, such as a half-wave radiator, the equiphase surfaces are ellipsoids[1] of revolution with the free ends of the half-wave radiator as foci. This can be understood when we begin first with an elementary linear radiator as in Fig. 109.

The details are then as follows: According to Table 2, the differential or elementary field contributions[2] are only

$$d\mathcal{E}_\theta = j\frac{\omega\mu_0}{4\pi}\frac{\sin\theta}{r}\epsilon^{j(\omega t-\beta r)}I\,dl = j\frac{10^{-7}\omega I\,dl}{r}\epsilon^{j(\omega t-\beta r)}\sin\theta \qquad \text{volts/m}$$

$$dH_\varphi = \frac{d\mathcal{E}_\theta}{376.7} \qquad \text{amp/m}$$

in the distant radiation region, since there are other terms that also play a part in the "near" region of the radiator and the "between" region. In these expressions, we find that we have only a j term in the radiation region but no real term and that the transverse $\mathcal{E}_\theta$ and H_φ fields are in time phase and space quadrature. Therefore, we have linear electric and linear magnetic polarizations, and the j factor has only

[1] ABRAHAM, M., *Ann. Physik*, **66**, 435 (1898). If l denotes the length of a linear radiator and $2R$ its diameter, the electromagnetic field can be found by means of an ellipsoid of revolution with a total axis length equal to l and the other axis equal to $2R$. This requires elliptic coordinates, which can be readily found from the general curvilinear orthogonal coordinates. Each meridian plane cuts the ellipsoid of revolution in an ellipse. The two foci are a distance $\sqrt{(\frac{1}{2}l)^2 - R^2}$ from the symmetry center. Confocal ellipses are then constructed in the meridian plane and confocal hyperbolas, which are orthogonal to the ellipses.

[2] I denotes the maximum time value I_0 in amperes, which is mostly done in this text for simplicity's sake.

mathematical significance and can be ignored as far as integrations over finite lengths of a linear radiator are concerned (as in Sec. 102). But in the complete expressions

$$d\mathcal{E}_\theta = j\frac{0.5Z_0 I\,dl}{r\lambda}A\sin\theta\left[1 - \frac{j}{\beta r} - \frac{1}{(\beta r)^2}\right]; \qquad d\mathcal{E}_r = \frac{0.5Z_0 I\,dl}{\pi r^2}A\cos\theta\left(1 - \frac{j}{\beta r}\right)$$

volts/m

$$dH_\varphi = j\frac{0.5I\,dl}{r\lambda}A\sin\theta\left(1 - \frac{1}{\beta r}\right) \quad \text{amp/m}; \qquad A = \epsilon^{j(\omega t - \beta r)}; \qquad Z_0 = \sqrt{\frac{\mu_0}{\kappa_0}} = 376.7$$

ohms for air

we have also a longitudinal $d\mathcal{E}_r$ component, which is a cos θ function, in addition to the transverse electric component $d\mathcal{E}_\theta$, which is a sin θ function, as is the transverse magnetic dH_φ field. These fields have, therefore, the forms

$$\left.\begin{aligned} d\mathcal{E}_\theta &= a\left[j + \frac{1}{\beta r} - \frac{j}{(\beta r)^2}\right]\sin\theta, \qquad d\mathcal{E}_r = b\left[1 - \frac{j}{\beta r}\right]\cos\theta; \\ &\qquad\qquad d\mathcal{E}_\theta \text{ and } d\mathcal{E}_r \text{ are in space quadrature} \\ dH_\varphi &= e\left(j + \frac{1}{\beta r}\right)\sin\theta \qquad \textit{only one } \text{field, but complex} \\ a &= \frac{0.5Z_0 I\,dl}{r\lambda}\epsilon^{j(\omega t - \beta r)}; \qquad b = \frac{0.5Z_0 I\,dl}{\pi r^2}\epsilon^{j(\omega t - \beta r)}; \qquad e = \frac{0.5I\,dl}{r\lambda}\epsilon^{j(\omega t - \beta r)} \end{aligned}\right\} \quad (808)$$

Now as far as the magnetic dH_φ field is concerned, the real and the imaginary terms state only that we have a complex elementary dH_φ contribution at a certain field point, whether very close to the elementary radiator or in the between region. This means that we deal only with a mathematical complex plane. Matters are different as far as the resultant $d\mathcal{E}$ contribution is concerned, since we have a complex $d\mathcal{E}_\theta$ and a complex $d\mathcal{E}_r$ component, which are in space quadrature. Even though these components by themselves represent linear polarizations, the resultant field $d\mathcal{E}$ is generally an elliptical polarization, since $d\mathcal{E}_\theta$ is generally *out of phase* with $d\mathcal{E}_r$, besides the components generally have different magnitudes. The magnitude effect, in the case of equal phases, would change only the direction of the linear polarization, but a time phase difference generally causes an elliptical electric polarization. Therefore, close to a linear radiator, we generally have an elliptically polarized electric field. Equations (808), according to Eqs. (163) and (164) and the numerical values of Table 3, can also be written as

$$\left.\begin{aligned} d\mathcal{E}_\theta &= m_e C\sin\theta\left(\frac{j}{r_e} + \frac{1}{r_e^2} - \frac{j}{r_e^3}\right); \qquad d\mathcal{E}_r = 2m_e C\cos\theta\left(\frac{1}{r_e^2} - \frac{j}{r_e^3}\right) \\ &\qquad\qquad\qquad\qquad \text{volts/m} \\ dH_\varphi &= m_e D\sin\theta\left(\frac{j}{r_e} + \frac{1}{r_e^2}\right) \qquad \text{amp/m} \\ C &= \frac{\beta^3}{4\pi\omega\kappa}; \qquad D = \frac{\beta^2}{4\pi}; \qquad r_e = \beta r = \frac{2\pi}{\lambda}r; \qquad m_e = I\,dl\epsilon^{j(\omega t - \beta r)} \quad \text{m}\cdot\text{amp} \end{aligned}\right\} \quad (808a)$$

which is a most convenient form if we want to superimpose linear polarizations in order to find out what the resultant polarization is at any distance of r meters from the radiation center. This is roughly indicated in the left side of Fig. 109 for an elementary linear radiator. Equation (808*a*) shows that the ratio of the differential transverse field contribution $d\mathcal{E}_\theta$ at any field point P to the longitudinal field contribution $d\mathcal{E}_r$ is

$$\frac{m_e C \sin\theta}{2m_e C \cos\theta}\,\frac{\frac{j}{r_e}+\frac{1}{r_e^2}-\frac{j}{r_e^3}}{\frac{1}{r_e^2}-\frac{j}{r_e^3}} = 0.5\tan\theta\,\frac{m+jn-jq}{m-jq} = 0.5\tan\left[\theta\,\frac{m+j(n-q)}{m-jq}\right]$$

$$m=\frac{1}{r_e^2};\qquad n=\frac{1}{r_e};\qquad q=\frac{1}{r_e^3}$$

$$m+j(n-q) = \sqrt{m^2+(n-q)^2}\ \Big/ \varphi_1 = \tan^{-1}\left(\frac{n-q}{m}\right);$$

$$m-jq = \sqrt{m^2+q^2}\ \Big/ \varphi_2 = \tan^{-1}\left(\frac{q}{m}\right)$$

$$\frac{m+j(n-q)}{m-jq} = \sqrt{\frac{m^2+(n-q)^2}{m^2+q^2}}\ \Big/\varphi_1-\varphi_2$$

Therefore,

$$d\mathcal{E}_\theta = \frac{\tan\theta}{2}\,\epsilon^{j(\varphi_1-\varphi_2)}\sqrt{\frac{m^2+(n-q)^2}{m^2+q^2}}\,d\mathcal{E}_r \qquad \text{volts/m} \tag{808b}$$

and as long as the phase difference $\varphi_1 - \varphi_2$ is not equal to zero, we *must expect elliptical electrical polarization*, which may, for some cases, yield a circular polarization. By means of Table 3, it is easy to find out what happens, since the m, n, and q values are given in this table for different electrical distances r_e to the field point under consideration. For instance, for the actual distance of r meters equal to 0.159λ meters, we have an electrical distance r_e equal to $\beta r = (6.28/\lambda)0.159\lambda = 1$, $m = n = q = 1$, and

$$\sqrt{\frac{m^2+(n-q)^2}{m^2+q^2}} = \sqrt{\frac{1^2+(1-1)^2}{1^2+1^2}}\sqrt{\frac{1}{2}} = \frac{1}{1.414} = 0.64;$$

$$\varphi_1 = \tan^{-1}\left(\frac{0}{1}\right) = 0 \text{ time degrees}$$

$$\varphi_2 = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ;\qquad d\mathcal{E}_\theta = 0.32\,d\mathcal{E}_r\,\epsilon^{-j45^\circ}\tan\theta \qquad \text{volts/m}$$

Now, since $d\mathcal{E}_r$ is a $\cos\theta$ function, we have a maximum value for the longitudinal or radial $d\mathcal{E}_r$ component for $\theta = 0$ deg, *i.e.*, along either direction of the linear $I\ dl$ radiator. This means that in the near region, there is a field of this type which is quite pronounced. But there is no such $d\mathcal{E}_r$ effect for $\theta = 90$ deg, since $\cos 90^\circ = 0$. Hence, along the equatorial plane there can be only the transverse $d\mathcal{E}_\theta$ field, since $\sin 90^\circ$ represents a maximum. Let us take now the effect along the $\theta = 15$ deg direction with respect to the dl direction. Then $\sin\theta = \sin 15^\circ = 0.2588$ and $\cos 15^\circ = 0.9659$,

while tan $\theta = 0.2679$. Hence,

$d\mathcal{E}_\theta = d\mathcal{E}_{15^\circ} = 0.32 \times 0.2679\epsilon^{-j45^\circ}\, d\mathcal{E}_r = 0.086\epsilon^{-j45^\circ}\, d\mathcal{E}_r$ volts/m For $\theta = 30^\circ$, tan $30^\circ = 0.5774$

$$d\mathcal{E}_{30^\circ} = 0.32 \times 0.5774\epsilon^{-j45^\circ}\, d\mathcal{E}_r = 0.184\epsilon^{-j45^\circ}\, d\mathcal{E}_r \quad \text{volts/m}$$

In the same way we have:

$$d\mathcal{E}_{45^\circ} = 0.32 \times 1.0\epsilon^{-j45^\circ}\, d\mathcal{E}_r = 0.32\epsilon^{-j45^\circ}\, d\mathcal{E}_r$$
$$d\mathcal{E}_{60^\circ} = 0.32 \times 1.7321\epsilon^{-j45^\circ}\, d\mathcal{E}_r = 0.554\epsilon^{-j45^\circ}\, d\mathcal{E}_r$$
$$d\mathcal{E}_{75^\circ} = 0.32 \times 3.7321\epsilon^{-j45^\circ}\, d\mathcal{E}_r = 1.19\epsilon^{-j45^\circ}\, d\mathcal{E}_r$$
$$d\mathcal{E}_{80^\circ} = 0.32 \times 5.6713\epsilon^{-j45^\circ}\, d\mathcal{E}_r = 1.81\epsilon^{-j45^\circ}\, d\mathcal{E}_r; \qquad d\mathcal{E}_{85^\circ} = 3.65\epsilon^{-j45^\circ}\, d\mathcal{E}_r$$

We find equal field amplitudes for $0.32 \tan\theta = 1$ or $\tan^{-1}(1/0.32) = \tan^{-1}(3.12) = 72.3^\circ = \theta$. Now we know that $\mathcal{E}_\theta$, in the equatorial plane of the elementary radiator, has a maximum value. Let us assume that at $r = 0.159\lambda$ meters in this plane, $\mathcal{E}_\theta = \mathcal{E}_{90^\circ} = 10$ volts/m; then generally at the same distance but a different angle, $\mathcal{E}_\theta = \mathcal{E}_{90^\circ} \sin\theta$ and

$\mathcal{E}_{85^\circ} = 10 \sin 85^\circ = 10 \times 0.9962 = 9.962$ volts/m; $\mathcal{E}_{80} = 9.848$; $\mathcal{E}_{75} = 9.659$
$\mathcal{E}_{60^\circ} = 8.66$; $\mathcal{E}_{45^\circ} = 7.071$; $\mathcal{E}_{30^\circ} = 5$; $\mathcal{E}_{15^\circ} = 2.588$ volts/m

These numerical values are out of proportion for a differential radiator, for which reason such evaluations are omitted. We are interested only in a numerical value so that we can compare the ratio of $\mathcal{E}_\theta$ and $\mathcal{E}_r$ values. Hence, the value of $\mathcal{E}_r$, in the $\theta = 15$ deg direction, is $2.588/0.086 = 30$ volts/m. Similarly,

θ°	15°	30°	45°	60°	72.3°	75°	80°	85°
$\mathcal{E}_r$, volts/m	30	27 2	22	15.6	9 578	8.1	5 42	2 72
$\mathcal{E}_\theta$, volts/m	2.588	5 0	7 071	8 66	9.578	9 659	9 848	9.962

Similar and more detailed computations are also given in Tables 4 to 6. The proportions for the $\mathcal{E}_\theta$ and $\mathcal{E}_r$ components, indicated in Fig. 109*a*, are based on the above calculations. We can use the equation for the general ellipse (page 555) in order to find out what the polarization will be at a distance of 0.159λ and in a certain direction θ with respect to a linear radiator. Suppose we apply it to the 60-deg direction ($\theta = 60$). We know that the $\mathcal{E}_\theta$ field is tangential to a meridian in a spherical coordinate system and the $\mathcal{E}_r$ field is radial. Hence, these components are in space quadrature and out of phase by 45 time degrees. Let us call u the displacement due to the $\mathcal{E}_\theta$ field and w the displacement due to the $\mathcal{E}_r$ field; then we have

$$u = \mathcal{E}_\theta \cos q; \qquad w = \mathcal{E}_r \cos(q - 45^\circ); \qquad q = \omega t;$$

$$w = \frac{1}{\mathcal{E}_\theta}[\mathcal{E}_r(u \cos 45^\circ + \sqrt{\mathcal{E}_\theta^2 - u^2}\sin 45^\circ)] \qquad \text{general ellipse}$$

$$w = \frac{1}{0.866}[15.6(0.7071u + \sqrt{8.66^2 - u^2}\,0.7071)]$$

$$= 1.82[0.7071u + \sqrt{8.66^2 - u^2}\,0.7071] = 1.28(u + \sqrt{74.5 - u^2})$$

is equation of resultant $\mathcal{E}$.

Figure 109 gives an idea of what happens close to a radiator in the case of a differential linear radiator and a half-wave radiator. For the half-wave radiator, the different radii in different θ directions are confocal hyperbolas, which are orthogonal to confocal ellipses. These ellipses take the place of the concentric meridian circles (in the case of an elementary linear radiator). The components of the electric vector for a half-wave radiator are then composed of a component that is tangent to an ellipse on which the field point is located. The meridian ellipse, if rotated about the half-wave radiator, produces the above-mentioned ellipsoid of revolution whose surface is an equiphase surface (wave front), just as the sphere of an elementary radiator represents an equiphase surface. The other electric component, corresponding to the $\mathcal{E}_r$ component of an elementary radiator, is normal to the ellipse, *i.e.*, tangent to the orthogonal hyperbola through the field point under consideration. Since we have, for an elementary radiator as well as for a half-wave radiator, only one magnetic component, namely, the H_φ field, the magnetic H_φ field is, in either case, tangent to circles around the radiator. In either case, the elliptical planes of polarization must be normal to the equiphase surface passing through the field point. If the electrical components are 90 time degrees out of phase, the resultant $\mathcal{E}$ vector must be elliptically polarized in a plane fixed by the field point and the half-wave radiator. The field points lying in a meridian plane on an elliptical contour refer to such polarizations, which differ along the contour, since for field points near the equatorial plane of a radiator, the electric component tangent to the elliptical contour is greater than the orthogonal electric component at the same field point and tangent to the corresponding hyperbola. On the other hand, the elliptical polarization is such that the component along the hyperbola gains as the component along the ellipse becomes smaller as we go to θ angles of smaller values. Hence, there must be an angle of θ for which both electric components are equal in magnitude and cause a circular polarization. The angle θ for equal values changes with the distance from the radiator. For angles θ still smaller than for equal magnitudes, the electric component along the hyperbola exceeds that along the corresponding ellipse. The electric component and the H_φ field that are transverse to the wave propagation, *i.e.*, in the equiphase surface, are the ones that will ultimately survive and cause the distant radiation field.

This discussion shows that near a linear radiator, we deal with an ellipsoidal wave front, which in the distant radiation region can be taken as a spherical wave front. For all wave fronts, whether close to a radiator or in the distant radiation region, all field vectors should be taken as retarded vectors, since, for instance, an electric displacement of $\kappa\mathcal{E}$

coulombs per square meter reaching one end of the radiator in the field, next to it, left the other end earlier and not simultaneously.

108. Notes on Space Fields and Patterns of Linear Radiators with Partial Radiation Suppression. Figure 108 shows arrangements where equal phasing is caused by means of quarter-wave lines, since such lines have a quarter-wave distribution on the go and a quarter-wave distribution on the return and essentially cancel strong field effects, leaving only small radiation fields in comparison with distributions along a single line with no return along it. The small radiation effects act besides in space quadrature to the pronounced radiation effects of the actual half-wave radiator sections. Figure 108 shows also partial radiation suppression by means of screening cylinders over half-wave distributions that would otherwise act in antiphase. Such screening was first suggested by H. E. Hollmann and A. Thoma.[1] Antiphase radiation is suppressed in order to radiate power toward the equatorial plane (cause beaming). If natural phasing with inphase and antiphase radiation sections were used, a spread-out fan-shaped radiation pattern, as in Fig. 106, would be obtained for radiators longer than a half-wavelength distribution. The treatment used in the preceding sections, by means of elementary $d\mathcal{E}_\theta$ contributions as in the distant radiation region together with the Poynting radiation vector, then leads to the following analysis:

Calling N the order of harmonic excitation of a linear radiator of length $2l$, we have, with respect to Fig. 110, a current distribution $\frac{1}{2}(N+1)\lambda$. A length $2l$ is used, so that the mid-point of the radiating conductor is a distance l from each end of the conductor and can be used, because of such a symmetry, as a reference point. Figure 110*b* gives the case where two half-wave ends of a conductor (is $2l$ meters long) radiate in phase, while the middle portion, which is in counterphase, is prevented from radiating. The integrations for the cases indicated in Fig. 110*a* are then extended only over the heavily drawn portions of the total length of $2l$. For any two symmetrically disposed radiation elements dz, we have for $\beta = 2\pi/\lambda$, the combined elementary field distribution for the magnitude $K = (188.35 \sin\theta)/(r\lambda)$

$$d\mathcal{E}_\theta = KI_z\epsilon^{j(\omega t-\beta r_1)}\,dz + KI_z\epsilon^{j(\omega t-\beta r_2)}\,dz \quad \text{volts/m}; \qquad dH_\varphi = \frac{d\mathcal{E}_\theta}{376.7} \quad \text{amp/m}$$

As far as the inverse distance effect is concerned, we can put $r_1 = r = r_2$ and also $\theta_1 = \theta = \theta_2$ in the K factor, but we *must* distinguish between r_1 and r_2 in the relations expressing the phase at the field point in the distant radiation region. Since $r_1 = r - z\cos\theta$ and $r_2 = r + z\cos\theta$, we find

$$\begin{aligned} d\mathcal{E}_\theta &= KI_z\,dz[\epsilon^{j(\omega t-\beta r+\beta z\cos\theta)} + \epsilon^{j(\omega t-\beta r-\beta z\cos\theta)}] = KI_z\epsilon^{j(\omega t-\beta r)}\,dz(\epsilon^{j\beta z\cos\theta} + \epsilon^{-j\beta z\cos\theta}) \\ &= KI_z\epsilon^{j(\omega t-\beta r)}\,dz[\cos(\beta z\cos\theta) + j\sin(\beta z\cos\theta) + \cos(\beta z\cos\theta) - j\sin(\beta z\cos\theta)] \\ &= 2KI_z\epsilon^{j(\omega t-\beta r)}\cos(\beta z\cos\theta)\,dz \qquad \text{volts/m} \end{aligned}$$

[1] *Zeit. Hochfrequenz und Akustik*, p. 195, 1938. Bontsch-Bruewitsch, *Ann. Physik*, **81**, 437 (1926).

The integration is to be carried out between the limits ql and l, since symmetrical $I_s\, dz$ current elements are combined in the $d\mathcal{E}_\theta$ expression. We have also the current

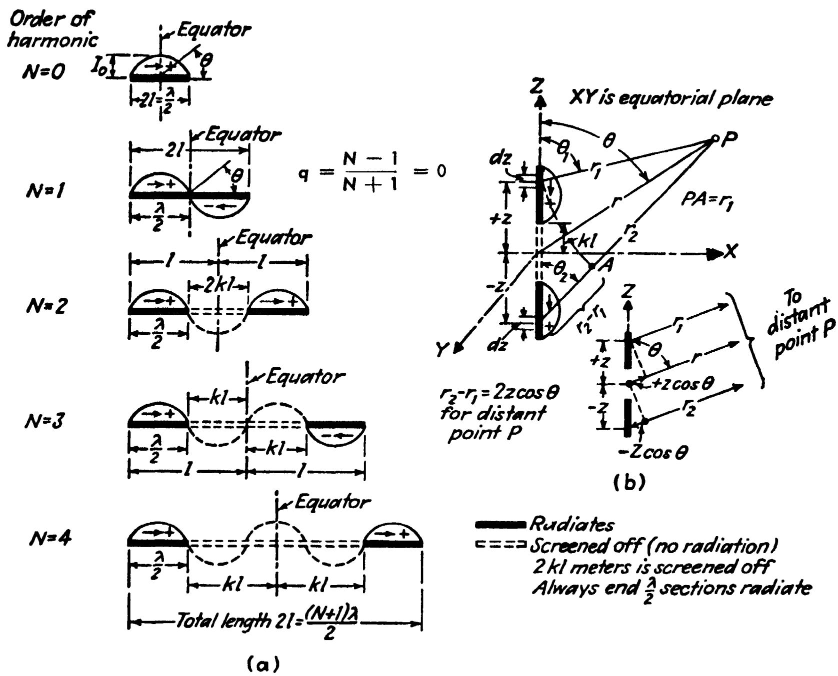


FIG. 110. Half-wave end radiators by means of partial radiation suppression.

distribution, for $N = 2, 4, 6, 8, \cdots$,

$$I_s = I_0 \cos\left[\frac{\pi(N+1)}{2l} z\right] = I_0 \cos\frac{nz}{l}$$

for $\frac{1}{2}(N+1)\pi = n$. Since the total linear conductor length is $2l = \frac{1}{2}(N+1)$, we have $\beta = n/l$ and find

$$\mathcal{E}_\theta = \underbrace{2KI_0\epsilon^{(j\omega t - \beta r)}}_{K_1} \int_{z=ql}^{z=l} \cos(\beta z\cos\theta)\cos\left(\frac{nz}{l}\right) dz$$

$$= K_1 \int_{ql}^{l} \cos\left(\frac{n}{l}z\right)\cos\left(\frac{n}{l}z\cos\theta\right) dz \qquad \text{for } a = \frac{n}{l} \text{ and } b = \frac{n}{l}\cos\theta$$

$$\int \cos(az)\cos(bz)\,dz = \frac{\sin(a-b)z}{2(a-b)} + \frac{\sin(a+b)z}{2(a+b)}$$

$$\mathcal{E}_\theta = K_1 \int_{z=ql}^{z=l} \cos\left(\frac{nz}{l}\right)\cos\left(\frac{nz}{l}\cos\theta\right) dz = \frac{K_1}{2}\left|\frac{\sin(a-b)z}{a-b} + \frac{\sin(a+b)z}{a+b}\right|_{z=ql}^{z=l}$$

$$\mathcal{E}_\theta = 0.5K_1\left[\frac{\sin(a-b)l - \sin(a-b)ql}{a-b} + \frac{\sin(a+b)l - \sin(a+b)ql}{a+b}\right]$$

$$= 0.5K_1\left[\frac{\sin(al)\cos(bl) - \cos(al)\sin(bl) - \sin(aql)\cos(bql) + \cos(aql)\sin(bql)}{a-b}\right.$$

$$\left. + \frac{\sin(al)\cos(bl) + \cos(al)\sin(bl) - \sin(aql)\cos(bql) - \cos(aql)\sin(bql)}{a+b}\right]$$

$$= \frac{0.5K_1}{a^2-b^2}[a\sin(al)\cos(bl) - a\cos(al)\sin(bl) - a\sin(aql)\cos(bql)$$
$$+ a\cos(aql)\sin(bql) + b\sin(al)\cos(bl) - b\cos(al)\sin(bl)$$
$$- b\sin(aql)\cos(bql) + b\cos(aql)\sin(bql) + a\sin(al)\cos(bl)$$
$$+ a\cos(al)\sin(bl) - a\sin(aql)\cos(bql) - a\cos(aql)\sin(bql)$$
$$- b\sin(al)\cos(bl) - b\cos(al)\sin(bl) + b\sin(aql)\cos(bql)$$
$$+ b\cos(aql)\sin(bql)]$$

$$= \frac{K_1}{a^2-b^2}[a\sin(al)\cos(bl) - b\cos(al)\sin(bl) - a\sin(aql)\cos(bql)$$
$$+ b\cos(aql)\sin(bql)]$$

$$= \frac{K_1 l^2}{n^2(1-\cos^2\theta)}\left[\frac{n}{l}\sin n\cos(n\cos\theta) - \frac{n}{l}\cos\theta\cos n\sin(n\cos\theta)\right.$$
$$\left. - \frac{n}{l}\sin(nq)\cos(nq\cos\theta) + \frac{n}{l}\cos\theta\cos(nq)\sin(nq\cos\theta)\right]$$

From the foregoing derivation, we find

$$\mathcal{E}_\theta = \frac{K_1 l}{n\sin^2\theta}[\overbrace{\sin n\cos(n\cos\theta) - \cos\theta\cos n\sin(n\cos\theta)}^{D}$$
$$\overbrace{-\sin(nq)\cos(nq\cos\theta) + \cos\theta\cos(nq)\sin(nq\cos\theta)}^{\text{part of } D}]$$
$$= \frac{376.7 I_0\epsilon^{j(\omega t-\beta r)}}{nr\lambda\sin\theta} lD = \frac{376.7 I_0 lD}{nr\lambda\sin\theta}\epsilon^{j(\omega t-\beta r)} = \frac{376.7 I_0 lD}{\beta lr\lambda\sin\theta}\epsilon^{j(\omega t-\beta r)}$$

Since $\beta = 2\pi/\lambda$, we have, for linear radiators with the partially suppressed radiation of Fig. 110, the formulas

$$\left.\begin{aligned}
\mathcal{E}_\theta &= \frac{60 I_0\epsilon^{j(\omega t-\beta r)}}{r} D_\theta \quad \text{volts/m}; \qquad H_\varphi = \frac{\mathcal{E}_\theta}{376.7} \quad \text{amp/m} \\
D_\theta &= D_{\theta,\varphi} = \frac{\left\{\begin{matrix}\sin n\cos(n\cos\theta) - \cos\theta\cos n\sin(n\cos\theta) \\ -\sin(nq)\cos(nq\cos\theta) + \cos\theta\cos(nq)\sin(nq\cos\theta)\end{matrix}\right\}}{\sin\theta} \\
n &= \beta l = \frac{1}{2}(N+1)\pi; \qquad q = \frac{N-1}{N+1}; \qquad nq = \frac{1}{2}(N-1)\pi; \\
&\text{total length } 2l = \frac{N+1}{\beta}\pi = \frac{1}{2}(N+1)\lambda; \quad N = 2, 4, 6, \cdots
\end{aligned}\right\} \quad (809)$$

where N is the order of harmonic excitation, as shown in Fig. 110, and D_θ the directivity in unobstructed space. The current I_0 in amperes denotes the maximum time value at the current *antinode* of the linear radiator, and r is the distance in meters to the *distant* field point. The correctness of this result can be readily checked by taking, for instance, the $N = 2$ case in Fig. 110*a*, and we find

$$n = \frac{1}{2}(N+1)\pi = \frac{1}{2}(2+1)\pi = \frac{3}{2}\pi \qquad \text{corresponding to } 270°;$$

$$q = \frac{N-1}{N+1} = \frac{2-1}{2+1} = \frac{1}{3}; \qquad nq = \frac{3\pi}{2}\frac{1}{3} = \frac{\pi}{2} \qquad \text{corresponding to } 90°$$

This yields a directivity

$$D_\theta = \frac{\left\{\begin{array}{l}\sin 270° \cos(270° \cos\theta) - \cos\theta \cos 270° \sin(270° \cos\theta) \\ \qquad - \sin 90° \cos(90° \cos\theta) + \cos\theta \cos 90° \sin(90° \cos\theta)\end{array}\right\}}{\sin\theta}$$

$$\sin 270° = -1; \qquad \cos 270° = 0; \qquad \sin 90° = +1; \qquad \cos 90° = 0$$

$$D_\theta = \frac{-\cos(270° \cos\theta) - \cos(90° \cos\theta)}{\sin\theta}$$

$$= -\frac{\cos(3 \times 90° \cos\theta)}{\sin\theta} - \frac{\cos(90° \cos\theta)}{\sin\theta}$$

We note that one directivity portion is the same as it would be for a half-wave dipole, while the other directivity term is the same as for a three-half-wave linear radiator, both directivity terms being additive and the sum being taken negative. The actual physical length of the current-carrying conductor is three half wavelengths long, except that for the $N = 2$ case, the radiation of the middle half-wavelength section is suppressed. This is the reason why there must be another interfering radiation effect in order to account for the suppressed radiation. In other words, because there are half-wave radiation sections, which by themselves radiate like a half-wave radiator, there must be a term that has the characteristic of the half-wave directivity. Because there are two half-wave sections that are in exact current phase relationship, as would occur for a three-half-wave radiator, there must be a characteristic term that represents the directivity of the three-half-wave radiator with natural phasing. Let us now inspect what the total D_θ directivity actually expresses. Suppose we look at the case in the essential plane of the radiator for $N = 2$ in Fig. 110*a*. For field points in the distant radiation region of this plane, $\theta = 90$ deg. Hence,

$$D_\theta = -\left[\frac{\cos(90° \cos\theta)}{\sin\theta} + \frac{\cos(3 \times 90° \cos\theta)}{\sin\theta}\right]$$

$$= -\left[\frac{\cos(90° \times 0)}{1} + \frac{\cos(3 \times 90° \times 0)}{1}\right] = -(1+1) = -2$$

and the corresponding electric and magnetic fields must be *twice* as strong as they would be for a single half-wave radiator. Hence, the advantage of the antiphase suppression is to *beam* the radiation more toward and along the equatorial plane.

It is then an easy matter to plot the directivity pattern from the D_θ formula simply by using values between $\theta = 0$ and 90 deg. Putting $90° \cos\theta = x$, we note that $D_\theta = -[\cos(x) + \cos(3x)]/\sin\theta$.

Moreover, in Eq. (809), we have, for $\sin(nq) = -\sin(n)$ and $\cos(nq) = 0$, the integral $\int I_z \cos(\beta z \cos\theta)\,dz$ simplified and find

$$\left.\begin{aligned}
\int_{z=ql}^{z=l} I_z \cos(\beta z\cos\theta)\,dz &= \pm\frac{I_0 l}{n\sin^2\theta} \\
&\quad\times\left\{\cos\left[(N+1)\frac{\pi}{2}\cos\theta\right] + \cos\left[(N-1)\frac{\pi}{2}\cos\theta\right]\right\} \\
&\qquad\text{for } N = 2, 4, 6, 8, \\
\int_{z=ql}^{z=l} I_z \cos(\beta z\cos\theta)\,dz &= \pm\frac{I_0 l}{n\sin^2\theta} \\
&\quad\times\left\{\sin\left[(N+1)\frac{\pi}{2}\cos\theta\right] + \sin\left[(N-1)\frac{\pi}{2}\cos\theta\right]\right\} \\
&\qquad\text{for } N = 1, 3, 5, 7, \cdots
\end{aligned}\right\} \tag{810}$$

since

$$I_z = I_0\cos\left[\frac{(N+1)\pi}{2l}z\right] \qquad \text{for } \textit{even} \text{ harmonic excitations } N = 2, 4, 6, \cdots$$

$$I_z = I_0\sin\left[\frac{(N+1)\pi}{2l}z\right] \qquad \text{for } \textit{odd} \text{ harmonic excitations } N = 1, 3, 5, \cdots$$

The corresponding directivities then become

$$\left.\begin{aligned}
D_\theta &= \frac{1}{\sin\theta}\left\{\cos[90°(N+1)\cos\theta] + \cos[90°(N-1)\cos\theta]\right\} \\
&\qquad\text{for } N = 0, 2, 4, 6, \cdots \\
D_\theta &= \frac{1}{\sin\theta}\left\{\sin[90°(N+1)\cos\theta] + \sin[90°(N-1)\cos\theta)\right\} \\
&\qquad\text{for } N = 1, 3, 5, \cdots
\end{aligned}\right\} \tag{810a}$$

For $N = 0$, we find

$$D_\theta = \frac{\cos(90°\cos\theta) + \cos(90°\cos\theta)}{\sin\theta} = 2\frac{\cos(90°\cos\theta)}{\sin\theta}$$

which holds for the half-wave radiator, where the factor 2 has nothing to do with the relative directivity and belongs properly to the field value. Putting the respective values for D_θ found from (810a) into the $\mathcal{E}_\theta$ formula of (809) gives the electric field intensity and the corresponding associated magnetic field intensity H_φ holding for the distant radiation region. The odd and even relations of (810) can be combined and

yield the *common* relation

$$\int_{z=ql}^{z=0} I_z \cos(\beta z \cos\theta)\, dz = \frac{I_0 l}{n \sin^2\theta} \left\{ \cos\left[(N+1)\frac{\pi}{2}\cos\theta + \frac{N\pi}{2}\right] - \sin\left[(N-1)\frac{\pi}{2}\cos\theta + (N-1)\frac{\pi}{2}\right] \right\} \quad (810b)$$

We have now a means of computing the energy flux perpendicular through a square-meter area during 1 sec in the distant radiation region. This is the well-known Poynting energy radiation vector. We can do this now for *odd and even* integer values of N. The instantaneous Poynting vector can be found from the instantaneous $\mathcal{E}_\theta$ and H_φ values of (809), since these field associations are in time phase in the radiation region. The D_θ value, due to solution (810b), is then conveniently used If $\mathcal{E}_e$ and H_e represent the corresponding effective values for $\mathcal{E}_\theta$ and H_φ, we have for $N = 1, 2, 3, 4, 5, \cdots$

$$\left.\begin{aligned}
\mathcal{E}_e &= \frac{60 I_e}{r} D_\theta \quad \text{volts/m}; \qquad H_e = \frac{\mathcal{E}_e}{376.7} \quad \text{amp/m} \\
D_\theta &= \frac{1}{\sin\theta}\left\{\cos\left[\frac{\pi}{2}(N+1)\cos\theta + \frac{\pi}{2}N\right] - \sin\left[\frac{\pi}{2}(N-1)\cos\theta + \frac{\pi}{2}(N-1)\right]\right\} \\
D_\theta &= \frac{1}{\sin\theta}\left\{\cos\left[90°(N+1)\cos\theta + 90°N\right] - \sin\left[90°(N-1)\cos\theta + 90°(N-1)\right]\right\}
\end{aligned}\right\} \quad (811)$$

where I_e amperes denotes the effective current value at a current *antinode* of the linear radiator and a distance r meters to the field point. The Poynting energy vector must then be the product of the *inphase* field associations, and

$$p = \mathcal{E}_e \cdot H_e = \frac{\mathcal{E}_e^2}{376.7} = \frac{60^2 D_\theta^2 I_e^2}{376.7 r^2} = 9.52\,\frac{D_\theta^2 I_e^2}{r^2} \qquad \text{watts/m}^2 \quad (812)$$

The total average power radiated in unobstructed radiation space is then the electromagnetic flux through the surface $S = 4\pi r^2$ square meters of a sphere of r meters radius, or

$$P = \int\!\!\int_S p \underbrace{2\pi r^2 \sin\theta\, d\theta}_{=dS} = 2\pi r^2 \int_0^\pi p \sin\theta\, d\theta = 60 I_e^2 \int_0^\pi D_\theta^2 \sin\theta\, d\theta \qquad \text{watts} \quad (813)$$

with the following details:

We find

$$\int_0^\pi D_\theta^2 \sin\theta \, d\theta =$$

$$\int_0 \frac{\cos^2\overbrace{\left[\frac{\pi}{2}(N+1)\cos\theta+\frac{\pi}{2}N\right]}^{A}+\sin^2\overbrace{\left[\frac{\pi}{2}(N-1)\cos\theta+\frac{\pi}{2}(N-1)\right]}^{B}-2\cos A\sin B}{\sin\theta}\,d\theta$$

$$= a + b + d$$

$$a = \int_0^\pi \frac{\cos^2\left[\frac{\pi}{2}(N+1)\cos\theta+\frac{\pi}{2}N\right]}{\sin\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi(N+1)} \frac{1-\cos\alpha}{\alpha}\,d\alpha$$

$$b = \int_0^\pi \frac{\sin^2\left[\frac{\pi}{2}(N-1)\cos\theta+\frac{\pi}{2}(N-1)\right]}{\sin\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi(N-1)} \frac{1-\cos\alpha}{\alpha}\,d\alpha$$

$$d = -2\int_0^\pi \frac{\cos\left[\frac{\pi}{2}(N+1)\cos\theta+\frac{\pi}{2}N\right]\sin\left[\frac{\pi}{2}(N-1)\cos\theta+\frac{\pi}{2}(N-1)\right]}{\sin\theta}\,d\theta$$

$$= \int \frac{\cos\alpha}{\alpha}\,d\alpha = \mathrm{Ci}\,(2\pi N) - \mathrm{Ci}\,(2\pi)$$

$$P = 60I_e^2\Big(\tfrac{1}{2}\{2\times 0.5772 + 2\log_\epsilon(2\pi) + \log_\epsilon(N+1) + \log_\epsilon(N-1)$$
$$- \mathrm{Ci}\,[2\pi(N+1)] - \mathrm{Ci}\,[2\pi(N-1)]\} - \mathrm{Ci}\,(2\pi) + \mathrm{Ci}\,(2\pi N)\Big) = I_e^2 60(\quad) \text{ watts;}$$
$$R_r = 60(\quad) \text{ ohms}$$

The radiation resistance, referred to I_e *amperes at a current antinode*, is then

$$R_r^{(\mathrm{ohms})} = 60\Big(\tfrac{1}{2}\{1.1544 + 2\log_\epsilon(6.28) + \log_\epsilon(N+1) + \log_\epsilon(N-1)$$
$$- \mathrm{Ci}\,[6.28(N+1)] - \mathrm{Ci}\,[6.28(N-1)]\} - \mathrm{Ci}\,(6.28) + \mathrm{Ci}\,(6.28N)\Big) \quad (813a)$$

and the total radiated power

$$P = R_r I_e^2 \quad \text{watts} \quad (813b)$$

With respect to the N values in these expressions, we note in Fig. 110a, that odd integer values such as $N = 1, 3, 5, 7, \cdots$ refer to radiation sections of a linear conductor that are in counterphase excitation. For instance, for $N = 3$, a full-wave section is prevented from radiating. Hence, in effect, we have two collinear half-wave radiators excited in

antiphase and separated by a full wavelength. For $N = 5$, we have two collinear half-wave radiators separated by 2 wavelengths and likewise excited in phase opposition. For $N = 1$, there is *no* separation at all, and we deal with a customary linear conductor radiating in the full-wavelength current distribution. From the same figure, we note that only for even values of N such as $N = 2, 4, 6, \cdots$, we obtain spaced inphase half-wave radiators in effect. For instance, for $N = 2$, the spacing between the two half-wave radiators is a half wavelength; while for $N = 4$, a 1.5λ spacing prevails. Therefore, *beaming* of electromagnetic field energy *toward the equatorial plane requires even values of N*, since only for such values are inphase current distributions effective in radiating. Even though for odd values of N no radiation effect can be caused in the equatorial plane, the partial suppression of the mid-section radiation causes the side lobes to beam more toward the equatorial plane, as can be noted in Fig. 111*b*. With increasing odd values of N, such as $5, 7, 9, 11, \cdots$, all radiation lobes point more and more toward the meridian, *i.e.*, away from the equatorial plane. If quarter-wave ends of a linear radiator only are exposed to space radiation effects, we have the conditions of Fig. 112. The integral that determines the direction effect is then

$$\int_{z=ql}^{z=l} I_z \cos(\beta z \cos\theta)\, dz = \frac{I_0 l}{n \sin^2\theta} D$$

$$D = \cos\left[\frac{\pi}{2}(N+1)\cos\theta + \frac{\pi}{2}N\right] - \cos\theta \cos\left[\frac{\pi}{2}N\cos\theta + \frac{\pi}{2}(N-1)\right]$$

for *odd* values of N

$$\mathcal{E}_\theta = \frac{376.7 I_0 l}{n r \lambda \sin\theta} D\epsilon^{j(\omega t - \beta r)} \qquad \text{volts/m}$$

The direction factor, therefore, becomes

$$D_\theta = \frac{D}{\sin\theta} = \frac{1}{\sin\theta}\{\cos[(N+1)90^\circ \cos\theta + N\,90^\circ] - \cos\theta\cos[N\,90^\circ\cos\theta + (N-1)90^\circ]\} \qquad \text{for } N = 0, 1, 2, 3, 4, \cdots \tag{814}$$

As before, we find a radiation resistance

$$R_r^{(\text{ohms})} = 60\Big(\tfrac{1}{2}\{1.1544 - 2\log_\epsilon(6.28) + \log_\epsilon(N+1) + \log_\epsilon(N) - \text{Ci}[6.28(N+1)] - \text{Ci}(6.28N)\} - \text{Ci}(3.14) + \text{Ci}[6.28(N+1)]\Big) \tag{814a}$$

holding for all values $N = 0, 1, 2, 3, 4, \cdots$. For $N = 0$, according to Fig. 112, we have a customary half-wave radiator with no partial

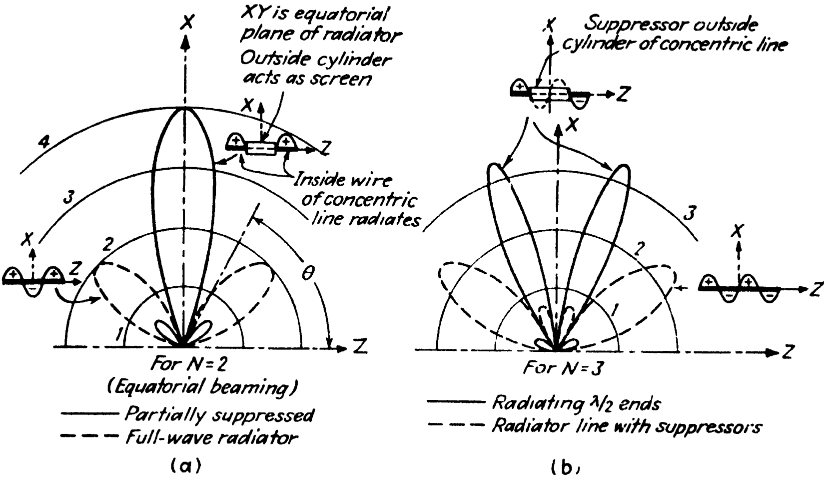
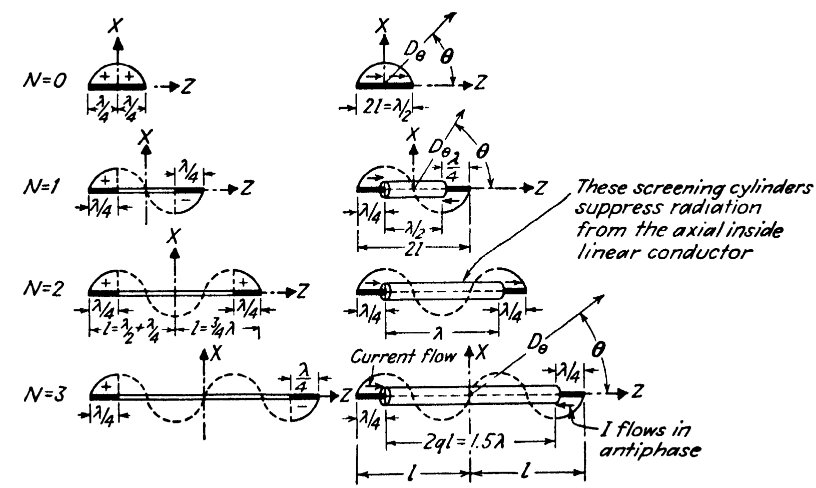


FIG. 111. Power directivity patterns (D_θ^2) for linear radiators with and without partial radiation suppression (end $\lambda/2$ radiators). D_θ as in Eq. (811).



—— radiates
-- prevented from radiating

$q = \frac{N}{N+1}$; $2ql$ meters is length of the screening cylinder.

For $N = 2$ we have, for instance, $q = \frac{N}{N+1} = \frac{2}{3}$; $l = \frac{3}{4}\lambda$, and $2ql = 2\frac{2}{3}\frac{3}{4}\lambda = \lambda$ meters of the total length $2l$ meters are prevented from radiating.

XY plane is equatorial plane of radiator.

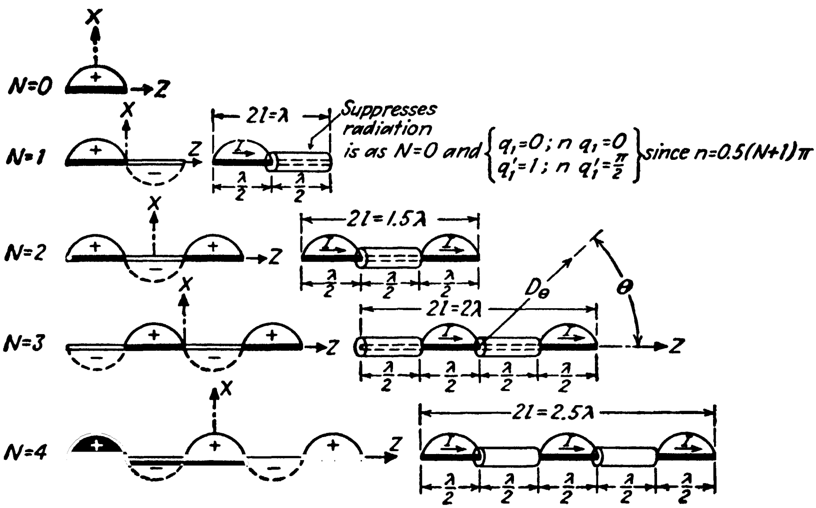
FIG. 112. Quarter-wave end radiators with partial radiation suppression.

radiation suppression, and

$$R_r = 30[0.5772 + \log_\epsilon (6.28) - \mathrm{Ci}\ (6.28)]$$
$$= 30(0.5772 + 1.83737) = 72.4 \text{ ohms}$$

which is close enough to the well-known value of a half-wave dipole.

So far, we have derived only expressions where either only $\frac{1}{2}\lambda$ ends or $\frac{1}{4}\lambda$ ends, respectively, are permitted to radiate while the middle por-



For $N = 2$, we have $\left\{ \begin{matrix} q_1 = \frac{1}{3};\ nq_1 = \frac{\pi}{2} \\ q_1' = 1;\ nq_1' = 3\frac{\pi}{2} \end{matrix} \right\}$ since $n = 0.5(2 + 1)\pi$

For $N = 3$, we have $\left\{ \begin{matrix} q_2 = \frac{3}{5};\ nq_2 = \frac{3\pi}{2} \\ q_2' = 1;\ nq_2' = 5\frac{\pi}{2} \end{matrix} \right\}$ since $n = 0.5(3 + 1)\pi$

For $N = 4$, we have $\left\{ \begin{matrix} q_1 = 0;\ nq_1 = 0 \\ q_1' = \frac{1}{5};\ nq_1' = \frac{\pi}{2} \end{matrix} \right\}$ since $n = 0.5(N + 1)\pi = 0.5(4 + 1)\pi$

FIG. 113. Method of securing inphase half-wave radiator sections by means of partial radiation suppression. XY plane is equatorial plane of radiator.

tion of a linear current-carrying conductor is prevented from radiating. For a conductor for which the order of harmonic excitation is larger than $N = 2$ (consult Fig. 110a), better beaming can be obtained when all equal-phased current distributions are utilized. We must then integrate over several sections and for different limits, which are $q_s l$ to $q_s' l$, for which s signifies that we deal with a summation. The electromagnetic flux, streaming during dt seconds normally across 1 sq m of the radiation field

space, is

$$dp = \mathcal{E}_\theta \cdot H_\varphi\, dt = \frac{K_1^2\, dt}{376.7}\left[\sum \int_{q_s l}^{q_s' l} I_z \cos\left(\frac{2\pi z}{\lambda}\cos\theta\right) dz\right]^2$$

which, for $N = 2, 4, 6, 8, \cdots$, yields an integral value

$$\sum \int_{q_s l}^{q_s' l} = \sum_s \{\sin(nq_s')\cos(nq_s'\cos\theta) - \sin(nq_s)\cos(nq_s\cos\theta) + \cos\theta\,[\cos(nq_s')\sin(nq_s'\cos\theta) - \cos(nq_s)\sin(nq_s\cos\theta)]\}$$

for $n = \frac{1}{2}(N + 1)\pi$. We have then conditions as shown in Fig. 113, since *no* antiphase radiating conductor sections are permitted to exist. Inserting the values of q and q' in the D_θ expression found from the summation integral, we find that the radiated power P in watts is proportional to

$$\int_0^\pi \frac{1}{\sin\theta}\left\{\sum_s \cos\left[\frac{\pi}{2}(s+1)\cos\theta\right]\right\}^2 d\theta$$

since

$$\left.\begin{aligned} P &= I_e^2 60 \int_0^\pi \frac{1}{\sin\theta}\left\{\sum_{s=0,2,4,6,\cdots} \cos\left[\frac{\pi}{2}(s+1)\cos\theta\right]\right\}^2 d\theta \quad \text{watts} \\ P &= I_e^2 R_r \quad \text{watts;} \quad R_r \text{ radiation resistance in ohms} \\ R_r &= 60\Bigg(\frac{1}{2}\sum_{s=0,2,4,6,\cdots}\{0.5772 + \log_\epsilon(6.28) + \log_\epsilon(s+1) - \text{Ci}\,[6.28(s+1)]\} \\ &\quad + \sum_{\substack{s,s'=0,2,4,6,\cdots \\ s \pm s'}} (-1)^{0.5(s-s')}\{\text{Ci}\,[3.14(s-s')] - \text{Ci}\,[(s+s'+2)3.14]\}\Bigg) \text{ohms} \end{aligned}\right\} \quad (815)$$

which again degenerates to the customary half-wave-radiator case, for $N = 0$ with $R_r = 72.4$ ohms as the result. We also note from (815) that that directivity factor is

$$[D_\theta = \frac{\displaystyle\sum_{s=0,2,4,6,\cdots} \cos\left[\frac{\pi}{2}(s+1)\cos\theta\right]}{\sin\theta} \quad (815a)$$

Figures 114 and 115 give directivity patterns and curves for the radiation resistance based on expressions derived in this section.

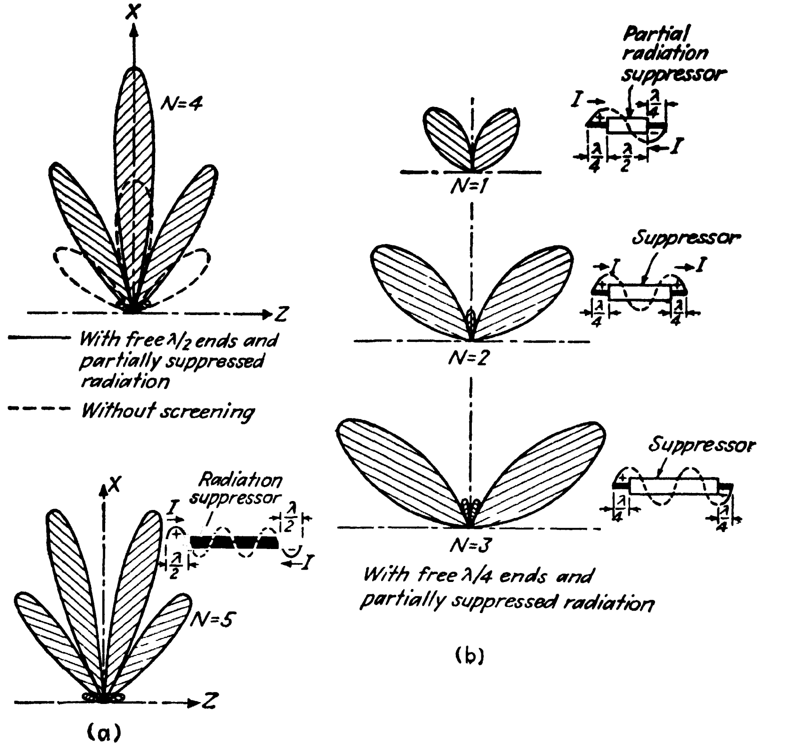


FIG. 114. Radiation patterns for free-end radiators and partial radiation suppression. XY plane is equatorial plane.

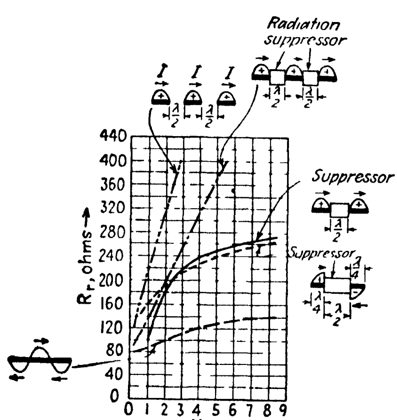


FIG. 115. Radiation resistance for quarter-wave and half-wave linear radiation sections.

109. Electromagnetic Fields and Directivity of Two Linear Elementary Radiators. This subject matter has already been applied in Sec. 38. It will now be discussed in more detail and with respect to different respective current values and different phases. For a *single* elementary radiator, we note, from (161) on page 165, that in the distant radiation region,

$$d\mathcal{E}_\theta^{(\text{volts/m})} = \frac{188.35 D_\theta I\, dl}{r\lambda}\,\epsilon^{j(\omega t-\beta r)}; \qquad dH_\varphi^{(\text{amp/m})} = \frac{d\mathcal{E}_\theta}{376.7};$$
$$D_\theta = D_{\theta,\varphi} = \sin\theta \quad (816)$$

since a fixed current distribution of I amperes prevails along the infinitesimal length dl, expressed in meters. Since circular symmetry prevails about this length, the space pattern $D_{\theta,\varphi}$ is identical with the D_θ pattern. The simplification $D_{\theta,\varphi} = D_\theta D_\varphi = D_\theta$ no *longer* holds for *two* elementary radiators, even if they are parallel at a certain spacing of s meters, because D_φ is no longer circular, *i.e.*, equal to unity. We have then the following details.

For two parallel differential radiators $\pm I_z\, dz$ as in Fig. 116, the radiation pattern in the equatorial XY plane is no longer circular. Let us discuss first antiphase currents in respective elementary dz conductors, which for $k = 188.35/(r\lambda)$ and $q = \sin\theta\cos\varphi$ leads to

$$\begin{aligned}
d\mathcal{E}_\theta &= kD_\theta I_z\, dz\, \epsilon^{j[\omega t-\beta(r-0.5qs)]} - kD_\theta I_z\, dz\, \epsilon^{j[\omega t-\beta(r+0.5qs)]} \\
&= \underbrace{kD_\theta I_z\, \epsilon^{j(\omega t-\beta r)}\, dz}_{k_1}\,(\epsilon^{j0.5\beta qs} - \epsilon^{-j0.5\beta qs}) \\
&= k_1 \left\{\left[\cos\left(\frac{\beta s}{2}q\right) + j\sin\left(\frac{\beta s}{2}q\right)\right] - \left[\cos\left(\frac{\beta s}{2}q\right) - j\sin\left(\frac{\beta s}{2}q\right)\right]\right\} \\
&= 2jk_1 \underbrace{\sin\left(\frac{\beta s}{2}q\right)}_{D_\varphi} = j\,\frac{376.7 D_\theta D_\varphi I_z\, dz}{r\lambda}\,\epsilon^{j(\omega t-\beta r)} \qquad \text{volts/m}
\end{aligned}$$

The field intensity is then

$$\left.\begin{aligned}
d\mathcal{E}_\theta^{(\text{volts/m})} &= j\,\frac{376.7 I_z\, dz}{r\lambda}\underbrace{\sin\theta}_{D_\theta}\,\underbrace{\sin\left(\frac{\beta s}{2}q\right)}_{D_\varphi}\,\epsilon^{j(\omega t-\beta r)} \\
&\qquad \text{for two parallel antiphase linear elementary radiators} \\
d\mathcal{E}_\theta^{(\text{volts/m})} &= j\,\frac{188.35 I_z\, dz}{r\lambda}\underbrace{\sin\theta}_{D_\theta}\cdot\underbrace{(1)}_{D_\varphi}\,\epsilon^{j(\omega t-\beta r)} \quad \text{for a single elementary radiator}
\end{aligned}\right\} \quad (817)$$

The alignment or group characteristic in space is

$$\left.\begin{array}{lll} D_g = \sin\left(180^\circ \frac{s}{\lambda} q\right) & \text{for } q = \sin\theta\cos\varphi & \\ & \text{alignment along } X \text{ axis as in Fig. 116} & \\ D_g = \sin\left(180^\circ \frac{s}{\lambda} q\right) & \text{for } q = \sin\theta\sin\varphi & \text{along the } Y \text{ axis} \end{array}\right\} \quad (817a)$$

Therefore, the space pattern is now $D_{\varphi,\theta} = D_g D_\theta$ at any field point r. φ, θ with a space directivity

$$\left.\begin{array}{ll} D_{\varphi,\theta} = D_\theta D_g = \sin\theta\sin\left(180^\circ q \frac{s}{\lambda}\right) & \text{for two parallel elementary radiators in current antiphase} \\ D_{\varphi,\theta} = D_\theta = \sin\theta & \text{for a single elementary linear radiator} \end{array}\right\} \quad (817b)$$

Figure 117 gives the equatorial direction pattern (D_φ pattern) in the XY plane for antiphase dz radiators. We note that for relatively large

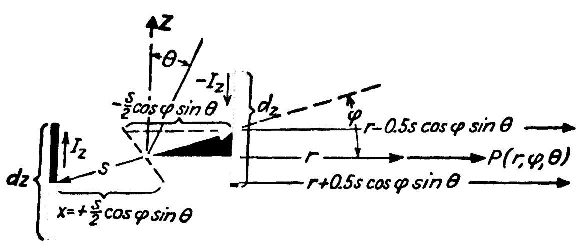


FIG. 116. Two elementary radiators dz a distance s m apart and excited in antiphase. I_z is fixed along dz.

spacings s in terms of the free-space wavelength, larger field intensities in certain directions are possible than for a single $I_z\,dz$ radiator, in spite of the differential action of respective radiators as far as their antiphase currents are concerned. For a spacing s equal to a half wavelength, we have $s/\lambda = 0.5$, yielding maximum reinforcement effect due to radiation interference and to the component field contributions of the two dz radiators with an amplitude value of $\mathcal{E}_{max} = 376.7 I_z\,dz/(r\lambda)$ volts per meter. This is twice the value which exists for a single elementary radiator. For a single elementary radiator, we should have a circular pattern of amplitude $188.35 I_z\,dz/(r\lambda)$. This statement can be readily confirmed, since for $s = \frac{1}{2}\lambda$

$$D_\varphi = \sin\left(180^\circ \frac{s}{\lambda}\cos\varphi\right) = \sin(90^\circ\cos\varphi)$$

is the only directivity in the equatorial XY plane, and $\sin \theta = D_\theta = \sin 90 \text{ deg} = 1$ and $D_{\theta,\varphi} = D_\theta D_\varphi = D_\varphi = \sin (90 \text{ deg} \cos \varphi) = \sin (90 \text{ deg} \cos 0 \text{ deg}) = \sin 90 \text{ deg} = 1$. Using this result in (817) yields in the $\varphi = 0$ deg directions ($\pm X$ directions) $\mathcal{E} = 376.7 I_z\, dz/(r\lambda)$. The $\varphi = 0$ deg direction represents the alignment direction of the two elementary dz radiators. It is also seen from the patterns of Fig. 117 that for small values of s/λ, the field interference effect causes more and more cancellation of the respective field contributions of both differential radiators. This then explains why, for Lecher wires whose spacing in wavelength is chosen small, the radiation effect must also be very small. Generally, two elementary radiators have a much changed field pattern as the spacing is changed. Along the $\varphi = 90$ deg direction ($\pm Y$ directions), no radiation occurs at all for any spacing, even for $s = 0$, since both dz radiators then coincide and the equal antiphase I_z currents cause zero current effect. For small spacings, the D_φ and, therefore, also the $D_{\theta,\varphi}$ patterns are the least distorted in comparison with the single radiator, but *at the expense* of a much *reduced* field intensity. What holds for differential radiators holds also for finite linear radiators, such as two half-wave radiators, frame aerials, and other multipole arrays.

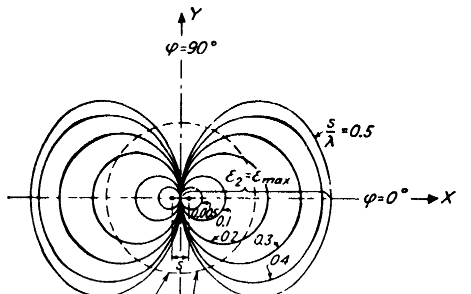


FIG. 117. Polar field patterns of two elementary linear radiators excited in antiphase and for different spacings s in the XY plane (equatorial plane). $\mathcal{E}_1 = 188.35 I\, dz/\lambda r$ represents the amplitude in volts/m for $I\, dz$ radiator. $\mathcal{E}_2 = 376.7 I\, dz/\lambda r$ is amplitude for two antiphase dz radiators.

110. Elementary and Small-loop Aerials. Figure 118 shows an elementary frame aerial, which consists of two dz antiphase current elements for a spacing s, which happens to be also the respective lengths that close the loop. Suppose a plane electromagnetic wave with a polarization of only $\mathcal{E} = \mathcal{E}_z$ volts per meter arrives. Then only the respective dz elements can be affected by the arriving wave. If it were not for the time delay occurring between the two dz conductor elements, voltages dE_1 and dE_2 would be induced along 1,2 and 4,3 that would be equal and in

phase. Hence, equal and opposite currents would cancel[1] themselves, In reality, the electric field arriving at the dz element equal to 4,2 is delayed with respect to the same field when arriving at the dz element equal to 1,2. The phase delay is $\psi = 360 \text{ deg } s/\lambda$ time degrees. Calling dE volts the magnitude of the induced voltages in the two conductor elements we have a resultant driving voltage

$$dE_r = 2\,dE \sin\left(\frac{\psi}{2}\right) = \left[2\,d\mathcal{E} \sin\left(\frac{\psi}{2}\right)\right] dz \qquad \text{volts}$$

since the corresponding field intensity must be equal to the voltage gradient dE/dz and it is assumed that the wave propagation occurs

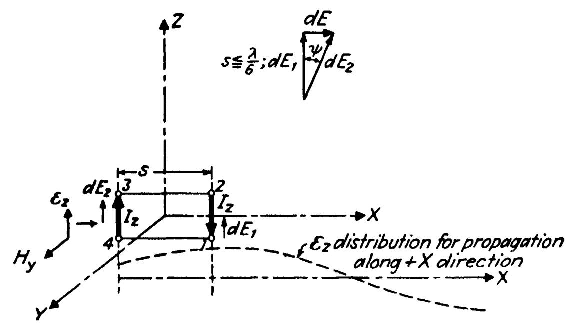


FIG. 118. Elementary loop and induced voltage due to an arriving electromagnetic field.

along the $-X$ direction. If the arriving $\mathcal{E}_z$ polarization were to arrive normal to the XZ plane, no current would circulate in the loop, since the induced voltages in the dz conductor elements would be in phase and of equal magnitude. If wave propagation occurs in a dissipative medium and the loop points toward the wave source, we also have to take the space dissipation into account.

For pronounced field dissipation and a well-insulated conductor, only the dz element closest to the arriving wave front would experience an induced current, since the other dz element would experience only a negligible induced current from the arriving much-weakened wave front. Hence, the circulating current in the receiving loop would be mostly due to the attenuation effect and not to the phase effect.

If the loop of Fig. 118 is in air (nondissipative medium) and makes an angle φ (azimuthal deviation) with the direction toward the radiation

[1] For other details, consult A. Hund, "High-frequency Measurements," pp. 521–523, 526–527, 532, 559–562, 2d ed., McGraw-Hill Book Company, Inc., New York, 1951, and "Phenomena in High-frequency Systems," pp. 466–473, 478–490, 496–499, 503–508, 512–517, McGraw-Hill Book Company, Inc., New York, 1936.

center, we have

$$\sin\left(\frac{\psi}{2}\right) = \sin\left(\frac{\pi s}{\lambda}\cos\varphi\right) \cong \frac{\pi s}{\lambda}\cos\varphi \qquad \text{for}$$

$$s \cong \lambda/6; \qquad dE_r = \left(\frac{6.28s\,dz}{\lambda}\cos\varphi\right) d\mathcal{E} \qquad \text{volts}$$

where $(6.28s\,dz/\lambda)\cos\varphi$ is equivalent height of the elementary loop. Since $s\,dz$ is the area of the loop, dE_r depends only on the area, but *not* on its contour.

111. Comparison of Two Elementary Radiators That Are Excited in Phase and in Antiphase. Figure 119 gives the comparison with representation (*a*) for the inphase excitation and (*b*) for the antiphase excitation, respectively. For inphase I_z excitations, we take I_z positive in each case and find, for $a = \frac{1}{2}\beta q s$ and $q = \sin\theta\cos\varphi$,

$$d\mathcal{E}_{\theta,\varphi} = k_1(\epsilon^{ja} + \epsilon^{-ja}) = k_1(\cos a + j\sin a + \cos a - j\sin a)$$
$$= 2k_1\cos\left(q\frac{\beta s}{2}\right)$$

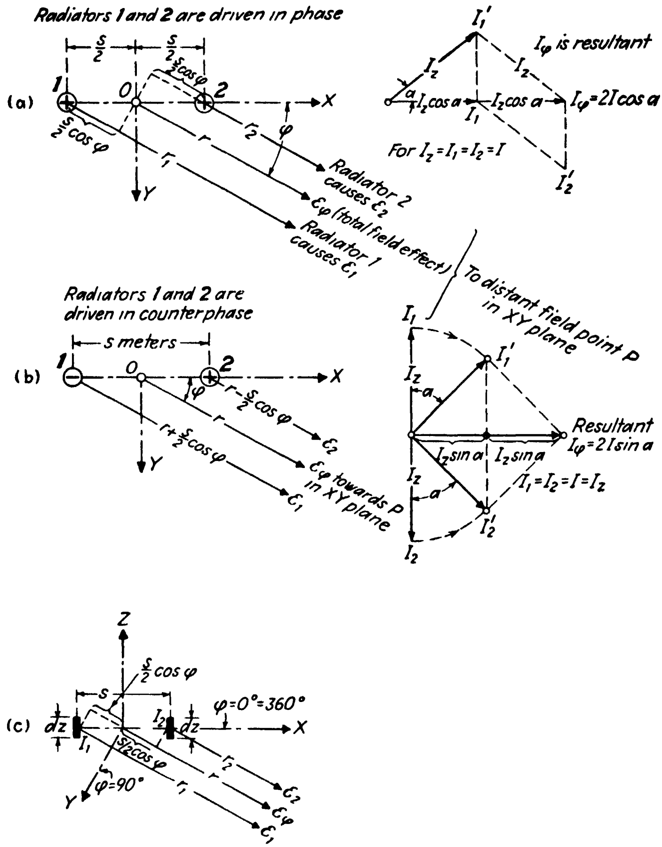
for an alignment along the X axis, we have

$$\left.\begin{aligned} d\mathcal{E}_{\theta\varphi}^{(\text{volts/m})} &= \frac{376.7I_z\,dz}{r\lambda}\sin\theta\cos\left(\frac{\beta s}{2}\sin\theta\cos\varphi\right)\epsilon^{j(\omega t-\beta r)} \\ &\qquad \text{for } inphase \text{ currents} \\ d\mathcal{E}_{\theta,\varphi}^{(\text{volts/m})} &= \frac{376.7I_z\,dz}{r\lambda}\sin\theta\sin\left(\frac{\beta s}{2}\sin\theta\cos\varphi\right)\epsilon^{j(\omega t-\beta r)} \\ &\qquad \text{for } antiphase \text{ currents}\end{aligned}\right\} \quad (818)$$

We have the directivity expressions

$$\left.\begin{aligned} D_{\varphi,\theta} &= D_\theta D_g = 2\underbrace{\sin\theta}_{D\theta}\cos\left(\frac{\beta s}{2}\sin\theta\cos\varphi\right) \quad \text{for inphase excitation} \\ D_{\varphi,\theta} &= D_\theta D_g = 2\overbrace{\sin\theta}\sin\left(\frac{\beta s}{2}\sin\theta\cos\varphi\right) \quad \text{for antiphase excitation}\end{aligned}\right\} \quad (818a)$$

With respect to Fig. 119, let us, for generality's sake, assume that $I_z = I_1$ is the fixed current distribution in the elementary radiator marked 1 and $I_z = I_2$ the fixed current distribution in the other elementary radiator. For the antiphase excitation in Fig. 119*b*, the currents I_1 and I_2 must be shown 180 deg out of phase and with equal lengths for equal I_z currents. As far as the distant field point is concerned, the path difference has to be taken into account with respect to the reference point, which is the origin O. The distance effect of $I_1\,dz$ at P in the equatorial plane is then taken care of by the delay $a = \frac{1}{2}\beta s\cos\varphi$. In the same way, the distance effect of the current element $I_2\,dz$ is taken care of by the *advance* $a = \frac{1}{2}\beta s\cos\varphi$ with respect to such a current element at O. Therefore, for the distant field point P, the counter-



Note: For any $P(r, \varphi, \theta)$ location

$a = \frac{\beta s}{2} \sin \theta \cos \varphi$ (general)

$a = \frac{\beta s}{2} \cos \varphi$ (in the equatorial XY plane)

$r_1 - r_2 = \begin{cases} s \cos \varphi \text{ in } XY \text{ plane path difference} \\ s \sin \theta \cos \varphi \text{ general } P \text{ location path difference} \end{cases}$

Resultant field at $P(r, \varphi, \theta)$ is $\mathcal{E}_{\theta,\varphi} = \mathcal{E}_1 + \mathcal{E}_2$; $\beta(r_1 - r_2)$ is phase lag at $P(r, \varphi, \theta)$

FIG. 119. Vector relations for two radiators driven in phase and in antiphase.

phase excitations I_1 and $-I_2$ no longer act in straight opposition, but with phases as shown by the current vectors I_1' and I_2'. Since we deal in this figure with the equatorial XY plane, the resultant current is called I_φ, and the geometrical additions of I_1' and I_2', yield

$$I_\varphi = 2I_z \sin a = 2I_z \sin\left(\frac{\beta s}{2} \cos \varphi\right) = 2I_z \sin\left(\frac{s}{\lambda} 180° \cos \varphi\right)$$

This resultant I_φ current acts now like an $I'_z\,dz$ current element, *i.e.*, like a *single* $I_\varphi\,dz$ element at the origin. We can, therefore, in (816) replace the fixed current distribution I by I_φ and find again the result of (817). In exactly the same way, we proceed in the vector diagram of Fig. 119*a* for the case of inphase current excitations and find $I_\varphi = 2I_z \cos a$, which, inserted in (816) in place of I, confirms the result of (818) holding for inphase exciter currents.

Such formulas for elementary current elements may seem academic. They are *not*, since we can readily *apply* such formulas to *finite* linear and other radiators. As a matter of fact, (818) can be directly applied without the use of integration if we have linear radiators not longer than a half wavelength. We can also do this without integrations for radiators that are longer if we deal only with *equal phasing*. In each case, we use then, in place of the elementary length dz, the equivalent or effective length l_e of the radiator of finite physical length l. This is dealt with on pages 166, 194, 518. The antiphase formulas apply to radiation from transmission lines (pages 191, 498–525).

112. Directivity Patterns of Two Linear Radiators in the Equatorial Plane. The D_θ pattern has already been dealt with in connection with Fig. 106, which has reference to one linear radiator. On account of the lack of circular symmetry in the case of *two* linear radiators, it is necessary to discuss only such patterns. We can, therefore, apply the elementary D_φ pattern, which holds in the equatorial XY plane. Hence, for antiphase excitations, we have $D_\varphi = \sin a$, which, for a spacing $s = \frac{1}{2}\lambda$, yields $a = 0.5\beta s \cos\varphi = 0.25\beta\lambda \cos\varphi = 0.25 \times 360° \cos\varphi = 90° \cos\varphi$. Therefore, $D_\varphi = \sin(90° \cos\varphi)$ is the equatorial directivity pattern. It

TABLE 26

φ, deg	$r_1 = r_2 + s\cos\varphi$, m	Path difference $r_1 - r_2$, m	Amplitude of $\mathcal{E}_\varphi$, directivity D_φ for a spacing $s = \frac{1}{2}\lambda$ — Inphase excitation: $D_\varphi = \cos a$	Inphase excitation: $\mathcal{E}_\varphi$	Antiphase excitation: $D_\varphi = \sin a$	Antiphase excitation: $\mathcal{E}_\varphi$
0	$r_1 = r_2 + s$	$+s$	0	$\mathcal{E}_1 - \mathcal{E}_2 = A - A = 0$	1	$\mathcal{E}_1 + \mathcal{E}_2 = 2A$
90	$r_1 = r_2$	0	1	$\mathcal{E}_1 + \mathcal{E}_2 = 2A$	0	$\mathcal{E}_1 - \mathcal{E}_2 = 0$
180	$r_1 = r_2 - s$	$-s$	0	$\mathcal{E}_1 - \mathcal{E}_2 = A - A = 0$	-1	$-2A$
270	$r_1 = r_2$	0	1	$2A$	0	0
360 = 0	$r_1 = r_2 + s$	$+s$	0	0	1	$2A$

Remarks: $A = \dfrac{188.35 I_z\,dz}{r\lambda}$; $a = 90° \cos\varphi$ For $r_1 - r_2 = 0$ inphase fields add directly.

would be $D_\varphi = \cos(90° \cos\varphi)$ for the *inphase* excitations. Table 26 gives the computations for the most conspicuous directivities. When the distant point P in the equatorial XY plane of Fig. 119c rotates around the origin O, the path difference $r_1 - r_2$ varies from a value $+s$ to zero value to $-s$ to zero and back to $+s$ as the azimuthal deviation from the plus X axis varies from $\varphi = 0$, to 90 to 180 to 270 and back to 0 deg. For both $\varphi = 90$ deg and $\varphi = 270$ deg, no path differ-

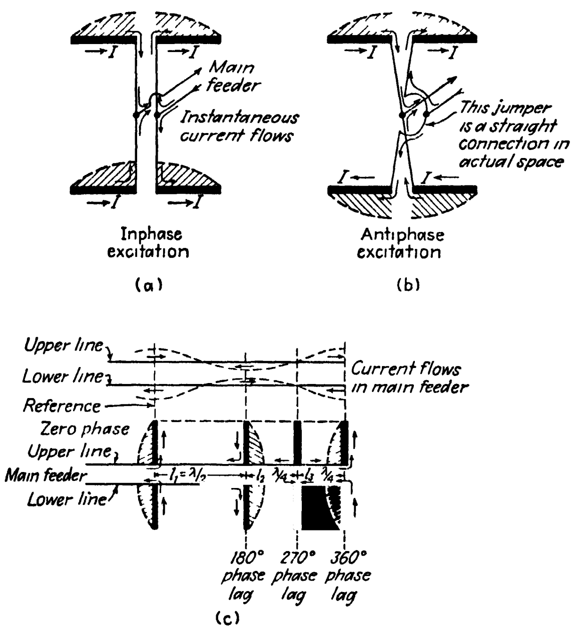


FIG. 120. Excitations of half-wave dipoles with respect to the relative phase of the exciter currents. For any length of x m we have a phase delay of $360° x/\lambda$ for λ in meters.

ence at all exists. This is equivalent to saying that the two dz radiators act as though located at the symmetry point O. Figure 120a and b shows how inphase and counterphase current excitations at the feed point can be produced by a common feeder, while Fig. 120c shows how any different phase excitation can be obtained. In Fig 121, a $\frac{1}{2}\lambda$ spacing is compared with a 1.5λ spacing and inphase excitations. The 1.5λ spacing causes more lobes, since for s larger than $\frac{1}{2}\lambda$ spacings, even for inphase excitations, of radiators 1 and 2, the resultant phase of the respective $\mathcal{E}_1$ and $\mathcal{E}_2$ field contributions cause, at the field point P, canceling effects in several sectors, because

$$D_\varphi = \cos(1.5 \times 180° \cos\varphi) = \cos(3 \times 90° \cos\varphi)$$

is a cosinoidal directivity that varies three times, while the cos (90° cos φ) function, holding for $s = \frac{1}{2}\lambda$, varies once. This means that since the one-half-wavelength spacing causes a single lobe in two quadrants (from $\varphi = 0$ deg to $\varphi = 180$ deg), the $s = (\frac{3}{2})\lambda$ spacing must cause three such lobes. Table 27 gives the computations for inphase and counterphase excitations and a half-wavelength spacing as well as for

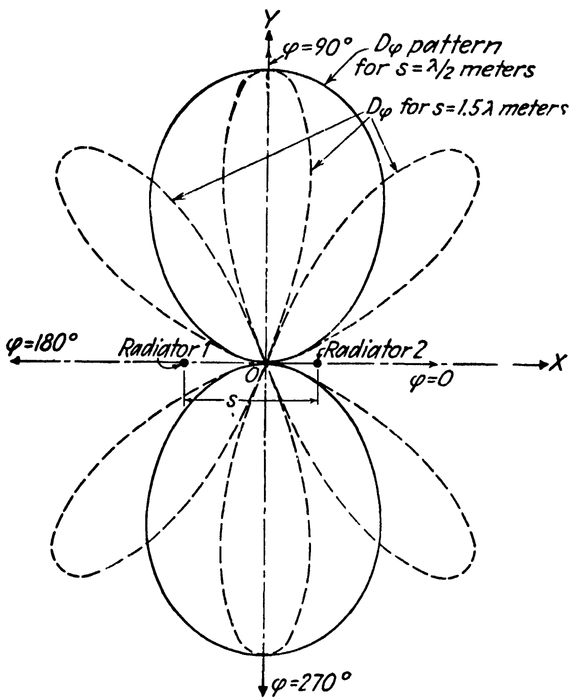


FIG 121 Field amplitude directivity D_φ in the equatorial XY plane for equal inphase currents in two equal linear radiators along the Z direction

a three-half-wavelength spacing From this table as well as from Figs. 117 and 121, we note, for instance, that for a spacing of a half wavelength, the lobes are along the $\pm X$ directions as symmetry axis in the case of antiphase excitations but along the $\pm Y$ directions for inphase excitations. Therefore, for a relative phase that is between 0 deg (inphase) and 180 deg (counterphase), the symmetry axis must be somewhere between the X and Y directions, depending on the value of the relative phase. It must be remembered that these patterns are computed for equal respective current values

The application to an actual radiator alignment is now an easy matter. Suppose we have two half-wave radiators parallel with the Z axis and

TABLE 27

$3a$, deg	φ, deg	$\cos\varphi$	$a = 90\cos\varphi$, deg	$\sin a$	$\cos a$	$\sin 3a$	$\cos 3a$
270	0	1.000	90	1.000	0.000	−1.000	0.000
268.8	5	0.9962	89.6	1.000	0.0070	−0.9998	−0.0209
265.8	10	0.9848	88.6	0.9997	0.0244	−0.9973	−0.0732
261	15	0.9659	87	0.9986	0.0523	−0.9877	−0.1564
253.5	20	0.9387	84.5	0.9954	0.0958	−0.9588	−0.2840
243.9	25	0.9063	81.3	0.9885	0.1513	−0.8980	−0.4399
234	30	0.8660	78	0.9781	0.2079	−0.8090	−0.5870
221.7	35	0.8192	73.9	0.9608	0.2773	−0.6653	−0.7466
207	40	0.7660	69	0.9336	0.3584	−0.4540	−0.8910
190.2	45	0.7071	63.4	0.8942	0.4478	−0.1771	−0.9842
174	50	0.6428	58	0.8480	0.5299	+0.1045	−0.9945
155.4	55	0.5736	51.8	0.7859	0.6184	+0.4163	−0.9092
135	60	0.5000	45	0.7071	0.7071	+0.7071	−0.7071
114	65	0.4226	38	0.6157	0.7880	+0.9135	−0.4067
92.4	70	0.3420	30.8	0.5120	0.8590	+0.9991	−0.0419
69.9	75	0.2588	23.3	0.3955	0.9184	+0.9391	+0.3437
16.8	80	0.1736	5.6	0.0976	0.9952	+0.2890	+0.9573
2.37	85	0.0872	0.79	0.0140	0.9999	+0.0419	+0.9991
0.00	90	0.0000	0.000	0.0000	1.0000	0.000	+1.000

Remarks: ($D_\varphi = \sin a$) is the relative pattern for *antiphase* excitations and $s/\lambda = 0.5$. ($D_\varphi = \cos a$) is the pattern for *inphase* excitations and $s/\lambda = 0.5$. ($D_\varphi = \sin 3a$) holds for *antiphase* excitations and $s/\lambda = 1.5$. ($D_\varphi = \cos 3a$) holds for *inphase* excitations and $s/\lambda = 1.5$, φ is the angle with the $+X$ axis. $\sin(180° + x) = -\sin x, \cos(180° + x) = -\cos x, \sin(90° + x) = \cos x, \cos(90° + x) = -\sin x$.

lined up along the X axis at points $\pm x = \frac{1}{2}s$, so that the origin of the XYZ system is the radiation center (symmetry center) of the two-linear radiator alignment. Suppose we want to know what the electric field intensity is in the distant radiation region at a space point r, θ, φ, for $\theta = 60$ deg against the Z axis, *i.e.*, also 60 deg against either half-wave radiator, and an angle $\varphi = 35$ deg against the alignment direction ($+X$ direction). We have then for the field intensity in any meridian plane, in the case of a *single* half-wave radiator along the Z axis,

$$\mathcal{E}_\theta^{(\text{volts/m})} = j\frac{60I_e}{r}\frac{\cos(90° \cos\theta)}{\sin\theta}$$

according to (807), since $l = \frac{1}{2}\lambda$;

$$D_\theta = \frac{\cos(90° \cos\theta)}{\sin\theta}$$

where I_e is the effective current in amperes at the current antinode of the half-wave radiator (at the mid-point of a linear radiator) and $\mathcal{E}_\theta$ is likewise

the *effective* electric field value in volts per meter. For two such half-wave radiators, we have an alignment or group characteristic of the form

$$D_g = 2\cos\left(180^\circ \frac{s}{\lambda}\sin\theta\cos\varphi\right) \quad \text{for } inphase \text{ excitations}$$

$$D_g = 2\sin\left(180^\circ \frac{s}{\lambda}\sin\theta\cos\varphi\right) \quad \text{for } antiphase \text{ excitations}$$

according to (818*a*) factor 2 affects field intensity value

From Table 25, on page 535, we find for $\theta = 60$ deg a directivity $D_\theta = 0.816$. Suppose the half-wave radiators are spaced one-quarter wavelength apart; then $s = \frac{1}{4}\lambda$ and $s/\lambda = \frac{1}{4}$. For $q = \sin\theta\cos\varphi = \sin 60^\circ \cos 35^\circ = 0.866 \times 0.8192 = 0.71$, we find

$$\cos\left(180^\circ \frac{s}{\lambda} q\right) = \cos(0.71 \times 45^\circ) = \cos 32^\circ = 0.848;$$

$$\sin\left(180^\circ \frac{s}{\lambda} q\right) = \sin 32^\circ = 0.5299$$

Therefore, the effective value of the electric field becomes $\mathcal{E}_e = 60I_eD_\theta D_g/r$ volts per meter, and

$$\mathcal{E}_e^{(\text{volts/m})} = \frac{60I_e}{r}\,0.816(2\cos 32^\circ) = 83\frac{I_e}{r} \qquad \text{for } inphase \text{ excitation}$$

$$\mathcal{E}_e^{(\text{volts/m})} = \frac{60I_e}{r}\,0.816(2\sin 32^\circ) = 52\frac{I_e}{r} \qquad \text{for } antiphase \text{ excitation}$$

for I_e effective amperes at the mid-point of each half-wave linear radiator and the distance r meters from the radiation center to the field point. It is only the absolute amounts of fifty-two and eighty-three times the effective antinode current divided by the distance that count.

Figure 122 shows the effect on the equatorial D_φ patterns for inphase excitations as the spacing s is varied from a zero value upward. In all the cases, equal exciter currents are assumed, so that for $s = 0$, for which both linear radiators coincide, we have a double current flow in one and the same conductor.

113. Directivities of an Alignment of Several Linear Radiators. This subject matter is described in detail in another publication[1] and the result is applied in Fig. 123 for n linear radiators of equal spacing of s meters along the X axis and energized by n equal currents of same amplitudes for any relative time phase step between consecutive radiator currents with the conductors parallel to the Z axis. The group factor

[1] HUND, A., "Phenomena in High-frequency Systems," pp. 517–530, McGraw-Hill Book Company, Inc., New York, 1936.

for any plane parallel to the equatorial XY plane is then

$$\left.\begin{aligned} D_g &= \frac{\sin(na)}{\sin a}; \qquad a = \underbrace{\pi\frac{s}{\lambda}q}_{\text{spacing}} + \underbrace{\pi\frac{\tau}{T}}_{\text{timing}}; \qquad T = \frac{1}{f}\ \text{sec} \\ q &= \sin\theta\cos\varphi \text{ for alignment along } X \\ q &= \sin\theta\sin\varphi \text{ for alignment along } Y \end{aligned}\right\} \tag{819}$$

The space directivity is then $D_{\theta,\varphi} = D_\theta D_g$, if D_θ denotes the meridian directivity for *only one radiator* along any θ direction with the Z axis,

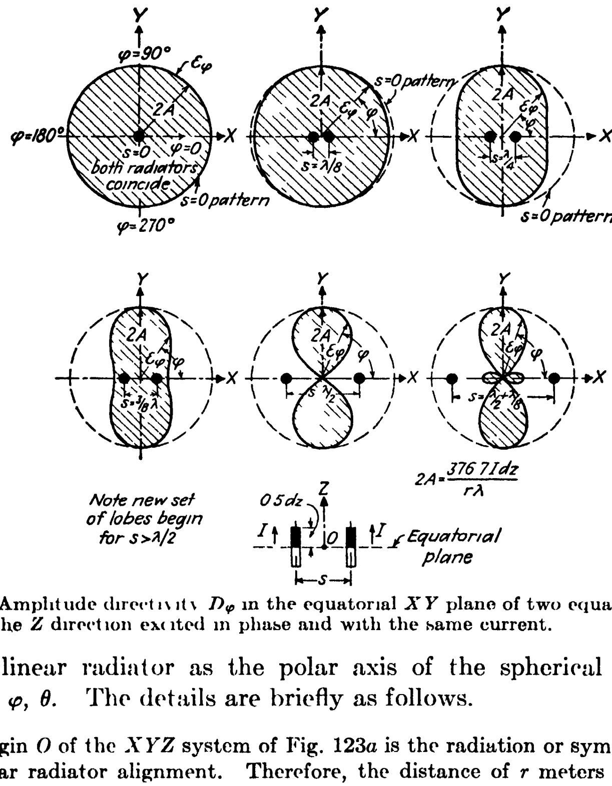


FIG. 122. Amplitude directivity D_φ in the equatorial XY plane of two equal linear radiators along the Z direction excited in phase and with the same current.

i.e., one linear radiator as the polar axis of the spherical coordinate system r, φ, θ. The details are briefly as follows.

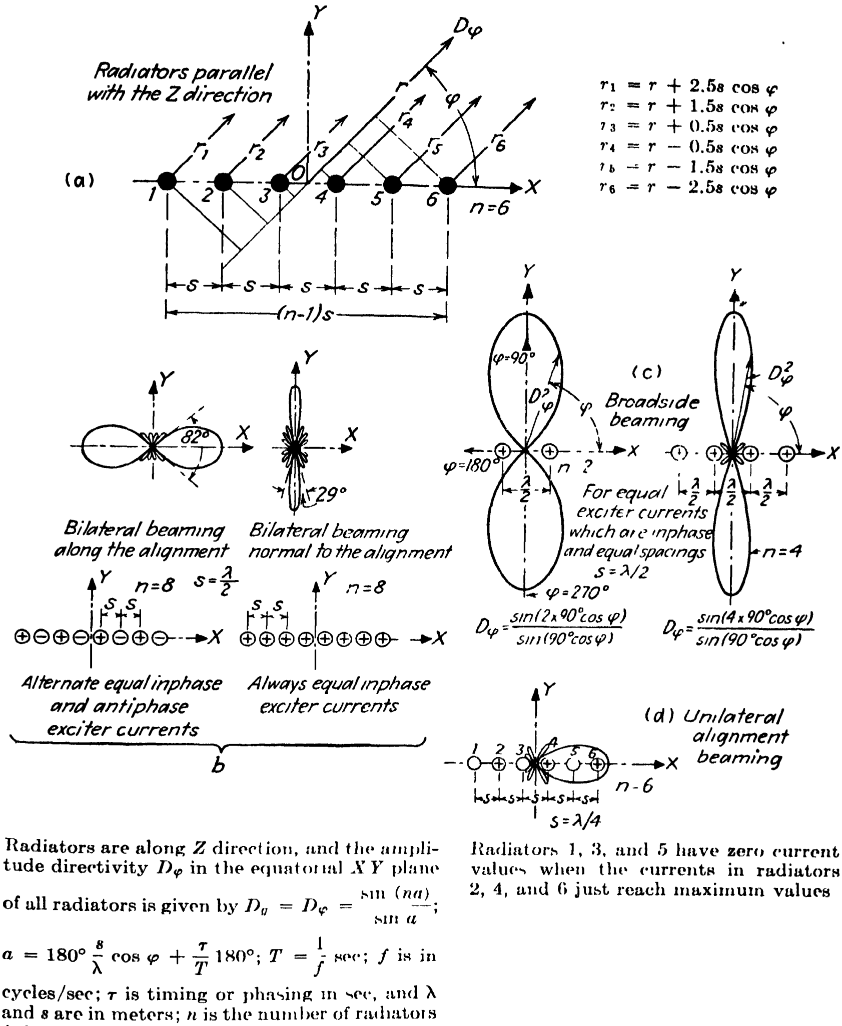
The origin O of the XYZ system of Fig. 123*a* is the radiation or symmetry center of the linear radiator alignment. Therefore, the distance of r meters from O to a field point $P(r, \varphi, \theta = 90 \text{ deg})$ in the equatorial XY plane is the reference distance. The equally spaced radiators 1, 2, 3, 4, 5, and 6, which are s meters apart, have then the inscribed distances r_1, r_2, r_3, r_4, r_5, and r_6, respectively, in terms of the reference distance r and the azimuthal deviation φ from the positive X axis. We find then

$$r_2 - r_1 = r_3 - r_2 = r_4 - r_3 = r_5 - r_4 = r_6 - r_5 = -s\underbrace{\sin\theta\cos\varphi}_{q}$$

for any plane parallel with the equatorial XY plane

$$r_2 - r_1 = r_3 - r_2 = r_4 - r_3 = r_5 - r_4 = r_6 - r_5 = -s\cos\varphi$$

for the equatorial XY plane



Radiators are along Z direction, and the amplitude directivity D_φ in the equatorial XY plane of all radiators is given by $D_u = D_\varphi = \frac{\sin(na)}{\sin a}$; $a = 180° \frac{s}{\lambda}\cos\varphi + \frac{\tau}{T}180°$; $T = \frac{1}{f}$ sec; f is in cycles/sec; τ is timing or phasing in sec, and λ and s are in meters; n is the number of radiators (which are equal)

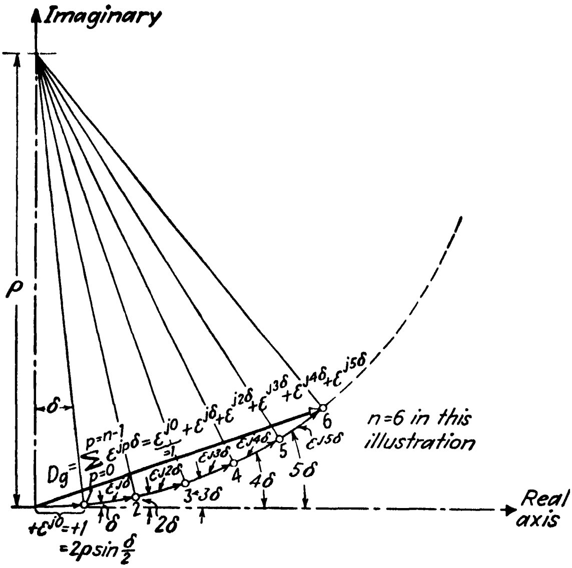
Radiators 1, 3, and 5 have zero current values when the currents in radiators 2, 4, and 6 just reach maximum values

FIG. 123. Radiation effects in the equatorial plane of an alignment of several equal linear radiators for different spacings s and timing τ.

Therefore, the electrical path difference between consecutive linear radiators is βsq electrical space degrees. If $\psi = 360° \tau/T = 360° f\tau$, for f in cycles per second and τ and $T = 1/f$ in seconds, represents the time phase lag of the current in radiator 6 with respect to that of radiator 5, the lag of 5 with respect to 4, the lag of radiator current in 4 with respect to the current in 3 etc., we deal with a total phase angle of

$$\delta = 360° \frac{s}{\lambda} q - 360° \frac{\tau}{T} = 360° \left(\frac{s}{\lambda} q - \frac{\tau}{T}\right) \quad \text{deg}$$

between consecutive radiators. Hence, for radiator 1, we have both $s = 0$ and $\tau = 0$ as far as the relative angles are concerned. Since at the distant field point $P(r, \varphi, \theta)$, the quantity $\epsilon^{j\delta}$ is the phase factor, we find, for radiator 1, a phase factor $\epsilon^{j\delta} = \epsilon^0 = 1$. For radiator 2, we have $360°[(s/\lambda)q - (\tau/T)] = \delta$ and consequently a



Resultant chord $D_g = \Sigma\epsilon^{jp\delta} = \dfrac{\sin(n\delta/2)}{\sin(\delta/2)}$ is known as the group characteristic of an alignment.

FIG. 124. Graphical solution for the group characteristic D_g of an alignment of linear radiators.

phase factor $\epsilon^{j\delta}$. For radiator 3, the phase factor is $\epsilon^{j2\delta}$; for radiator 4, it is $\epsilon^{j3\delta}$; etc. This yields the *group* space factor of the alignment as

$$D_g = 1 + \underbrace{\epsilon^{j\delta}}_{=k} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta} + \underbrace{\cdots + \epsilon^{j(n-1)\delta}}_{\substack{\text{for } n \text{ equal}\\ \text{linear}\\ \text{radiators}}}$$

$$= \underbrace{k^0 + k^1 + k^2 + k^3 + k^4 + \cdots + k^{(n-1)}}_{\text{geometrical progression}}$$

$$= 1\,\frac{k^n - 1}{k - 1} = \frac{k^n - 1}{k - 1} = \frac{\epsilon^{jn\delta} - 1}{\epsilon^{j\delta} - 1} = \frac{2\sin(n\delta/2)}{2\sin(\delta/2)} = \frac{\sin(n\delta/2)}{\sin(\delta/2)}$$

or

$$\frac{\delta}{2} = 180°\,\frac{s}{\lambda}\,q - 180°\,\frac{\tau}{T} = a \qquad \text{deg}$$

relation (819) is confirmed. Figure 124 gives the graphical solution. In the geometric series $1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta} + \epsilon^{j5\delta}$, we have $(n - 1) = (6 - 1) = 5$ for six linear radiators as in Fig. 123*a*, and the unity value of the series stands for $\epsilon^{j0\delta} = \epsilon^0 = 1$. Hence, the value 1 of this geometric progression is plotted along the positive real axis

and all other terms have the same length, but rotated by an angle δ for every succeeding phase factor. The closing vector then represents our D_g value. Putting $j\delta = x$, we have series $D_g = 1 + \epsilon^x + \epsilon^{2x} + \epsilon^{3x} + \cdots + \epsilon^{(n-1)x}$, which, for six radiators, reads $D_g = 1 + \epsilon^x + \epsilon^{2x} + \epsilon^{3x} + \epsilon^{4x} + \epsilon^{5x}$. By dividing $\epsilon^{jn\delta} - 1 = \epsilon^{nx} - 1$ by $\epsilon^{j\delta} - 1 = \epsilon^x - 1$, we obtain $\epsilon^{nx-x} + \epsilon^{nx-2x} + \epsilon^{nx-3x} + \epsilon^{nx-4x} + \epsilon^{nx-5x} + \epsilon^{nx-6x} + \cdots$ or for $n = 6$

$$\epsilon^{6x-x} + \epsilon^{6x-2x} + \epsilon^{6x-3x} + \epsilon^{6x-4x} + \epsilon^{6x-5x} + \epsilon^{6x-6x} = \epsilon^{j5\delta} + \epsilon^{j4\delta} + \epsilon^{j3\delta} + \epsilon^{j2\delta} + \epsilon^{j\delta} + 1$$

which shows that the geometric series for D_g can be expressed by the ratio $(\epsilon^{jn\delta} - 1)/(\epsilon^{j\delta} - 1)$. We also find

$$\underbrace{|\epsilon^{jn\delta} - 1|}_{\text{absolute amount}} = \sqrt{(\epsilon^{jn\delta} - 1)(\epsilon^{-jn\delta} - 1)}$$

$$= \sqrt{[\cos(n\delta) + j\sin(n\delta) - 1][\cos(n\delta) - j\sin(n\delta) - 1]}$$

$$= \sqrt{[\cos(n\delta) - 1]^2 + \sin^2(n\delta)}$$

$$= \sqrt{\underbrace{\cos^2(n\delta) + \sin^2(n\delta)}_{=1} - 2\cos(n\delta) + 1} = \sqrt{2[1 - \cos(n\delta)]}$$

$$= 2\sqrt{\frac{1 - \cos(n\delta)}{2}} = 2\sin\left(\frac{n\delta}{2}\right)$$

Hence,

$$\left|\frac{\epsilon^{jn\delta} - 1}{\epsilon^{j\delta} - 1}\right| = \frac{2\sin\left(\frac{n\delta}{2}\right)}{2\sin\left(\frac{\delta}{2}\right)} = \frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)} = D_g \quad \text{group characteristic confirming} \quad (819a)$$

114. Equatorial Directivity Patterns for Different Spacing and Phasing. As far as the D_φ pattern is concerned, *i.e.*, the pattern in the equatorial XY plane with the origin at the radiation center of the alignment, formula (819*a*) can be applied to any radiator length, *i.e.*, to an elementary length dl, a length l with a half-wave distribution, or any other excitation. It is the half-wave distribution, however, that concerns us mostly in VHF and UHF work. In Fig 123*b*, *c*, and *d*, the equatorial patterns are given. All ⊕ radiators are in phase while all ⊖ radiators are in antiphase. It is seen that the relative phase step between consecutive radiator excitations determines not only the beaming directions but also the spacing of the alignment. This is obvious, since the phase $\delta = 360° [(s/\lambda)\cos\varphi + f\tau]$ degrees in the XY plane contains space and time terms. It is half this angle which is used in (819*a*), since $a = \frac{1}{2}\delta$ and $f\tau$ may be positive or negative (leading or lagging). For Fig. 123*c*, all radiators use inphase current excitations; hence, $\tau = T = 1/f$ seconds. Since $s = \frac{1}{2}\lambda$ in this figure, we have $s/\lambda = \frac{1}{2}$ and $a = 180° (0.5\cos\varphi + 1) = 180° + 90°\cos\varphi$. Actually, this angle comes from δ degrees, since $a = \delta/2$. Because $\delta = 360° + 180°\cos\varphi =$

$180° \cos \varphi$, in effect, our phase coefficient may be taken as $a = 90° \cos \varphi$. Since we have $n = 8$ equal-phased radiators in Fig. 123c, the equatorial directivity becomes

$$D_\varphi = \frac{\sin (8 \times 90° \cos \varphi)}{\sin (90° \cos \varphi)} = \frac{\sin (720° \cos \varphi)}{\sin (90° \cos \varphi)}$$

yielding the indicated broadside beaming to the radiator line-up. On account of the large amplitude of 720 deg in the numerator, we obtain the several indicated side lobes, which are by no means desirous in most practical applications. For Fig. 123b, we have also a $\frac{1}{2}\lambda$ spacing, *i.e.*, $s/\lambda = \frac{1}{2}$, but *anti*phasing. Since we start with a positive linear radiator and go to a negative to a positive to a negative, etc., radiator, we have a succession of inphase, counterphase, inphase, counterphase, etc., excitations, which means a phase *lag* of a half period of $T/2 = \tau = 1/(2f)$ seconds. We must, therefore, use a *negative* sign for the time phase and find

$$a = 180° \underbrace{\frac{s}{\lambda} \cos \varphi}_{0.5 \cos \varphi} - 180° \underbrace{f\tau}_{0.5} = 90° \cos \varphi - 90° = (\cos \varphi - 1)90°$$

$$D_\varphi = \frac{\sin [8(\cos \varphi - 1)90°]}{\underbrace{\sin (90° \cos \varphi - 90°)}_{-(90° - 90° \cos \varphi)}} = \frac{\sin [720°(\cos \varphi - 1)]}{-\sin (90° - 90° \cos \varphi)}$$

$$= -\frac{\sin [720°(\cos \varphi - 1)]}{\cos (90° \cos \varphi)} = -\frac{\sin [720° \cos \varphi - 720°]}{\cos (90° \cos \varphi)}$$

$$= +\frac{\sin (720° - 720° \cos \varphi)}{\cos (90° \cos \varphi)} = -\frac{\sin (720° \cos \varphi)}{\cos (90° \cos \varphi)}$$

This means that the *broadside bilateral beaming* of Fig 123c is, for the alignment of Fig. 123b, *rotated* into the $\pm X$ axes, and we have so-called "end-firing," since there is bilateral alignment beaming. Figure 123d represents a 90-time-degree phase lead of every linear radiator excitation with respect to the preceding one; *i.e*, $\tau = +T/4 = 1/(4f)$ seconds. Since we have also $s = \frac{1}{4}\lambda$ spacing, we have $s/\lambda = \frac{1}{4}$, and

$$a = 180° \frac{s}{\lambda} \cos \varphi + 180° f\tau = 45°(\cos \varphi + 1)$$

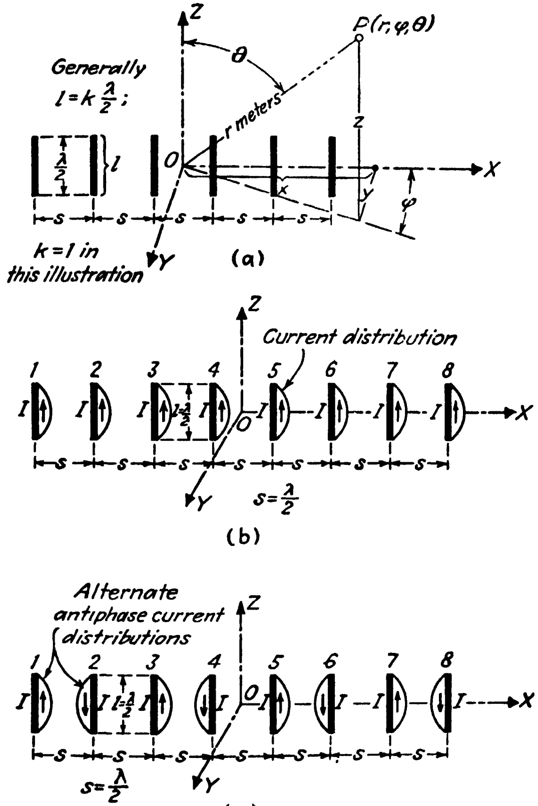
since $(s/\lambda) \cos \varphi = 0.25 \cos \varphi$ and $f\tau = 0.25$;

$$D_\varphi = \frac{\sin [8 \times 45°(1 + \cos \varphi)]}{\sin [45°(1 + \cos \varphi)]} = \frac{\sin [360°(1 + \cos \varphi)]}{\sin [45°(1 + \cos \varphi)]}$$

$$= \frac{\sin (360° + 360° \cos \varphi)}{\sin [45°(1 + \cos \varphi)]} = \frac{\sin (360° \cos \varphi)}{\sin [45°(1 + \cos \varphi)]}$$

causing *unilateral end-firing.*

115. Application of the Alignment Formula to Several Half-wave Radiators. For formula (819) and the arrangement of Fig. 123*a*, we have, in space, the half-wave radiator alignment as in Fig. 125. If I_e denotes the effective current at each driven current antinode (each radiator is conveniently driven at the mid-point of the linear half-wave



$x = r \sin\theta \cos\varphi$; $y = r \sin\theta \sin\varphi$; $z = r \cos\theta$

Origin O is taken as the radiation center of the alignment, since O is the symmetry center.

Inphase excitations

$$D_\varphi = \frac{\sin(720° \cos\varphi)}{\sin(90° \cos\varphi)}$$

for the D_φ pattern (*amplitude* directivity; *power* directivity is D_φ^2) in the equatorial XY plane for which $z = 0$. The group characteristic D_g for the field *amplitude* is

$$D_g = \frac{\sin(720° \sin\theta \cos\varphi)}{\sin(90° \sin\theta \cos\varphi)}$$

holding for any field point $P(r, \varphi, \theta)$ and an alignment *along* the X axis.

$a = 90°(1 + \cos\varphi)$; $D_\varphi = \dfrac{\sin(8a)}{\sin a}$ in XY plane;

$D_\theta = \dfrac{\cos(90° \cos\theta)}{\sin\theta}$ for *one* $\dfrac{\lambda}{2}$ radiator in any meridian plane.

FIG. 125. Alignment of half-wave radiators along the X axis with the radiators normal to the XY equatorial plane.

radiator, making it a half-wave dipole) and the distance r in meters refers to a field point $P(r, \varphi, \theta)$ in the *distant* radiation region and unobstructed radiation space, we have, for the electric *effective* field intensity,

$$\left.\begin{aligned} \mathcal{E}_P &= j60 \frac{I_e}{r} D_{\theta,\varphi} \quad \text{volts/m}; \qquad H_P = \frac{\mathcal{E}_P}{376.7} \quad \text{amp/m} \\ D_{\theta,\varphi} &= D_g D_\theta = \frac{\sin(na)}{\sin a} \frac{\cos(90° \cos\theta)}{\sin\theta}; \\ a &= 180° \left(\frac{s}{\lambda} \sin\theta \cos\varphi + \frac{\tau}{T}\right) \end{aligned}\right\} \qquad (820)$$

This means that we have only to multiply the customary radiation formula for the electric field intensity of a half-wave radiator by D_g. We have then for the useful harmonic excitations

$$\left.\begin{aligned} \mathcal{E}_P &= j60\,\frac{I_e}{r}\,\frac{\sin (na)}{\sin (a)}\,\frac{\cos (k\ 90^\circ \cos \theta)}{\sin \theta} \\ &\qquad\qquad \text{for } k = 1, 3, 5, \cdots \\ \mathcal{E}_P &= 60\,\frac{I_e}{r}\,\frac{\sin (na)}{\sin (a)}\,\frac{\sin (k\ 90^\circ \cos \theta)}{\sin \theta} \\ &\qquad\qquad \text{for } k = 2, 4, 6, \cdots \end{aligned}\right\} \text{volts/m} \qquad (820a)$$

n is the number of equal linear radiators of l m in length; $k = l/2$ [in Eqs. (806) and (807)]; s/λ is the spacing in terms of the operating wavelength along the radiator; τ/T is the timing between consecutive radiator exciting currents per period. We see, therefore, that we have only to multiply the electric field characteristic of *one* linear *radiator* (the $\mathcal{E}_\theta = j60D_\theta I_e/r$ for odd $\lambda/2$ current distributions or $60D_\theta I_e/r$ for even $\lambda/2$ distributions) by the alignment or group characteristic D_g. The group characteristic D_g, for the alignment of Fig. 125, is *only* the *same* as the equatorial D_φ directivity in the *equatorial* XY plane.

In Sec. 114, we have already seen that trigonometric simplifications can be made, in some cases, in the group characteristic formula. Thus, two antiphase half-wavelength radiators ($n = 2$) give $\tau = \frac{1}{2}T$; and for a half-wavelength spacing, we have $s/\lambda = 0.5$ and $\tau/T = 0.5$, and

$$a = 180^\circ(0.5 \cos \varphi + 0.5) = 90^\circ + 90^\circ \cos \varphi$$

$$D_\varphi = \frac{\sin (2a)}{\sin (a)} = \frac{2 \sin (a) \cos (a)}{\sin (a)} = 2 \cos (a) = 2 \cos (90^\circ + 90^\circ \cos \varphi)$$
$$= 2 \sin (90^\circ \cos \varphi)$$

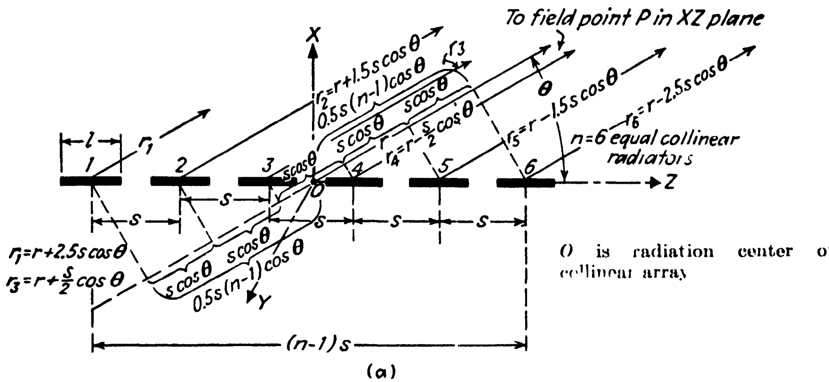
This result checks formula (818), which is derived for two elementary radiators. If we have four equal linear radiators, we find the group characteristic

$$D_g = \frac{\sin (4a)}{\sin (a)} = \frac{2 \sin (2a) \cos (2a)}{\sin (a)} = \frac{2 \times 2 \sin (a) \cos (a) \cos (2a)}{\sin (a)}$$
$$= 4 \cos (a) \cos (2a)$$

where the respective 2 and 4 factors show that the corresponding field values are two and four times the value holding for one radiator, where the expression for a is given in (819).

116. Collinear Antennas, Group Characteristics, and Alignment Formulas. The group characteristic D_g is similar to the equatorial D_φ characteristic holding for a parallel antenna alignment, since the angle θ replaces the angle φ in the group formula. This is evident from Fig. 126, where collinear radiators are arranged along the Z axis. We have,

therefore, again a circular D_φ pattern for which $D_\varphi = 1$. But in any meridian plane with the Z axis as the polar axis, we must first consider the $\mathcal{E}_\theta$ pattern, *i.e.*, the directivity D_θ pattern of one radiator, and multiply by the group pattern D_g, taking the alignment into consideration. This means that a certain field point receives $\mathcal{E}_\theta$ contributions from each



$$r_2 - r_1 = r_3 - r_2 = r_4 - r_3 = r_5 - r_4 = r_6 - r_5 = -s\cos\theta$$

$$D_{\text{group}_\theta} = 1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta} + \epsilon^{j5\delta} + \cdots + \underbrace{\epsilon^{j(n-1)\delta}}_{\text{if we have } n \text{ equal collinear radiators}} = \frac{\epsilon^{jn\delta} - 1}{\epsilon^{j\delta} - 1} = \frac{\sin(n\delta/2)}{\sin(\delta/2)}$$

$$\delta = 360\left(\frac{s}{\lambda}\cos\theta \pm \frac{\tau}{T}\right)\text{deg} \begin{cases} + \text{ sign for a time phase } lead \text{ of } \tau \text{ sec} \\ - \text{ sign for a time phase } lag \text{ of } \tau \text{ sec} \end{cases}$$

For *in*phase conditions $\tau = T = 1/f$ sec, where f is in cycles/sec
For *anti*phase conditions $\tau = T/2 = 1/2f$ sec

For a 90-time-degree phase *lag*, $\tau = -T/4 = -\dfrac{1}{4f}$ sec

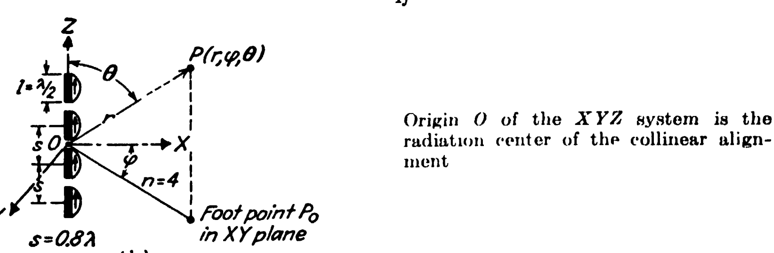


FIG. 126. Conditions for an alignment of collinear radiators along the Z axis.

individual linear antenna, which must be added at the field point with respect to magnitude and phase. If we compare Fig. 126 with Fig. 123*a*, we note that we work now in the XZ instead in the XY plane and the angle θ takes the place of angle φ, *i.e.*, the bearing with respect to all collinear antennas. We have, therefore, the formulas for the effective

values at field point P

$$\mathcal{E}_P = j60 \frac{I_e}{r} D_\theta D_g \quad \text{volts/m}; \qquad H_P = \frac{\mathcal{E}_P}{376.7} \quad \text{amp/m};$$

$$D_\theta D_g = \frac{\cos(90° \cos\theta)}{\sin\theta} \frac{\sin(na)}{\sin a}; \qquad a = 180°\left(\frac{s}{\lambda}\cos\theta + \frac{\tau}{T}\right) \quad (821)$$

The factor D_g is the group characteristic for the alignment of Fig. 126. It is to be multiplied with the meridian directivity D_θ of one linear radiator of length l. The length l is then conveniently equal to $\frac{1}{2}\lambda$ at VHF's and UHF's, for which reason the D_θ for a half-wave radiator is incorporated in the above formula. There is one limitation to the alignment of Fig. 126. We can *never* go to smaller values than $s = \frac{1}{2}l$; as a matter of practical fact, this value can be met only approximately, since we must have some insulation, clearance space, between adjacent ends of consecutive collinear radiators. The limiting case brings us to the linear radiators with equal phasing, and either formula will hold.

The application is then simple. Suppose we have four half-wave dipoles as in Fig. 126*b* driven in phase ($\tau = T = 1/f$ seconds) and spaced $s = 0.8\lambda$ along the Z axis. Then

$$\delta = 360°\left(\frac{s}{\lambda}\cos\theta + \frac{\tau}{T}\right) = 360°(0.8\cos\theta + 1) = \underbrace{0.8 \times 360°\cos\theta}_{\text{in effect}} = 288°\cos\theta;$$

$$a = 144°\cos\theta$$

$$D_g = \frac{\sin(4 \times 144°\cos\theta)}{\sin(144°\cos\theta)} = \frac{\sin(576°\cos\theta)}{\sin(144°\cos\theta)}; \qquad D_\theta = \frac{\cos(90°\cos\theta)}{\sin\theta}$$

Suppose we want to find the electric field intensity in the distant radiation field for 10-amp effective current passing into respective dipoles for a 200-Mc excitation, at a distance of 5 miles from the radiation center, with a lateral bearing of 60 deg with the X axis and a meridian bearing of 80 deg with respect to the polar Z axis. The frequency enters only to that extent, in order to find out if we are in the distant radiation region. The distance of $r = 5$ miles $= 5 \times 1{,}609 = 8{,}045$ m. The free-space wavelength is $\lambda_0 = 300/200 = 1.5$ m. We are, therefore, well in the distant radiation region, since $8{,}045/1.5 \cong 5{,}363$ is the distance in wavelength. All rays are practically parallel (another requirement, due to the derivation), since the entire collinear antenna system comprises a total length of only $4(\frac{1}{2}\lambda) + 3(0.8\lambda - \frac{1}{2}\lambda)$ meters. Even if we take the approximately 5 per cent longer wavelength, holding in free space, as a measure, the total length is negligible compared with 8,045 m. The lateral angle is $\varphi = 60$ deg and does not enter, since for a collinear alignment, we have circular symmetry around the alignment in unobstructed radiation space (always the same value for $P(\theta, r)$). We have, therefore, to compute only functions that depend upon $\theta = 80°$. From Table 25 on page 535, we find $D_\theta = D_{80} = 0.9604$. Moreover, $\cos 80° = 0.1736$, rendering

$\sin(576° \times 0.1736) = \sin(100°) = \sin(90° + 10°) = \cos 10° = 0.9848;$

$$144 \cos \theta = 144 \times 0.1736 = 25$$

$$\sin 25° = 0.4226$$

$$D_\sigma = \frac{\sin(576° \cos 80°)}{\sin(144° \cos 80°)} = \frac{0.9848}{0.4226} = 2.332$$

Formula (821) yields

$$\mathcal{E}_P = j60 \frac{I_e}{r} D_\theta D_\sigma = j60 \frac{5}{8{,}045} 0.9601 \times 2.332 = j83.6 \times 10^{-3} \text{ volt/m}$$

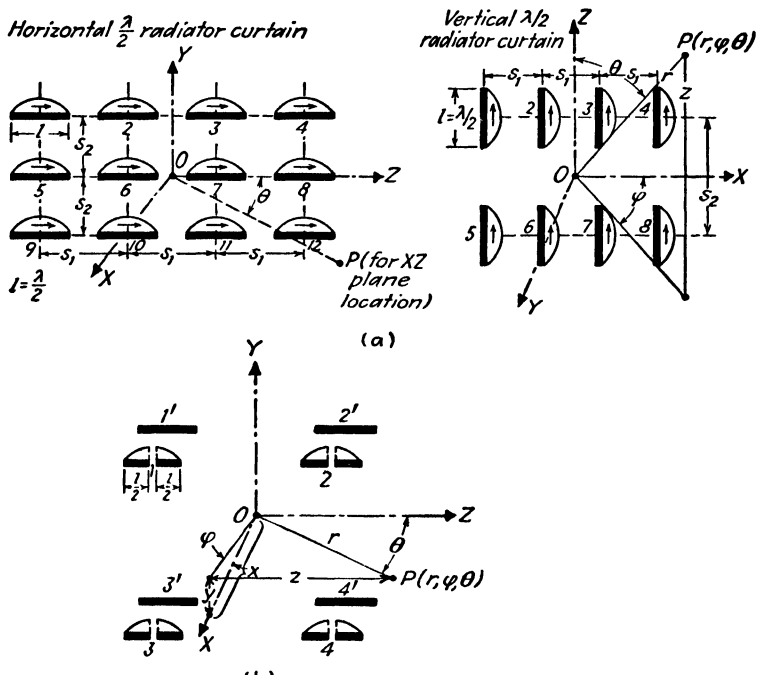
which is 83.6 mv/m, since the j factor has nothing to do with the absolute amount. The associated H_P field is $83.6 \times 10^{-3}/376.7 = 2.22 \times 10^{-4}$ amp/m.

The foregoing is a computation for unobstructed radiation propagation. For UHF work, such a propagation can be realized if we screen off the indirect ray toward ground and with the field pickup close to the radiator and far from other wave obstacles, but in the radiation region.

117. Remarks on Dipole Curtains. We have found that a linear radiator with equal-phased half-wave sections concentrates the radiation energy toward the equatorial plane. But as far as a linear radiator (one radiator only) is concerned, it has circular symmetry, and we have equal wave propagation all around the radiator in the equatorial plane and parallel planes above and below it. Such an energy spread may not be desired. We also found that in the equatorial plane we can get energy concentration in a preferred direction by using several linear radiators in parallel and properly spaced and phased. By arranging, therefore, several equal-phased linear radiators in parallel and in a plane, we can cause, for inphase excitations, more or less sharp forward and backward special beaming, since we bring about radiation concentration in two perpendicular planes. The plane formed by several parallel equal-phased linear radiators that are several half wavelengths long may be called a "radiator curtain." Such a curtain can also conveniently be made up by several half-wave dipoles that are driven and equal phased. We may have then four parallel dipole alignments with four half-wave dipoles in one alignment. The 16 dipoles form then a dipole curtain. It does not take much space for such a curtain in the case of VHF's and UHF's; and in many practical cases, we need fewer dipoles as mentioned in the above example.

Figure 127*a* shows two types of half-wave radiator curtains that are in practical applications half-wave dipoles, meaning that each pole has a quarter-wavelength distribution along it and is center fed. For one representation, we have horizontal half-wave radiators in a plane; while in the other representations, all half-wave radiators are vertical and in a

plane. For several like linear radiators (by "like" is meant of the same electrical length βl), we multiply the field intensity formula holding for one radiator by the group characteristic D_g, which takes care of the array disposition. It is convenient to be consistent with our nomenclature. We therefore take θ as the angle with the direction of l, where l



θ is the angle of any propagation direction with respect to the $+Z$ axis. φ is the angle with respect to the $+X$ axis and the projection of the propagation direction into the equatorial XY plane. The origin O is the radiation center of the half-wave-dipole curtain, and the mid-points 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 are the individual respective radiation centers of the elementary $\lambda/2$ radiators.

Space radiator array. Driven half-wave radiators 1, 2, 3, and 4 form the front curtain, and the parasitic linear radiators 1′, 2′, 3′, and 4′ form the rear curtain; the YZ plane is halfway between these curtains. The curtain with 1′, 2′, 3′, and 4′ acts as a reflector if properly spaced behind the 1, 2, 3, 4 curtain (causing only forward beaming).

FIG. 127. Half-wave-radiator curtains. (It is convenient to place the linear radiators along the Z direction.)

denotes the length direction and magnitude of one radiator. In the same way, the angle φ refers always to the azimuthal bearing that is taken around the l direction wherever possible. In Fig. 127a, we have the radiation centers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, respectively, for the respective radiators that make up the curtain, while the origin O represents the symmetry center and therefore the radiation center of the

curtain. Let us take, first, the curtain with the horizontal half-wave radiators. The collinear alignment of radiators 1, 2, 3, and 4 produces a resultant field intensity $\mathcal{E}_P$ as given by formula (821). The same is also the case for the respective collinear alignments 5, 6, 7, 8 and 9, 10, 11, 12. Therefore, each collinear alignment by itself concentrates its radiated power toward the equatorial XY plane. The more the equal-phased half-wave radiators are collinear the more this occurs. Now the vertical column of half-wave radiators 1, 5, and 9 represents an alignment of parallel linear radiators. Therefore, formula (820) applies, with the corresponding group characteristic D_g. For each vertical alignment such as 1, 5, 9 or 2, 6, 10, or 3, 7, 11 or 4, 8, 12, respectively, $\mathcal{E}_P$ of (820) is the field intensity for each column. This formula and the corresponding patterns show that we obtain, for each column, radiation power concentration toward the XZ plane. Now the energy concentration toward the XZ plane, due to the vertical alignments, and the energy concentration toward the XY plane, due to respective horizontal alignments, *must* cause a resultant energy concentration toward the intersection $\pm X$ of the XY and XZ planes, and we obtain energy beaming toward these $\pm X$ directions. Therefore, beaming along $\pm X$ is more pronounced if we use more equal-phased $\lambda/2$ radiators in respective alignments. Hence, we have a means of adjusting the width of the space beam in the respective XY and XZ planes. For a sharper horizontal pattern, we simply use more half-wave radiators in the horizontal alignments than in the vertical alignments for the curtain with horizontal radiators. The opposite is the case for the vertical half-wave radiators in the curtain. From the above-mentioned formulas, we note that the resultant electric field due to all half-wave radiators in the curtain must contain the group characteristics of both alignments besides the elementary D_θ directivity holding for one half-wave radiator only.

Even though the curtain has the advantage of causing beaming toward a direction instead of toward a plane, as is the case for an alignment, we have still the inherent disadvantage of a bilateral space beaming. This is remedied by using a double-curtain arrangement as in Fig. 127*b*. The front curtain then consists of inphase driven half-wave dipoles, while the rear curtain is used as a reflector in order to beam all the electromagnetic field energy only toward the $+X$ direction. This is then unilateral beaming and causes additional radiation gain along the $+X$ direction. The reflector curtain can be made up of parasitic half-wave radiators, which are not driven at all. In practical installations, they are about a quarter wavelength behind the actual radiator curtain, and the linear dimension of each parasitic radiator is somewhat longer than given by the actual length of the driven half-wave dipole. The

forward beaming along the $+X$ direction can be emphasized still more by also using a director curtain in front of the curtain that contains the actual radiators. The director half-wave radiators may be driven or parasitic. The use of two and three curtains brings us to radiation centers of half-wave radiators distributed in space, and it is convenient then to use a space lattice concept in order to obtain a general radiation formula that can be reduced to a curtain formula, still more reduced to

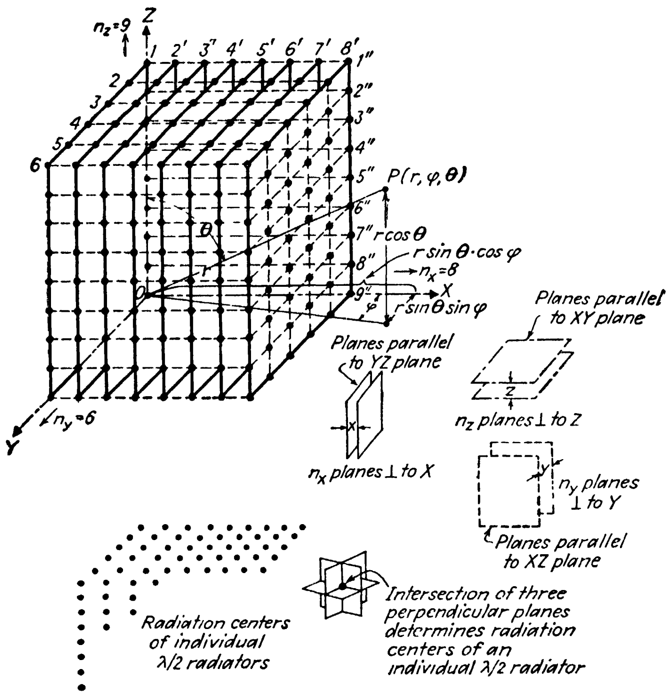


FIG. 128. Lattice structure for the individual radiation centers of $\lambda/2$ radiators in space. Radiation center of the space array of $\lambda/2$ radiators is the symmetry point of the space concerned.

alignment formulas, and finally simplified to the formula of a single linear radiator.

118. Radiation Centers Distributed in Space and a General Radiation Formula. In order to find the direction pattern $D_{\theta,\varphi}$ of several equal radiators distributed in space as in Fig. 128, it is convenient to arrange the radiation centers of each radiator at intersection points of three perpendicular planes. The resultant radiation center is then, the symmetry point O of all individual radiation centers. A distant field point P is then again characterized by spherical coordinates r, φ, and θ, which

expressed in Cartesian coordinates read $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, and $z = r \cos\theta$. These are the well-known Laplacian transformations. For such a system, an arrangement of half-wave dipoles seems the only practical solution, even though there is no limitation to such dipoles. Since half-wave dipoles (in effect, a half-wave linear pole or radiator), when along the Z direction, have a characteristic directivity

$$D_\theta = \frac{\cos(90^\circ \cos\theta)}{\sin\theta}$$

because $D_\varphi = 1$ for one half-wave radiator and $D_{\theta,\varphi} = D_\theta D_\varphi = D_\theta$, we have two choices in the radiation center space mosaic of Fig. 128. One is that all individual radiators are vertical, and the other is that all radiators are horizontal. It therefore does not matter which is the case, since we can always rotate the XYZ system into such a space orientation that the radiators are along the Z axis and parallel thereto. We have then *three* group characteristics D_x, D_y, and D_z, and when they are multiplied by the directivity D_θ characteristic of each individual radiator, we have the space characteristic of all radiators (radiators are parallel with Z axis). Hence,

$$D_{\varphi,\theta} = D_x D_y D_z D_\theta; \qquad D_x D_y D_z = D_g \quad \text{group factor} \tag{822}$$

and the resultant effective electric field intensity becomes

$$\mathcal{E}_P = 60\frac{I_e}{r} D_\theta D_x D_y D_z \quad \text{volts/m} \tag{822a}$$

where I_e denotes the effective current in amperes at the mid-point of one half-wave radiator (feeder current) and r is the distance in meters. The details are then as follows:

According to Fig. 128, we have component planes n_x, n_y, n_z, which determine the various individual radiation centers. We have then only to resolve the group characteristic D_g similar to the ones used for parallel and collinear radiator alignments into the three Cartesian directions and find

$$\left.\begin{aligned}
&D_x = \frac{\sin(n_x a)}{\sin(a)}; \qquad D_y = \frac{\sin(n_y b)}{\sin(b)}; \qquad D_z = \frac{\sin(n_z c)}{\sin(c)} \\
&a = \underbrace{180^\circ \frac{x}{\lambda} \cos\varphi \sin\theta}_{\substack{\text{spacing of consecu-}\\ \text{tive radiators along}\\ X \text{ direction}}} + \underbrace{180^\circ \frac{\tau_x}{T}}_{\text{phasing}} \\
&b = \underbrace{180^\circ \frac{y}{\lambda} \sin\varphi \sin\theta}_{\text{spacing along } Y} + \underbrace{180^\circ \frac{\tau_y}{T}}_{\substack{\text{relative}\\ \text{timing or}\\ \text{phasing}}} \qquad c = \underbrace{180^\circ \frac{z}{\lambda} \cos\theta}_{\substack{\text{spacing along}\\ Z}} + \underbrace{180^\circ \frac{\tau_z}{T}}_{\text{phasing}}
\end{aligned}\right\} \tag{822b}$$

Using these results for D_x, D_y, and D_z in (822), we arrive at exactly the same expression as given by G. C. Southworth,[1] since D_x, D_y, and D_z denote the effective disposition of the individual radiators along the Cartesian coordinate directions and

$$D_{\theta,\varphi} = \frac{\sin(n_x a)}{\sin(a)} \frac{\sin(n_y b)}{\sin(b)} \frac{\sin(n_z c)}{\sin(c)} D_0; \quad D_0 \text{ is directivity of individual radiator} \tag{822c}$$

The respective effective field intensities are then

$$\left.\begin{aligned} \mathcal{E}_P &= 60 \frac{I_e}{r} \frac{\sin(n_x a)}{\sin(a)} \frac{\sin(n_y b)}{\sin(b)} \frac{\sin(n_z c)}{\sin(c)} D_0 \quad \text{volts/m}; \\ H_P &= \frac{\mathcal{E}_P}{376.7} \quad \text{amp/m} \end{aligned}\right\} \tag{823}$$

The application of this formula is not difficult. Suppose we take the radiation-center mosaic of Fig. 128 as an example. We have nine planes normal to the Z axis; hence, there must be $n_z = 9$ radiation centers along any of the three normal lattice intersections along the Z axis. We have eight planes normal to the X axis; therefore $n_x = 8$. We have six planes normal to the Y axis, and $n_y = 6$. Suppose we use practical linear current distributions, *i.e.*, vertical half-wave radiators. Then $D_0 = D_\theta$ with the D_θ formula as given at the beginning of this section. Suppose we choose the spacing $x = 0.75\lambda$ between n_x planes, $y = 0.8\lambda$ between n_y planes, and the spacing $z = 0.9\lambda$ between the n_z planes and assume equal-phased half-wave radiators; then $\tau_x = \tau_y = \tau_z = T = 1/f$ seconds. We find then

$$a = 180° \cdot 0.75 \cos\varphi \sin\theta + 180°; \qquad b = 180° \cdot 0.8 \sin\varphi \sin\theta + 180°$$
$$c = 180° \cdot 0.9 \cos\theta + 180°; \qquad \sin(180° + \alpha) = -\sin(\alpha)$$
$$\sin(8 \times 180° + 8\alpha) = +\sin(8\alpha); \quad \sin(6 \times 180° + 6\alpha) = +\sin(6\alpha)$$
$$\sin(9 \times 180° + 9\alpha) = -\sin(9\alpha); \quad 0.75 \times 180° = 135°$$
$$0.8 \times 180° = 144°; \quad 0.9 \times 180° = 162°$$

$$D_x = \frac{\sin(n_x a)}{\sin(a)} = \frac{\sin(8a)}{\sin(a)} = \frac{\sin(8 \times 135° \cos\varphi \sin\theta)}{-\sin(135° \cos\varphi \sin\theta)} = -\frac{\sin(1080° \cos\varphi \sin\theta)}{\sin(135° \cos\varphi \sin\theta)}$$

$$D_y = \frac{\sin(n_y b)}{\sin(b)} = \frac{\sin(6b)}{\sin(b)} = \frac{\sin(6 \times 144° \sin\varphi \sin\theta)}{-\sin(144° \sin\varphi \sin\theta)} = -\frac{\sin(864° \sin\varphi \sin\theta)}{\sin(144° \sin\varphi \sin\theta)}$$

$$D_z = \frac{\sin(n_z c)}{\sin(c)} = \frac{\sin(9c)}{\sin(c)} = \frac{-\sin(9 \times 162° \cos\theta)}{-\sin(162° \cos\theta)} = +\frac{\sin(1458° \cos\theta)}{\sin(162° \cos\theta)}$$

[1] Certain Factors Affecting the Gain of Directive Antennas, *Proc. IRE*, **18**, 1502 (1920). R. M. Foster, Directive Diagrams of Antenna Arrays, *Bell Sys. Tech. Jour.*, 1926, p. 292.

The total group characteristic $D_g = D_xD_yD_z$ is, therefore, a positive result. We find, for the resultant effective electric field intensity at a space point $P(r, \varphi, \theta)$,

$$\mathcal{E}_P = 60\frac{I_e}{r}\frac{\cos(90°\cos\theta)}{\sin\theta}\frac{\sin(1080°\cos\varphi\sin\theta)}{\sin(135°\cos\varphi\sin\theta)} \times \frac{\sin(864°\sin\varphi\sin\theta)}{\sin(144°\sin\varphi\sin\theta)}\frac{\sin(1158°\cos\theta)}{\sin(162°\cos\theta)} \quad \text{volts/m}$$

For horizontal $\lambda/2$ radiators along the X direction, we have, in the above expression,

$$\frac{\cos(90°\cos\varphi)}{\sin\varphi} \quad \text{instead of} \quad \frac{\cos(90°\cos\theta)}{\sin\theta}$$

For horizontal $\lambda/2$ radiators along the Y directions, we must use

$$\frac{\cos[90°\cos(90-\varphi)]}{\sin(90-\varphi)} = \frac{\cos(90°\sin\varphi)}{\cos\varphi} \quad \text{instead of} \quad \frac{\cos(90°\cos\theta)}{\sin\theta}$$

119. Radiation Formulas for Linear Curtain Radiators. These formulas are simplified versions of formula (823), since for a curtain, *i.e.*, a plane, we have only two dimensions instead of three space dimensions. We then have to put any one of the three component D_x, D_y, and D_z group characteristics of D_g equal to unity. The heavily drawn n_x planes of Fig. 128 are curtains. Let us take any one of these curtains. We note that since we deal with a plane normal to the X axis, there can be only z and y separations between respective consecutive individual radiation centers. We then find $D_g = D_yD_z$, since $D_x = 1$, and

$$b = 180°\frac{y}{\lambda}\sin\varphi\sin\theta + 180°\frac{\tau_y}{T}; \qquad c = 180°\frac{z}{\lambda}\cos\theta + 180°\frac{\tau_z}{T};$$

$$D_y = \frac{\sin(n_yb)}{\sin(b)}; \qquad D_z = \frac{\sin(n_zc)}{\sin(c)}; \qquad \mathcal{E}_I = 60\frac{I_e}{r}D_yD_zD_0 \quad \text{volts/m} \quad (824)$$

where the directivity D_0 depends upon the *orientation* of the individual half-wave radiators, since

$$\left.\begin{aligned} D_0 &= \frac{\cos(90°\cos\theta)}{\sin\theta} && \text{for individual radiators } \left(l = \frac{\lambda}{2}\right) \text{ parallel with } Z \text{ axis} \\ D_0 &= \frac{\cos(90°\cos\varphi)}{\sin\varphi} && \text{parallel with } X \text{ axis} \\ D_0 &= \frac{\cos(90°\sin\varphi)}{\cos\varphi} && \text{parallel with } Y \text{ axis} \end{aligned}\right\} \quad (824a)$$

In exactly the same way, we can find the formulas for an n_y and an n_z curtain, respectively. Nothing is gained by doing this, since we can always turn our XYZ system in such a way that the individual parallel radiators are parallel to the Z axis and (824) applies with the customary D_θ formula for D_0 [first of Eqs. (824*a*)].

120. Application of the General Space Lattice Radiation Formula to Linear Antenna Alignments. This represents an additional simplification made for the formulas of Sec. 119. The individual radiation centers are then no longer in a curtain (a plane) but along a definite direction. For collinear individual half-wave antennas, we rotate the XYZ system into such an orientation that the collinear direction coincides with the Z axis, so that the $D_0 = D_\theta$ directivity formula for the individual half-

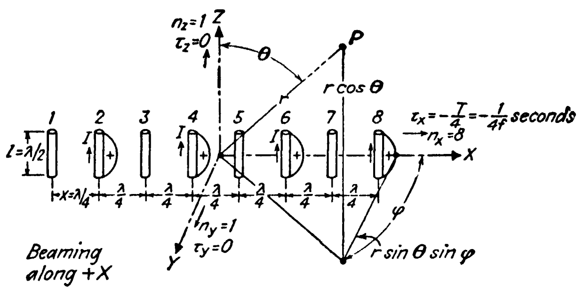


FIG. 129. Unilateral beaming by proper phasing of odd radiators 1, 3, 5, and 7 with respect to even half-wave radiators 2, 4, 6, and 8. Maximum current effect in even radiators such as 2, 4, 6, and 8 at times when *no* currents in 1, 3, 5, and 7.

wave radiator can be used. The group characteristic then degenerates to $D_g = D_z$, since $D_x = 1$ and $D_y = 1$. Hence,

$$c = 180°\frac{z}{\lambda}\cos\theta + 180°\frac{\tau_z}{T}; \qquad D_g = D_z = \frac{\sin(n_z c)}{\sin(c)};$$
$$\mathcal{E}_P = 60\frac{I_c}{r}\frac{\cos(90°\cos\theta)}{\sin\theta}D_z \qquad \text{volts/m} \qquad (825)$$

This is exactly the same result as found in (821), since c is the same quantity as a in (821) and z denotes the spacing s between consecutive individual radiation centers; n_z represents the number of half-wave radiators. For the alignment of parallel $\lambda/2$ radiators, we choose conveniently the alignment along the X axis. Then $D_0 = D_\theta$, $D_g = D_x$, $D_y = 1 = D_z$, and

$$a = 180°\frac{x}{\lambda}\cos\varphi\sin\theta + 180°\frac{\tau_x}{T}$$
$$\mathcal{E}_P = 60\frac{I_c}{r}\frac{\cos(90°\cos\theta)}{\sin\theta}D_g \quad \text{volts/m}; \qquad D_g = D_x = \frac{\sin(n_x a)}{\sin(a)} \qquad (826)$$

which checks the result of (820), since $x = s$ and $\tau_s = \tau$. A numerical example is then as follows:

Suppose we have an alignment as in Fig. 129 along the X axis, where we have a fixed spacing of $x = \frac{1}{4}\lambda$ and such phasing that all inphase odd radiators 1, 3, 5, and 7 are 90 time degrees ahead of the inphase even radiators 2, 4, 6, and 8. We have then

$$\tau = -\frac{T}{4} = -\frac{1}{4f}; \qquad x = \frac{\lambda}{4}; \qquad a = \frac{180°}{4}\cos\varphi\sin\theta - \frac{180°}{4} = 45°\cos\varphi\sin\theta - 45°$$

for any space point φ, θ; and $a = 45°\cos\varphi\sin 90° - 45° = 45°\cos\varphi - 45°$ for any point φ, $\theta = 90°$, *i.e.*, in the equatorial XY plane. We have also $n_x = 8$.

$$D_x = \frac{\sin(8 \times 45°\sin\theta\cos\varphi - 8 \times 45°)}{\sin(45°\sin\theta\cos\varphi - 45°)} = \frac{\sin(360°\sin\theta\cos\varphi - 360°)}{\sin(45°\sin\theta\cos\varphi - 45°)}$$

Since there is no alignment along the Y or along the Z direction, we have $n_y = 1$ and $n_z = 1$ (always one radiator for no alignment, since any radiator along the X direction represents also radiator 1 along the Y direction and radiator 1 along the Z direction). Therefore,

$$D_y = \frac{\sin(n_y b)}{\sin(b)} = \frac{\sin(b)}{\sin(b)} = 1; \quad D_z = \frac{\sin(n_z c)}{\sin(c)} = \frac{\sin(c)}{\sin(c)} = 1; \quad D_g = D_x D_y D_z = D_x$$

$$D_{\theta,\varphi} = D_\theta D_g = D_\theta D_x = \frac{\cos(90°\cos\theta)}{\sin\theta}\,\frac{\sin(360°\sin\theta\cos\varphi - 360°)}{\sin(45°\sin\theta\cos\varphi - 45°)}$$

$$\mathcal{E}_\theta = 60\frac{I_e}{r}D_{\theta,\varphi} \quad \text{volts/m;} \qquad H_\varphi = \frac{\mathcal{E}_\theta}{376.7} \quad \text{amp/m at any field point } r, \varphi, \theta$$

For the polar pattern in the equatorial XY plane, we have $\theta = 90°$, and

$$D_{\varphi,\theta=90°} = D_\varphi = \frac{\cos(90°\cos 90°)}{\sin 90°}\,\frac{\sin(360°\sin 90°\cos\varphi - 360°)}{\sin(45°\sin 90°\cos\varphi - 45°)}$$
$$= \frac{\sin(360°\cos\varphi - 360°)}{\sin(45°\cos\varphi - 45°)}$$

Along the *positive* X axis, we have $\varphi = 0°$, and

$$D_{\varphi=0°} = \frac{\sin(360° - 360°)}{\sin(45° - 45°)} = \frac{0}{0}$$

which is indeterminate. This expression can, however, be transformed as follows:

$$45°\cos\varphi - 45° = \alpha; \qquad 360°\cos\varphi - 360° = 8\alpha; \qquad D_\varphi = \frac{\sin(8\alpha)}{\sin(\alpha)}$$
$$= \frac{2\sin(4\alpha)\cos(4\alpha)}{\sin(\alpha)} = \frac{2 \times 2\sin(2\alpha)\cos(2\alpha)\cos(4\alpha)}{\sin(\alpha)}$$
$$= \frac{2 \times 2 \times 2\sin(\alpha)\cos(\alpha)\cos(2\alpha)\cos(4\alpha)}{\sin(\alpha)} = 8\cos(\alpha)\cos(2\alpha)\cos(4\alpha)$$
$$D_{\varphi=0°} = 8\underbrace{\cos(45° - 45°)}_{=1}\underbrace{\cos(90° - 90°)}_{=1}\underbrace{\cos(180° - 180°)}_{=1} = +8$$

which is now determinate. We have, therefore, eightfold maximum effect along the $+X$ direction. This does not necessarily mean that we have also eightfold amplitude effect along the $-X$ direction, which has an angle $\varphi = 180$ deg. It yields zero field effect, since

$$D_{\varphi=180°} = 8 \cos \underbrace{\underbrace{(-45° - 45°)}_{-90°}}_{=0} \underbrace{\cos (-180°)}_{=-1} \underbrace{\cos (-360°)}_{=+1} = 0$$

This result shows that we have *unilateral beaming* along the $+X$ direction, known as "end firing." We have no radiation along the $\pm Y$ directions, since

$$D_{\varphi=90°} = 8 \cos (\underbrace{45° \cos 90°}_{=0°} - 45°) \underbrace{\cos (\underbrace{90° \cos 90°}_{=0°} - 90°)}_{=0} \times \cos (\underbrace{180° \cos 90°}_{=0°} - 180°) = 0$$

$$D_{\varphi=270°} = 8 \cos (\underbrace{45° \cos 270°}_{=0°} - 45°) \underbrace{\cos (\underbrace{90° \cos 270°}_{=0°} - 90°)}_{=0} \times \cos (\underbrace{180° \cos 270°}_{=0°} - 180°) = 0$$

In the same way, we can find all other equatorial D_φ directivities and find a pattern as shown in Fig. 123*d*. We note from this pattern that we have only to investigate the meridian pattern in the XZ plane, since there can be no radiation effects along the $\pm Y$ and $-X$ directions. Therefore, the meridian XZ pattern gives us all the information about the broadness of the vertical pattern. Such information is important for some applications. For instance, for communication along the horizon, as from steamer to steamer, we are concerned only with a well-defined and suitable D_φ pattern. We may want to have this pattern reasonably sharp, so that we have a large radiation gain and do not interfere with other surface craft. On the other hand, we may want a broader vertical pattern in the optimum vertical surface region (the XZ region), for the sake of contacting any aircraft at different altitudes. For the significant XZ pattern, we have the following details:

In the XZ plane, $\varphi = \theta$ degrees; and for $\lambda/2$ radiator elements,

$$D_{\theta\,\varphi=0°} = \frac{\cos (90° \cos \theta)}{\sin \theta} \frac{\sin (360° - 360°)}{\sin (45° - 45°)}$$

$$= 8 \frac{\cos (90° \cos \theta)}{\sin \theta} \underbrace{\cos (45° - 45°)}_{=1} \underbrace{\cos (90° - 90°)}_{=1}$$

This outcome shows that we have to consider only the fixed meridian D_θ pattern of one individual radiator making up the alignment. This pattern is shown on page 534. It is fairly broad, as a matter of fact, almost as good as the $\sin\theta$ pattern holding for an elementary linear radiator, and is the best condition we can ever obtain for a linear radiator. If a sharper vertical pattern is desired, we have to go to a curtain system.

121. Notes on Numerical Data on Curtain Half-wave Radiator Arrays. A curtain radiator is a means by which we can control and adjust the beam concentration in the equatorial XY plane and any of the meridian planes. The linear alignment can sharpen the beam width only in one plane direction unless we use individual radiators longer than a half wavelength and convenient equal phasing. But such a remedy is also, in effect, a curtain array, since, for instance, for two equal-phased half-wave sections, we have the number of half-wave radiators equal to 2 for each individual radiator that are aligned and require at least two such radiators. Hence, two such radiators along the alignment direction and in parallel and two half-wave sections per individual radiator (perpendicular thereto) form a plane or a curtain array.

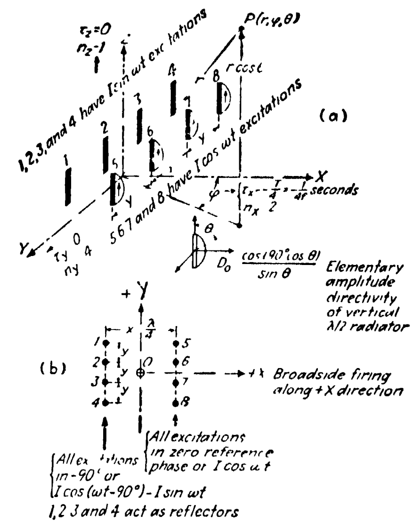


FIG. 130. Conditions for unilateral broadside beaming of an array.

Figure 130*b* shows the radiation centers 1, 2, 3, 4, 5, 6, 7, and 8 of the individual half-wave radiators in the XY plane. As Fig. 130*a* shows, we have vertical half-wave radiators parallel with the Z direction of an XYZ system. Therefore, the equatorial XY plane 1-4-8-5-1 represents a curtain. According to Fig. 130, we have $n_x = 2$, since there are only two parallel radiator rows 1, 2, 3, 4 and 5, 6, 7, 8, respectively, along the X direction. We have $n_y = 4$, since in each row that is along the Y direction, we have four radiators. Since the curtain is in the XY plane and no other curtain planes are parallel thereto, the XY curtain is the only radiation-center plane perpendicular to the Z direction and n_z must be unity.

According to formulas (822*b*), we find $x/\lambda = \frac{1}{4}$, $\tau_x = -\frac{1}{4}T$ or $\tau_x/T = -\frac{1}{4}$, $\tau_y/T = 1$ for each row and y/λ, for the time being any value; $n_x = 2$, $n_y = 4$, and $n_z = 1$, yielding

$$a = 180° \frac{x}{\lambda} \sin \theta \cos \varphi + 180° \frac{\tau_x}{T} = \frac{180°}{4} \frac{x}{\lambda} \sin \theta \cos \varphi + \left(-\frac{180°}{4}\right)$$
$$= 45° (\sin \theta \cos \varphi - 1)$$

$$b = \underbrace{180° \frac{y}{\lambda} \sin \theta \sin \varphi}_{=0 \text{ since } z = 0} + 180° \frac{\tau_y}{T} = 180° \frac{y}{\lambda} \sin \theta \sin \varphi + 180°$$

$$c = 180° \frac{z}{\lambda} \cos \theta + 180° \frac{\tau_z}{T} = 180°$$

$$D_{\varphi,\theta} = D_g D_0 = D_x D_y D_z D_\theta = \frac{\sin(n_x a)}{\sin(a)} \frac{\sin(n_y b)}{\sin(b)} \frac{\sin(n_z c)}{\sin(c)} D_\theta$$
$$= \underbrace{\frac{\sin(2a)}{\sin(a)}}_{D_x} \underbrace{\frac{\sin(4b)}{\sin(b)}}_{D_y} \underbrace{\frac{\sin(c)}{\sin(c)}}_{=1} D_\theta$$

$$D_\theta = \frac{\cos(90° \cos \theta)}{\sin \theta}$$

$$\mathcal{E}_P = 60 \frac{I_e}{r} D_x D_y D_\theta \qquad \text{volts/m;}$$

$$H_P = \frac{\mathcal{E}_P}{376.7} \qquad \text{amp/m, at any point } P(r, \varphi, \theta) \text{ for } \mathcal{E}_P = \mathcal{E}_\theta \text{ and } H_P = H_\varphi$$

Let us first investigate again the equatorial directivity pattern D_φ. We have, in the equatorial XY plane, $\theta = 90°$, $\sin 90° = 1$, $\cos 90° = 0$, $D_\theta = 1$, and

$$D_{\varphi,\theta=90°} = D_\varphi = \frac{\sin(2a)}{\sin(a)} \frac{\sin(4b)}{\sin(b)}; \qquad \frac{\sin(2a)}{\sin(a)} = \frac{2 \sin a \cos a}{\sin a} = 2 \cos a$$

$$\frac{\sin(4b)}{\sin(b)} = \frac{2 \sin(2b) \cos(2b)}{\sin(b)} = \frac{2 \times 2 \sin(b) \cos(b) \cos(2b)}{\sin(b)} = 4 \cos(b) \cos(2b)$$

$$D_\varphi = 8 \cos a \cos b \cos(2b)$$
$$= 8 \cos(45° \underbrace{\sin 90°}_{=1} \cos \varphi - 45°) \cos\left(180° \frac{y}{\lambda} \sin 90° \sin \varphi + 180°\right)$$
$$\times \cos\left(360° \frac{y}{\lambda} \sin \varphi + 360°\right);$$

$$\cos(45° \cos \varphi - 45°) = \cos[-(45° - 45° \cos \varphi)] = \cos(45° - 45° \cos \varphi);$$

$$\cos\left(360° + 360° \frac{y}{\lambda} \sin \varphi\right) = \cos\left(360° \frac{y}{\lambda} \sin \varphi\right);$$

$$\cos\left(180° + 180° \frac{y}{\lambda} \sin \varphi\right) = -\cos\left(180° \frac{y}{\lambda} \sin \varphi\right)$$

The directivity in the equatorial XY plane then becomes

$$D_\varphi = -8 \cos(45° - 45° \cos \varphi) \cos\left(180° \frac{y}{\lambda} \sin \varphi\right) \cos\left(360° \frac{y}{\lambda} \sin \varphi\right)$$

On account of the factor $\cos(45° - 45° \cos\varphi)$, *no* radiation will take place in the $-X$ direction, since for such a direction $\varphi = 180$ deg and $\cos(45° - 45° \underbrace{\cos 180}_{=-1})$ $= \cos 90°$ yields this factor equal to zero. For the $+X$ direction ($\varphi = 0$), we have

$$D_\varphi = -8 \cos[45° - 45° \cos(0°)] \cos\left(180° \frac{y}{\lambda} \sin 0°\right) \cos\left(360° \frac{y}{\lambda} \sin 0°\right) = -8$$

This outcome shows that we have broadside beaming along the $+X$ axis. This is also evident from the current excitations. We note in Fig. 130*b* that the front row with inphase radiators 5, 6, 7, and 8 is leading with a time phase of 90 deg while the rear row with inphase radiators 1, 2, 3, and 4 are all 90 time degrees behind the front-row excitations. But it takes the fields from the rear row $T/4$ second to reach the front row, since they are $\frac{1}{4}\lambda$ apart. Hence, the retarded fields from the rear row are at the front row in time phase with the fields just starting out at the front row and cause an additive effect or a forward beaming along the $+X$ axis. We have, therefore, maximum beaming also in the vertical ZX plane along the positive X direction. For the meridian ZX plane, we find $\varphi = 0$ along $+X$ so $\varphi = 0$, and $\cos\varphi = 1$. Hence,

$$D_{\theta,\varphi=0} = \underbrace{D_\theta\, 2\cos a}_{D_x}\; \underbrace{4 \cos b \cos 2b}_{D_y}$$

$$= -8 \frac{\cos(90° \cos\theta)}{\sin\theta} \cos(45° - 45° \underbrace{\sin\theta \cos 0°}_{\sin\theta}) \underbrace{\cos\left(180 \frac{y}{\lambda} \sin\theta \cdot \overbrace{\sin 0}^{=0}\right)}_{\text{hence } 1} \times \underbrace{\cos\left(360 \frac{y}{\lambda} \sin\theta \sin 0°\right)}_{=1}$$

Therefore,

$$D_{\theta,\varphi=0°} = -8 \frac{\cos(90° \cos\theta)}{\sin\theta} \cos[45°(1 - \sin\theta)]$$

From these expressions, we note that for the phasing chosen, we cause only unilateral broadside beaming and the horizontal beam width can be adjusted by means of a suitable y/λ ratio, which is the spacing y in wavelengths. The vertical pattern is again principally affected by the D_θ characteristic of each individual radiator, but modified by the value $\cos a \cos b \cos 2b$. This modification has the forward "thrust" term $\cos(a) = \cos[45° (\sin\theta \cos\varphi - 1)]$, while the other two factors $\cos(b)$ and $\cos(2b)$ have to do only with the horizontal width of the forward beam along the $+X$ direction. This outcome can be noted from Fig. 130, since the alignment along the vertical Z axis consists only of one radiation center per row ($n_z = 1$), and we must expect principally the broadness of the D_θ characteristic in the vertical meridian

planes. It takes more than one radiation center along the Z directions to cause vertical radiation concentration, which is narrower than predicted from the individual D_θ pattern. To do this, we have to use a vertical line-up as in Fig. 131, where we have again a front row that is now vertical as far as the radiation centers are concerned and a vertical rear row that is excited 90 time degrees, lagging with respect to the half-

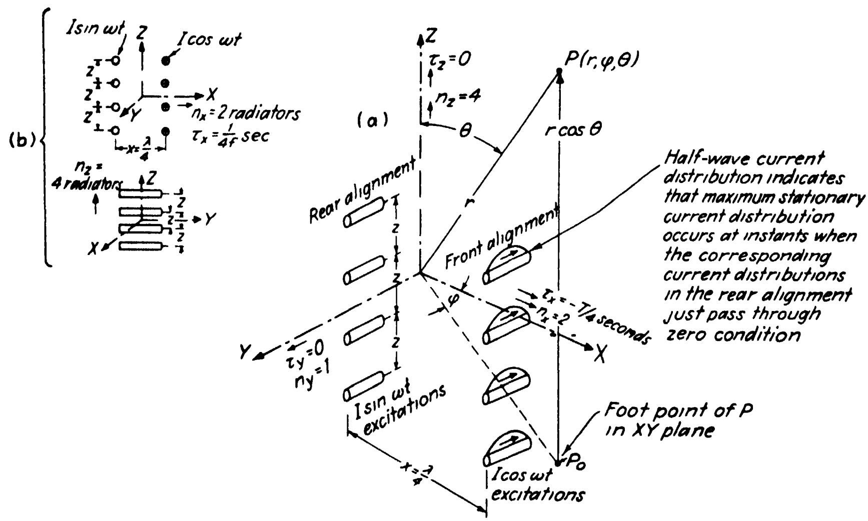


FIG. 131. A front alignment and a backup alignment of half-wave radiators with respective alignments along the Z direction.

wave antenna currents of the front row. We have now $n_z = 4$, $n_y = 1$, and $n_x = 2$ with the following details:

According to formulas (822b), we find

$$\frac{\tau_x}{T} = f\tau_x = -\frac{1}{4}; \qquad \frac{\tau_y}{T} = k \qquad \text{for } k = 0, 1, 2, 3, 4, \cdots$$

where $k = 0$ strictly applies, even though $k = 1, 2, 3, 4, 5, \cdots$ gives the same results, since a vector rotation, in a complex plane, of 360,2 × 360,3 × 360 ⋯ deg coincides with 0 time degrees, which is an inphase condition. We also find in Fig. 131

$$D_{\theta,\varphi} = \underbrace{\frac{\sin(2a)}{\sin(a)}}_{\substack{\text{since}\\ n_x = 2}} \underbrace{\frac{\sin(b)}{\sin(b)}}_{\substack{\text{since}\\ n_y = 1}} \underbrace{\frac{\sin(4c)}{\sin(c)}}_{\substack{\text{since}\\ n_z = 4}} D_0; \qquad D_0 = \underbrace{\frac{\cos[90^\circ \cos(90^\circ - \varphi)]}{\sin(90^\circ - \varphi)}}_{\substack{\text{since each radiator along}\\ \text{the } Y \text{ direction}}} = \frac{\cos(90^\circ \sin\varphi)}{\cos\varphi}$$

$$D_{\theta,\varphi} = D_0 \frac{\sin(2a)}{\sin(a)} \frac{\sin(4c)}{\sin(c)} = 8D_0 \cos a \cos c \cos(2c)$$

$$a = \tfrac{1}{4}180^\circ \sin\theta \cos\varphi - \tfrac{1}{4}180^\circ = 45^\circ \sin\theta \cos\varphi - 45^\circ$$

$$c = 180° \frac{z}{\lambda} \cos\theta + 180° \times 0 = 180° \frac{z}{\lambda} \cos\theta; \quad \cos a = \cos\left[-(45° - 45° \sin\theta\cos\varphi)\right]$$

$$\cos a = \cos(45° - 45° \sin\theta\cos\varphi)$$

$$\cos(c) = \cos\left(180° \frac{z}{\lambda} \cos\theta\right); \qquad \cos(2c) = \cos\left(360° \frac{z}{\lambda} \cos\theta\right)$$

$$D_{\theta,\varphi} = 8 \frac{\cos(90° \sin\varphi)}{\cos\varphi} \cos(45° - 45° \sin\theta\cos\varphi) \cos\left(180° \frac{z}{\lambda} \cos\theta\right) \times \cos\left(360° \frac{z}{\lambda} \cos\theta\right)$$

$$\mathcal{E}_P^{(\text{volts/m})} = 60 \frac{I_e}{r} D_{\theta,\varphi}; \qquad H_P^{(\text{amp/m})} = \frac{\mathcal{E}_P}{376.7} \qquad \text{for effective values}$$

In the equatorial XY plane, we have $\theta = 90$ deg, $\sin\theta = 1$, $\cos\theta = 0$, and

$$D_{\varphi,\theta} = 8 \frac{\cos(90° \sin\varphi)}{\cos\varphi} \cos(45° - 45° \sin 90° \cos\varphi) \times \cos\left(180° \frac{z}{\lambda} \cos 90°\right) \cos\left(360° \frac{z}{\lambda} \cos 90°\right)$$

$$D_{\varphi,\theta=90°} = D_\varphi$$

$$D_\varphi = 8 \frac{\cos(90° \sin\varphi)}{\cos\varphi} \cos(45° - 45° \cos\varphi) \qquad \text{for } \varphi = 0° \text{ (along } +X \text{ direction)}$$

$$D_{\varphi=0°} = 8 \frac{\cos(90° \sin 0°)}{\cos 0°} \cos(45° - 45° \cos 0°) = 8$$

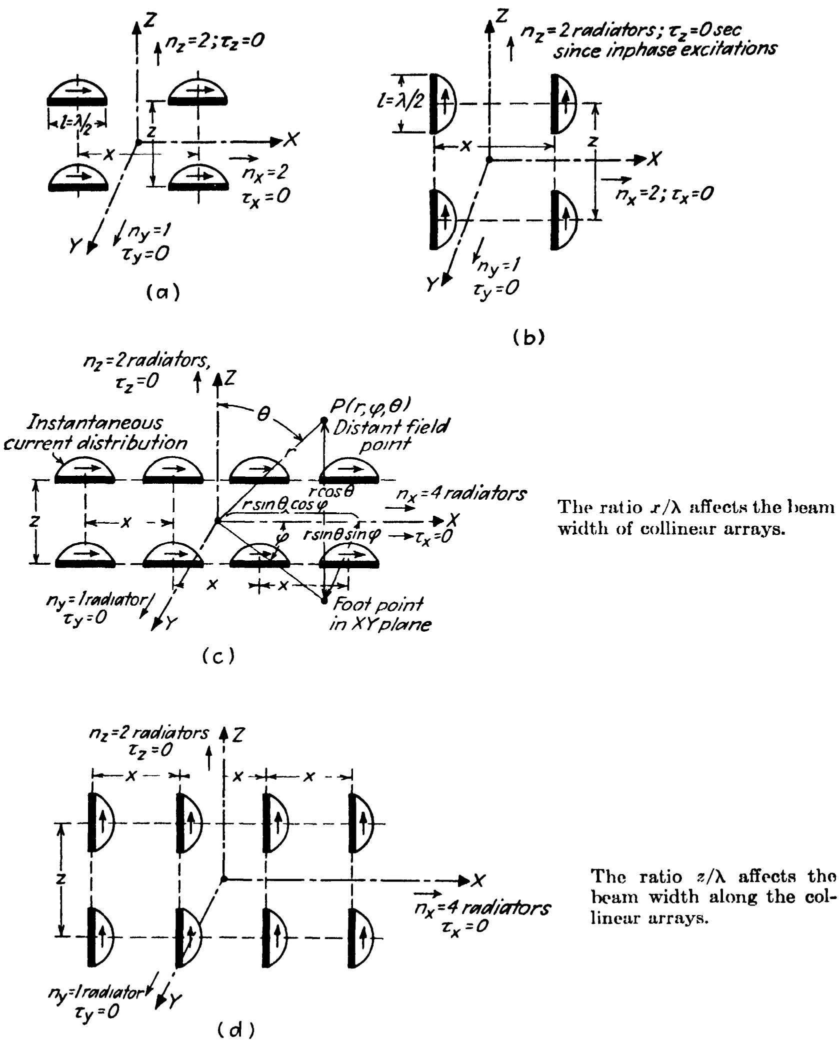
For $\varphi = 180°$ (along the $-X$ direction),

$$D_{\varphi=180°} = \frac{\cos(90° \sin 180°)}{\cos 180°} \cos(45° - 45° \underbrace{\cos 180°}_{=-1}) = 0$$

Along the $\pm Y$ directions, $\varphi = 90$ and 270 deg, $\sin 90° = 1$, $\sin 270° = -1$, rendering, in each case, the numerator $\cos(90° \sin\varphi) = 0$, since it is equal to $\cos(\pm 90°)$. Hence, we have no radiation effect at all along the $\pm Y$ directions and along the $-X$ direction. This is obvious, since the individual half-wave radiators are along $\pm Y$ directions, in effect, and the phasing indicated in Fig. 131 causes at the front radiator plane an inphase effect due to the rear radiators, *i.e.*, a forward field thrust along the $+X$ axis, *at the expense of no* field along the $-X$ direction. It is, therefore, necessary only to examine the vertical ZX plane along the $+X$ direction. We find then for $\varphi = 0$ deg

$$D_{\theta,\varphi=0} = \underbrace{8 \frac{\cos(90° \sin 0°)}{\cos 0°}}_{=8} \cos(45° - 45° \sin\theta\cos 0°) \cos\left(180° \frac{z}{\lambda} \cos\theta\right)$$

$$\times \cos\left(360° \frac{z}{\lambda} \cos\theta\right) = 8 \underbrace{\cos(45° - 45° \sin\theta)}_{\text{forward thrust factor}} \underbrace{\cos\left(180° \frac{z}{\lambda} \cos\theta\right) \cos\left(360° \frac{z}{\lambda} \cos\theta\right)}_{\text{determines broadness of vertical pattern}}$$



The ratio x/λ affects the beam width of collinear arrays.

The ratio z/λ affects the beam width along the collinear arrays.

FIG. 132. Half-wave curtain arrays driven inphase and for different beam widths in the vertical and horizontal planes. We have to deal with a horizontal as well as a vertical beam width in such curtain arrays.

We note that the curtain array of Fig. 131 is then the complementary case to the curtain array of Fig. 130, since for Fig. 131 we have now a means of changing the width of the vertical beam by changing the z/λ ratio just as in Fig. 130 we have a means of changing the horizontal beam width by changing the ratio y/λ. It must be realized that in both figures, the curtain is used for *unilateral* broadside beaming and,

therefore, only an alignment either for horizontal or for vertical beam sharpening is available in the respective figures. If, however, the curtain is used in such a way that we have, as in Fig. 132*a*, at least two horizontal rows, with two collinear half-wave inphase radiators or, as in Fig. 132*b*, at least two vertical columns with two collinear half-wave inphase radiators, we can adjust the beam width readily by appropriate x/λ and z/λ ratios. It does not take many more half-wave radiators to produce satisfactory beaming. For instance, Fig. 132*c* and *d* shows useful curtains. Since we are usually not interested in backward beaming, we need only a reflector curtain behind the radiator curtain. It is also customary to locate respective collinear half-wave radiators along the focal axes of parabolic cylinders in order to obtain a most satisfactory forward radiation. We obtain then, in effect, in the aperture plane a most ideal curtain, as is taken up under diffraction.

122. Directivity of Antenna Arrays with Unequal Spacings between Consecutive Individual Antennas. Such irregular spacings are normally not used; but for the sake of generality, the derivation is given in connection with the symmetrical radiation centers 1 and 2, 3 and 4, 5 and 6, and 7 and 8 of Fig. 133. We have then the unequal separations s_1 of radiation centers 1 and 2, s_2 of centers 3 and 4, s_3 of couples 5 and 6, and s_4 of couples 7 and 8, respectively. The consecutive spacings between 7 and 5, 5 and 3, 3 and 1, and 1 and 2 are then generally unequal. Inasmuch as the respective spacings s_1, s_2, s_3, and s_4 are symmetrical with respect to the origin O of the XYZ system, we can treat the effects for all radiation center couples by using the law of superposition. Therefore, if *in*phase as well as *anti*phase excitations of the *same* amplitude exist for all couples, we must have the resultant effect $D_\theta = D$.

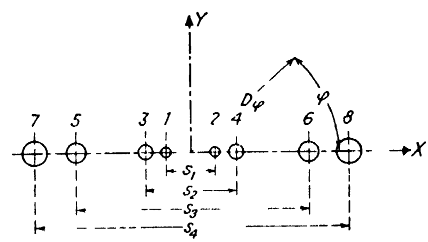


FIG. 133. Directivity for an alignment with unequal spacings $(s_4 - s_3)/2$, $(s_3 - s_2)/2$; $(s_2 - s_1)/2$; and s_1.

$$
\left.
\begin{aligned}
D &= \cos\left(180^\circ \frac{s_1}{\lambda} \underbrace{\sin\theta\cos\varphi}_{q}\right) + \cos\left(180^\circ \frac{s_2}{\lambda} q\right) \\
&\quad + \cos\left(180^\circ \frac{s_3}{\lambda} q\right) + \cos\left(180^\circ \frac{s_4}{\lambda} q\right) \qquad \text{for inphase currents} \\
D &= \cos\left(180^\circ \frac{s_1}{\lambda} q\right) + \sin\left(180^\circ \frac{s_2}{\lambda} q\right) + \cos\left(180^\circ \frac{s_3}{\lambda} q\right) \\
&\quad + \sin\left(180^\circ \frac{s_4}{\lambda} q\right) \qquad \text{for alternate and antiphase currents}
\end{aligned}
\right\} \quad (827)
$$

The most general case exists for exciter currents of different amplitudes and any phasing. Therefore, when the s_1 couple of radiators has a time difference of τ_1 seconds, the s_2 couple of τ_2 seconds, the s_3 couple of τ_3 seconds, and the s_4 couple of τ_4 seconds, we have, for the amplitudes of the successive s_1, s_2, s_3, and s_4 couples in the proportion 1:3:4:6, the group characteristic

$$\left.\begin{aligned} D_g = \cos\left[180°\left(\frac{s_1}{\lambda}q + \frac{\tau_1}{T}\right)\right] + 3\cos\left[180°\left(\frac{s_2}{\lambda}q + \frac{\tau_2}{T}\right)\right] \\ + 4\cos\left[180°\left(\frac{s_3}{\lambda}q + \frac{\tau_3}{T}\right)\right] + 6\cos\left[180°\left(\frac{s_4}{\lambda}q + \frac{\tau_4}{T}\right)\right] \end{aligned}\right\} \quad (827a)$$

This relation can be applied to any of the previous sections, where it was always assumed that the current amplitudes of all individual radiators were the same at the current antinode. It is then necessary only to choose $s_2 = 2s_1$, $s_3 = 3s_1$, and $s_4 = 4s_1$. We note from this expression that we can phase by spacing, by timing, or by both. We can also change the pattern by using different amplitudes. The above D_g characteristic is then to be multiplied by the meridian conductivity of the individual radiator, which is

$$\left.\begin{aligned} D_0 &= \frac{\cos(90° \cos\theta)}{\sin\theta} && \text{for } \frac{\lambda}{2} \text{ radiators parallel with } Z \text{ axis} \\ D_0 &= \frac{\cos(90° \cos\varphi)}{\sin\varphi} && \text{for radiators parallel with } X \text{ axis} \\ D_0 &= \frac{\cos(90° \sin\varphi)}{\cos\varphi} && \text{for radiators parallel with } Y \text{ axis} \end{aligned}\right\} \quad (827b)$$

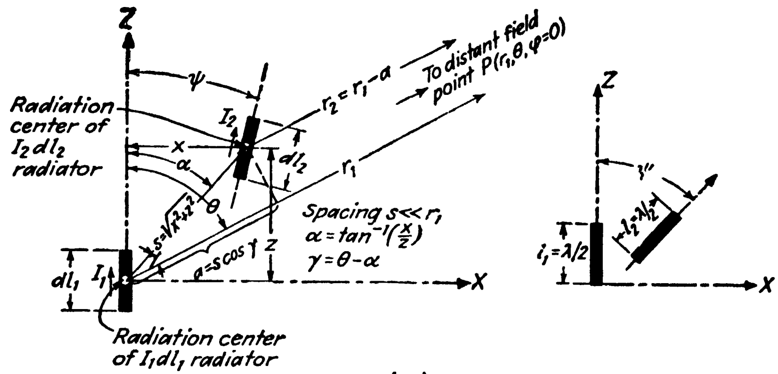
We find then the effective values of the electric and magnetic field intensities at a space point $P(r, \varphi, \theta)$, as

$$\left.\begin{aligned} \mathcal{E}_\theta^{(\text{volts/m})} = 60\frac{I_e}{r}D_0\left\{\cos\left[180°\left(\frac{s_1}{\lambda}q + \frac{\tau_1}{T}\right)\right]\right. \\ + k_1\cos\left[180°\left(\frac{s_2}{\lambda}q + \frac{\tau_2}{T}\right)\right] + k_2\cos\left[180°\left(\frac{s_3}{\lambda}q + \frac{\tau_3}{T}\right)\right] \\ \left. + k_3\cos\left[180°\left(\frac{s_4}{\lambda}q + \frac{\tau_4}{T}\right)\right]\right\} \\ H_\varphi^{(\text{amp/m})} = \frac{\mathcal{E}_\theta}{376.7} \end{aligned}\right\} \quad (827c)$$

The quantity I_e then denotes the effective current value at the current antinode of the first half-wave radiator couple separated a distance of s_1

meters. The consecutive s_1, s_2, s_3, and s_4 half-wavelength couples are then in the proportion of $I_e:k_1:k_2:k_3$.

123. Remarks on Linear Radiator Elements That Are Not Parallel and Not Collinear. This has reference to radiator elements as are shown



(a)

$$\underbrace{r = \sqrt{x^2 + y^2 + z^2}}_{\text{Cartesian}} = \underbrace{\sqrt{(r \sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 + (r\cos\theta)^2}}_{\text{spherical}}$$

$$dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \text{ in Cartesian coordinates}$$

$$dl = \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\varphi)^2} \text{ in spherical coordinates}$$

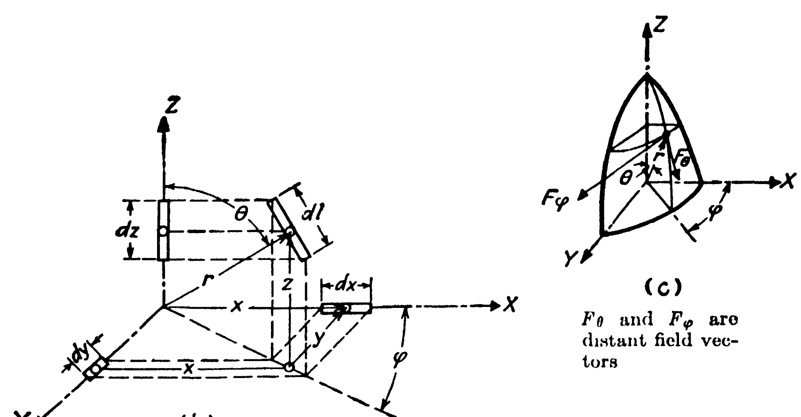


FIG. 134. Relations for a differential radiator length dl in the general space orientation.

in Fig. 134a, where the I_1dl_1 radiator element has a different direction from the I_2dl_2 radiator element and also the condition that we generally have unequal electric current moments I_1dl_1 and I_2dl_2. If ψ denotes the angle between both directions, we can find the resultant field in the

distant radiation region by the law of superposition, as was done in Sec. 122. We know from Table 2 on page 163 that the respective instantaneous field contributions of an I_1dl_1 elementary radiator, at a distant field point $P(r, \varphi, \theta)$, are

$$d\mathcal{E}_\theta = A_1\epsilon^{j(\omega t-\beta r_1)} \text{ volts/m}; \qquad dH_\varphi = \frac{d\mathcal{E}_\theta}{376.7} \text{ amp/m};$$

$$A_1 = j\,\frac{188.35 \sin\theta}{r_1\lambda}\,I_1dl_1$$

The function $\sin\theta$ represents then the meridian directivity D_θ of the elementary linear $I_1\,dl_1$ radiator. For the sake of simplicity, the radiation center of this radiator coincides with the origin of the XYZ system. We note that the other elementary radiator I_2dl_2 has its radiation center a certain distance s from the radiation center of the first radiator and makes an angle ψ with the positive direction of it. As far as the inverse distance effect is concerned, we can put $r_2 \cong r_1$, but we must use the electrical angle βr_2, as far as the phase delay due to the path to the field point is concerned. As far as the field contribution at P due to this radiator alone is concerned, the direction r_2 with the dl_2 length now makes an angle $\theta - \psi$, and we have only a directivity $D_\theta = \sin(\theta - \psi)$ from the $I_2\,dl_2$ radiator in the r_2 direction and the expressions

$$(d\mathcal{E}_\theta)_2 = A_2\epsilon^{j(\omega t-\beta r_2)} \text{ volts/m}; \qquad (dH_\varphi)_2 = \frac{(d\mathcal{E}_\theta)_2}{376.7} \text{ amp/m}$$

$$A_2 = j\,\frac{188.35 \sin(\theta - \psi)}{r_1\lambda}\,I_2dl_2$$

where $(\quad)_2$ states that it is the field due to the second elementary radiator.

As far as the radiation centers are concerned, instead of a line-up along the X, the Y, or the Z direction, we have a line-up along the s direction, besides the condition that the radiators are neither parallel nor collinear. This condition is taken care of by the $\sin(\theta - \psi)$ factor for one contribution and the $\sin\theta$ factor for the other contribution. The resultant field contribution at the distant field point is then the superposition of the respective field contributions. In the same way, we can proceed when dealing with two linear half-wave radiators, except that all the I_1dl_1 elements are integrated over the finite length l_1 and all the I_2dl_2 effects over the length l_2. In connection with the discussions of Secs. 38 and 39, we found that we can also use the emf method (pages 18–190, 192, 235, 515–516) to account for the radiation effects. In the sections referred to, relations are given for parallel elementary current elements. For the current elements of Fig. 134*a*, we can use the same procedure, since the $d\mathcal{E}_1$ radiation field effect of the I_1dl_1 radiator induces a voltage $d\mathcal{E}_1dl_2 \cos\psi$ volts in the dl_2 element, which must be compensated by the I_2dl_2 radiator by an impressed voltage of *minus* $d\mathcal{E}_1dl_2 \cos\psi$ volts, defining the *mutual* impedance $Z_{12} = -d\mathcal{E}_1dl_2 (\cos\psi)/I_1$ between respective dl_1 and dl_2 radiator elements. We have then

$$
\left.
\begin{aligned}
Z_{12}^{(\text{ohms})} &= \frac{d\mathcal{E}_1 dl_2 \cos\psi}{I_1} = \frac{1}{I_1} j\,\frac{188.35 \sin 90^\circ I_1 dl_1}{x\lambda}\, dl_2 \cos 0^\circ \overset{a}{\epsilon^{j(\omega t - \beta x)}} \\
&= j\,\frac{188.35\, dl_1 dl_2}{x\lambda}\,\epsilon^{ja} \quad \text{for two parallel } dl_1 \text{ and } dl_2 \text{ elements along the } X \text{ axis; } s = x \text{ m} \\
Z_{12}^{(\text{ohms})} &= \frac{d\mathcal{E}_\theta dl_2 \cos\psi}{I_1} = \frac{1}{I_1} j\,\frac{188.35 \sin\alpha I_1 dl_1}{s\lambda}\, dl_2 \cos\psi\, \overset{b}{\epsilon^{j(\omega t - \beta s)}} \\
&= j\,\frac{188.35\, dl_1 dl_2 \sin\alpha \cos\psi}{s\lambda}\,\epsilon^{jb} \quad \text{for Fig. 134}a
\end{aligned}
\right\} \quad (828)
$$

since for an alignment of two parallel elementary radiators along the X axis of Fig. 134a, we have $s = x$ meters because $z = 0$, $\sin\alpha = \sin 90^\circ = \sin\theta = 1$, $\cos\psi = \cos 0^\circ = 1$. According to pages 187–193, we have to deal with a self-impedance Z_{11} of radiator $I_1 dl_1$ and a mutual impedance Z_{21} due to radiator $I_2 dl_2$. In a similar way, Z_{22} is the self-impedance of radiator $I_2 dl_2$ and Z_{12} its mutual impedance due to radiator $I_1 dl_1$, defined by

$$
\left.
\begin{aligned}
Z_{11} &= \frac{dE_{11}}{I_1}; \qquad Z_{12} = \frac{d\mathcal{E}_1 dl_2 \cos\psi}{I_1} = \frac{dE_{12}}{I_1}; \qquad Z_{22} = \frac{dE_{22}}{I_2} \\
& \qquad\qquad\qquad\qquad\qquad\qquad Z_{21} = \frac{d\mathcal{E}_2 dl_1 \cos\psi}{I_2} = \frac{dE_{21}}{I_2} \\
dE_1 &= Z_{11} I_1 + Z_{12} I_2; \qquad dE_2 = Z_{22} I_2 + Z_{21} I_1
\end{aligned}
\right\} \quad (828a)
$$

where dE_1 and dE_2 are in volts and represent the impressed voltages of the respective elementary radiators. This is dealt with in detail in Secs. 130–139.

We need a much more general solution if we expect to cover all possible commercial radiation systems. Figure 134a refers only to current elements lying both in the ZX plane, *i.e.*, in a plane. We have to deal with radiation elements that may be oriented like the dl element of Fig. 134b, so that the effects of any number of such elements oriented in any geographical direction can be added up at the distant field point $P(r, \varphi, \theta)$ of Fig. 134c, with respect to amplitude and phase. This is then conveniently done by means of the concept of an electric radiation vector.

124. The Schelkunoff Radiation Vectors. Such vectors can be electric as well as magnetic radiation vectors and *differ* from the well-known Poynting vector, which in literature is sometimes also called a "radiation" vector, since it refers to the cross product of the electric and magnetic field vectors. S. A. Schelkunoff[1] has given a general radiation formula for which the power P_e radiated by a known *electric* current

[1] *Proc. IRE*, **27**, 660, October, 1939. "Electromagnetic Waves," pp. 331–335, D. Van Nostrand Company, Inc., New York.

distribution can be found from an electric radiation vector N. The power P_m radiated from a known *magnetic* current distribution can similarly be computed from a magnetic radiation vector L. We have then

$$\left.\begin{array}{l} \left.\begin{array}{rl} P_r &= 5.232 \times 10^{-4} f^2 \int_0^\pi \int_0^{2\pi} S = P_e \\ S &= \underbrace{(N_\theta N_\theta^* + N_\varphi N_\varphi^*) \sin\theta\, d\theta\, d\varphi}_{\text{solid angle}} \\ P_r &= 3.68 \times 10^{-9} f^2 \int_0^\pi \int_0^{2\pi} S' = P_m \\ S' &= (L_\theta L_\theta^* + L_\varphi L_\varphi^*) \sin\theta\, d\theta\, d\varphi \end{array}\right\} \text{watts} \\ \dfrac{5.232 \times 10^{-4}}{3.68 \times 10^{-9}} = 1.42129 \times 10^5 = 376.7^2 = Z_0^2 \end{array}\right\} \qquad (829)$$

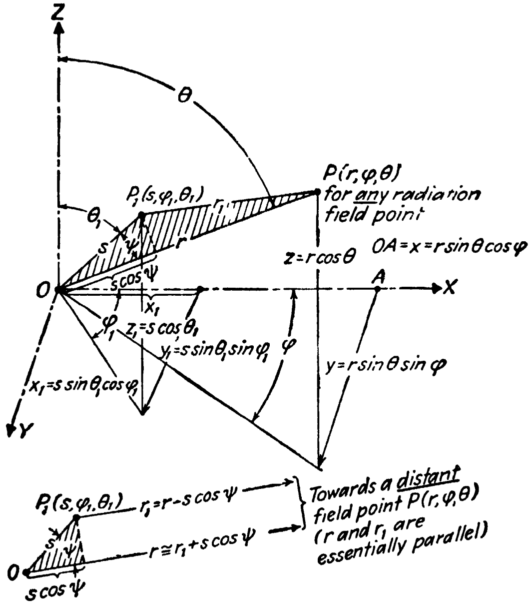
where f is in megacycles per second. The respective θ components of the electric radiation vector N and the magnetic radiation vector L take then the place of the tangential F_θ component of Fig. 134c, while the φ components of N and L take the place of the tangential F_φ component. The values N^* and L^* designate the conjugate complex values of N and L, respectively. For customary antennas, we deal only with P_e watts, since we have only *electric* current distributions. The hypothetic magnetic current concept is of value when we deal with wave-front radiators, where we can employ magnetic and electric current sheets (pages 350–356) to advantage. We can also use the P_m expression when we substitute a known electric current distribution by an equivalent magnetic current distribution, as can be done, for instance, when we compute the radiated power from an electric current loop. Both relations, namely, the expression for P_e and the expression for P_m, play an important part when it is easier to assume a possible field distribution than an electric current distribution, as at the opening of a wave guide, at apertures, or generally electromagnetic horns. We then work with both equivalent electric and magnetic current sheets, which replace the assumed field effects. In such a case, we need not only the P_e and P_m expressions but also the expression

$$\left.\begin{array}{rl} P_{e,m} &= 6.95 \times 10^{-7} f^2 \int_0^\pi \int_0^{2\pi} [(N_\theta L_\varphi^* + N_\theta^* L_\varphi) \\ &\qquad - (N_\varphi L_\theta^* + N_\varphi^* L_\theta) \sin\theta\, d\theta\, d\varphi] \\ &= 1.39 \times 10^{-6} f^2 \,\mathrm{Re}\left[\int_0^\pi \int_0^{2\pi} (N_\theta L_\varphi^* - N_\varphi L_\theta^*) \sin\theta\, d\theta\, d\varphi\right] \quad \text{watts} \\ P_{\text{total}} &= P_e + P_m + 2P_{e,m} \qquad \text{watts;} \\ & \qquad 2P_{e,m} = \text{respective mutual watts} \end{array}\right\} \qquad (829a)$$

This means that we also have to take into account the reaction $P_{e,m}$ from an electric current distribution upon a magnetic current distribu-

tion and the reaction $P_{m,e} = P_{e,m}$ from a magnetic current distribution upon an electric current distribution. These would be the mutual effects of electric on magnetic current sheets and vice versa. These formulas play an important part in electromagnetic diffraction phenomena (pages 1087–1089, 1092, 1129, 1154, 1157, 1160, and 1167). We have then the following details:

In Fig. 135, we deal with two important space points. One is the field point P ith spherical coordinates r, φ, and θ; the other is the point P_1 with coordinates s,



$$r^2 = x^2 + y^2 + z^2;\ s^2 = x_1^2 + y_1^2 + z_1^2$$
$$r_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$
$$r_1 = \sqrt{r^2 + s^2 - 2rs\cos\psi}$$

FIG. 135. Relations for the space point P_1 of an electric or a magnetic current element, respectively, and the distant field point P. Since ψ is the angle between two directions, φ, θ and φ_1, θ_1, we have the *important* relation $\cos\psi = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\varphi - \varphi_1)$.

φ, and θ, while the origin O of the XYZ system is the fixed reference point. The point P_1 designates, therefore, the location of any current element that contributes to the radiation effect at any point P lying in the distant radiation region. We found already that not all r, φ, θ locations are the seat of electromagnetic field energy, since only an ideal and hypothetical point radiator (isotropic radiator) covers all possible r, φ, θ locations in an unobstructed region about the source. The locations P_1 refer then to either electric current or equivalent magnetic current elements. The dimension s of any current element is very small compared with respective distances r and r_1 to the field point P, for which reason the $r = r_1 + s\cos\psi$ holds. We have from spherical trigonometry

$$\cos\psi = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\varphi - \varphi_1)$$

Since s meters denotes the spacing between a typical current element and the reference point O, the quantity $s \cos \psi$ must be the projection of this distance from the reference point upon a typical r, φ, θ direction, connecting the reference point O to a field point P. Inasmuch as $s \cos \psi$ denotes the path difference between the reference distance r and the current element distance r_1 from a distant field point P, the retarded time phase is expressed by the factor

$$\epsilon^{j(\omega t - \beta r_1)} = \epsilon^{j[\omega t - \beta(r - s \cos \psi)]} = \underbrace{\epsilon^{j(\omega t - \beta r)}}_{\text{fixed}}\underbrace{\epsilon^{j\beta s \cos \psi}}_{\text{varies}}$$

We have, therefore, a fixed factor and a factor that changes its value, depending upon the location of a typical current element. This factor must remain under the integral, expressing our retarded field effects at a typical space point P. Such field effects are, according to Eqs. (241) and (242) for the magnetic vector potential Λ_m and electric vector potential Λ_e, defined by the relations

$$\left.\begin{aligned} \Lambda_m = \Lambda &= \frac{\mu}{4\pi}\int\int\int_{\tau_1} \frac{J_1\epsilon^{j(\omega t - \beta r)}\epsilon^{j\beta s \cos \psi}}{r_1}\,d\tau_1 \\ &\cong \frac{\mu\epsilon^{j(\omega t - \beta r)}}{4\pi r}\int\int\int_{\tau_1} J_1\epsilon^{j\beta s \cos \psi}\,d\tau_1 \qquad \text{volt}\cdot\text{sec/m} \\ \Lambda_e &= \frac{\kappa}{4\pi}\int\int\int_{\tau_1} \frac{J_{m_1}\epsilon^{j(\omega t - \beta r)}\epsilon^{j\beta s \cos \psi}}{r_1}\,d\tau_1 \\ &\cong \frac{\kappa\epsilon^{j(\omega t - \beta r)}}{4\pi r}\int\int\int_{\tau_1} J_{m_1}\epsilon^{j\beta s \cos \psi}\,d\tau_1 \qquad \text{amp}\cdot\text{sec/m} \end{aligned}\right\} \tag{830}$$

where the integrations have to be taken over the volume of τ_1 cubic meters occupied by all *current* elements. For the Λ_m expression, we deal with *electric* current elements; while for the Λ_e expression, we deal with *magnetic* current elements (hypothetical). Hence, τ_1 must include all P_1 points of Fig. 135. The product $J_1\,d\tau_1$ is the electrical space moment for the elementary volume $d\tau_1$, while the product $J_{m_1}\,d\tau_1$ is the hypothetical magnetic space moment. If, as often the case, we deal with surface moments $J_1\,dS_1$ amperes, where J_1 is in amperes per square meter, the above triple integral reduces to a double integral. For the corresponding electric vector potential, we have then $J_{m_1}\,dS_1$ volts, since the magnetic current density J_{m_1} is in volts per square meter. For thin radiators, we deal with electric and magnetic current moments as discussed under electric and magnetic dipoles. We have then only single integral in (830). The retarded vector potentials act at the field point P in place of the actual radiator a distance r away. As far as the inverse r effect is concerned, we can put all r_1 distances equal to the reference distance of r meters. As mentioned on page 223, the current density J_1 acts, as it were, at the symmetry point of the volume element $d\tau_1$, and its magnitude can be expressed by $\sqrt{|J_x^2| + |J_y^2| + |J_z^2|}$ amperes per square meter if we use Cartesian coordinates, for instance. The subscript 1 in formula (830) expresses that we deal with a typical current element at a typical location P_1. Inasmuch as the quantities that are not under the volume integral in (830) are constant, we can proceed with only the integral expressions and have

$$N = \int\int\int_{\tau_1} J_1\epsilon^{j\beta s \cos \psi}\,d\tau_1 \ \text{m}\cdot\text{amp};\quad L = \int\int\int_{\tau_1} J_{m_1}\epsilon^{j\beta s \cos \psi}\,d\tau_1 \ \text{m}\cdot\text{volts} \tag{831}$$

which are the definitions for the *electric* radiation vector N and the *magnetic* radiation vector L.

According to (830) and (831), we have

$$\Lambda_m = \Lambda = \frac{\mu\epsilon^{j(\omega t-\beta r)}}{4\pi r} N \quad \text{volt} \cdot \text{sec/m};$$

$$\Lambda_e = \frac{\kappa\epsilon^{j(\omega t-\beta r)}}{4\pi r} L \quad \text{amp} \cdot \text{sec/m} \qquad (832)$$

Since the respective vector potentials can generally be resolved into components, such as the radial r component, the meridian θ component, and the azimuthal φ component, we must also have N_r, N_φ, N_θ and L_r, L_φ, L_θ components in a spherical coordinate system. The expressions of (831) are exactly the integrals that we have used in connection with radiation effects of linear radiators, since $d\tau_1$ then reduces to the line element dl_1 and J_1 to the line current I_1 at a place 1 along a linear radiator.

Figure 3 shows the radial r component in addition to the meridian θ component and the azimuthal φ components, which are both tangential components of the curl H, since, as far as the general case is concerned, all three components exist. It was found, however, that *only* terms which do *not decrease* by more than an inverse distance law ($1/r$ law) can play a part in the distant radiation region. It is this region for which the above relations hold, since we assumed in Fig. 135 that r_1 is parallel with r, so that $r_1 = r - s\cos\psi$, instead of being equal to $\sqrt{r^2 + s^2 - 2rs\cos\psi}$. We also assumed that we work in a pure dielectric region, so that $Z_0 = \sqrt{\mu/\kappa} \cong \sqrt{\mu_0/\kappa_0} = 376.7$ ohms holds for the unobstructed radiation space. Hence, we have at field points $P(r, \varphi, \theta)$ of Figs. 3 and 135, only $\text{curl}_\varphi H$ and $\text{curl}_\theta H$ components that can play a part. But curl $\Lambda_m = \mu H$, according to (231); and therefore, we can expect only Λ_φ and Λ_θ components at the distant field point. (It must be realized that these vector potential functions act, in effect, at $P(r, \varphi, \theta)$ of the distant radiation region.) These components are shown in Fig. 17, because the resultant Λ magnetic vector potential represents the geometrical addition. Since for the magnetic vector potential Λ_m, we have the relations (231), *i.e.*, curl $\Lambda_m = B$ and $\mathcal{E} = -\text{grad}\ V - \partial\Lambda_m/\partial t$ on account of Eq. (1) from which curl $\mathcal{E} = -dB/dt$, the vector $\mathcal{E} + (\partial\Lambda_m/\partial t)$ must be a potential vector, since its curl is zero.

Therefore, either expression of (832) has for point P of the distant radiation region only φ and θ components, since we also have curl $\Lambda_e = \kappa\mathcal{E}$ for the electric vector potential. This means that we also have only N_θ and N_φ as well as only L_θ and L_φ components in the distant radiation region. It is this region which is basic for relations (829) and (829*a*), since the Poynting integration was used in the derivation

of these expressions. Such an integration is over a sphere reaching into the *far distant* region.

Even though expressions (831) have already been used in this text without the definition of the radiation vectors N and L, we shall find that their use greatly simplifies electromagnetic diffraction problems. Both N and L must be used then.

125. Details on the Application of the Radiation Vector Concept. If v_r, v_θ, and v_φ represent unit vectors along the radial r, the meridian θ, and azimuthal φ direction, respectively, of a spherical coordinate system, we have generally

$$\left.\begin{aligned} \text{Vector} = N &= v_rN_r + v_\varphi N_\varphi + v_\theta N_\theta \\ &\cong v_\varphi N_\varphi + v_\theta N_\theta \qquad \text{m}\cdot\text{amp} \\ \text{Vector} = L &= v_rL_r + v_\varphi L_\varphi + v_\theta L_\theta \\ &\cong v_\varphi L_\varphi - v_\theta L_\theta \qquad \text{m}\cdot\text{volts} \end{aligned}\right\} \text{approximations hold in distant radiation region}$$

where, in the future, we shall omit the unit vectors but have them understood just the same. Since (832) holds, we find the following details.

For the magnetic vector potential, we have $B = \mu H = \text{curl } \Lambda_m$, and according to (250)

$H_r = 0$ in distant radiation region

$$H_\varphi = \frac{\text{curl}_\varphi \Lambda_m}{\mu} = \frac{\dfrac{\partial(r\Lambda_\theta)}{\partial r} - \dfrac{\partial(\Lambda_r)}{\partial\theta}}{\mu r} = +\frac{1}{\mu r}\frac{\partial(r\Lambda_\theta)}{\partial r}, \qquad \text{since } \Lambda_r \cong 0 \text{ (negligible)}$$

$$H_\theta = \frac{\text{curl}_\theta \Lambda_m}{\mu} = \frac{\dfrac{\partial(\Lambda_r)}{\partial\varphi} - \sin\theta\dfrac{\partial(r\Lambda_\varphi)}{\partial r}}{\mu r\sin\theta} \cong -\frac{\sin\theta\dfrac{\partial(r\Lambda_\varphi)}{\partial r}}{\mu r\sin\theta} = -\frac{1}{\mu r}\frac{\partial(r\Lambda_\varphi)}{\partial r}, \text{ since } \Lambda_r \cong 0$$

Hence,

$$\underline{H_\varphi^{(\text{amp/m})}} = \frac{1}{\mu r}\frac{\partial(r\Lambda_\theta)}{\partial r} = \frac{1}{\mu r}\frac{\partial\left(r\dfrac{\mu\epsilon^{j(\omega t-\beta r)}}{4\pi r}N_\theta\right)}{\partial r} = \frac{1}{\mu r}\frac{\partial\left(\dfrac{\mu\epsilon^{j\omega t}\epsilon^{-j\beta r}}{4\pi}N_\theta\right)}{\partial r}$$

$$= \frac{\mu\epsilon^{j\omega t}}{4\pi\mu r}N_\theta\frac{\partial(\epsilon^{-j\beta r})}{\partial r} = \frac{N_\theta\epsilon^{j\omega t}}{4\pi r}(-j\beta)\epsilon^{-j\beta r} = -j\beta\frac{N_\theta\epsilon^{\overbrace{j(\omega t-\beta r)}^{=a}}}{4\pi r}$$

$$= -j\frac{2\pi}{\lambda}\frac{N_\theta\epsilon^{ja}}{4\pi r} = \underline{-j\frac{N_\theta\epsilon^{ja}}{2r\lambda}}$$

In exactly the same way, we find

$$\underline{H_\theta^{(\text{amp/m})}} = -\frac{1}{\mu r}\frac{\partial(r\Lambda_\varphi)}{\partial r} = -\frac{1}{\mu r}\frac{\partial\left(r\dfrac{\mu\epsilon^{ja}}{4\pi r}N_\varphi\right)}{\partial r} = +j\beta\frac{N_\varphi\epsilon^{ja}}{4\pi r} = \underline{+j\frac{N_\varphi\epsilon^{ja}}{2r\lambda}}$$

Since λ meters $= 300/f^{(\text{Mc/sec})}$, we have the instantaneous values

$$\left.\begin{aligned} H_\varphi &= -j1.67 \times 10^{-3} f \frac{N_\theta}{r} \epsilon^{j(\omega t - \beta r)} \quad \text{amp/m}; \\ H_\theta &= +j1.67 \times 10^{-3} f \frac{N_\varphi}{r} \epsilon^{j(\omega t - \beta r)} \quad \text{amp/m}; \qquad a = \omega t - \beta r \end{aligned}\right\} \tag{833}$$

where r is in meters and f in megacycles per second; $\omega/6.28$ is in cycles per second. Formulas (833) are the respective tangential instantaneous magnetic field components yielding the corresponding maximum values if we omit the time factor ϵ^{ja}. The corresponding effective values are found by dividing 1.67×10^{-3} by $\sqrt{2}$, for which $1.67 \times 10^{-3}/1.4141 = 1.18 \times 10^{-3}$. We also know that in the distant radiation region any H_φ component is associated with an *inphase* electrical component $\mathcal{E}_\theta$, which is in space quadrature, and that $\mathcal{E}_\theta/H_\varphi = \sqrt{\mu/\kappa} \cong 376.7$ ohms for propagation in air. In exactly the same way the H_θ field has an associated $\mathcal{E}_\varphi$ field equal to $376.7H_\theta$ volts per meter. We have, therefore, for the *effective* field values, the components

$$\left.\begin{aligned} H_\varphi &= -j1.18 \times 10^{-3} \frac{fN_\theta}{r} \quad \text{amp/m}; \\ &\qquad H_\theta = +j1.18 \times 10^{-3} \frac{fN_\varphi}{r} \quad \text{amp/m} \\ H_\varphi^* &= +j1.18 \times 10^{-3} \frac{fN_\theta^*}{r} \quad \text{amp/m}; \\ &\qquad H_\theta^* = -j1.18 \times 10^{-3} \frac{fN_\varphi^*}{r} \quad \text{amp/m} \\ \mathcal{E}_\theta &= 376.7H_\varphi = -j4.44 \times 10^{-1} \frac{fN_\theta}{r} \quad \text{volts/m}; \\ &\qquad \mathcal{E}_\varphi = 376.7H_\theta = +j4.44 \times 10^{-1} \frac{fN_\varphi}{r} \quad \text{volts/m} \end{aligned}\right\} \tag{833a}$$

where f is in megacycles per second. The Poynting vector, which combines both the electric and the magnetic inphase field associations in the *distant* wave front, then becomes

$$\begin{aligned} p &= H_\varphi^* \cdot \mathcal{E}_\theta - H_\theta^* \cdot \mathcal{E}_\varphi \\ &= 5.232 \times 10^{-1} \frac{f^2}{r^2} (|N_\theta|^2 + |N_\varphi|^2) \qquad \text{watts/m}^2 \end{aligned} \tag{834}$$

where it is understood that the respective electric radiation factor components represent the absolute values or magnitudes $|N_\theta|$ and $|N_\varphi|$. The total radiated power is then obtained by integrating the p energy

flux per square meter and per second over the area of a sphere of radius of r meters, which reaches into the *distant* radiation region. We find then

$$P_r = P_e = 5.232 \times 10^{-4}\frac{f^2}{r^2}\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}(|N_\theta|^2 + |N_\varphi|^2)r^2 \sin\theta\, d\theta\, d\varphi$$

$$= 5.232 \times 10^{-4}f^2\int_0^{\pi}\int_0^{2\pi}(N_\theta \cdot N_\theta^* + N_\varphi \cdot N_\varphi^*)\overbrace{\sin\theta\, d\theta\, d\varphi}^{\text{solid angle}} \text{ watts} \quad (834a)$$

since a complex vector N multiplied by its conjugate N^* yields the $|N|^2$ value, which is the square of its magnitude. If $N = a + jb$, then $N^* = a - jb$ and NN^* yields $a^2 + b^2 = |N|^2$ meter^2 · ampere^2. The outcome of (834a) checks (829). In exactly the same way, the P_m relation of (829) for the magnetic radiation vector L in meter · volts can be proved. The product pr^2 of the Poynting vector p yields

$$\left.\begin{aligned} D_p^{(\text{watts})} &= pr^2 = 5.232 \times 10^{-4}f^2(N_\theta \cdot N_\theta^* + N_\varphi \cdot N_\varphi^*) \\ &\qquad \text{for } \textit{electric} \text{ radiation vector} \\ D_p^{(\text{watts})} &= pr^2 = 3.68 \times 10^{-9}f^2(L_\theta \cdot L_\theta^* + L_\varphi \cdot L_\varphi^*) \\ &\qquad \text{for } \textit{magnetic} \text{ radiation vector} \\ N_\theta \cdot N_\theta^* &= |N_\theta|^2; \quad N_\varphi \cdot N_\varphi^* = |N_\varphi|^2 \cdots \end{aligned}\right\} \quad (835)$$

which is known as the radiation intensity. It represents, in reality, the *power directivity* D_p. It will be noted that this directivity increases with the square of the operating frequency f. For a fixed frequency, it is necessary only to plot $|N_\theta|^2 + |N_\varphi|^2$ for different θ and φ values where only an electric radiation vector N is acting. It is then convenient to make a plot in the equatorial XY plane for which $\theta = 90$ deg in order to find out where we obtain maximum effects. Then it may be necessary only to make a plot in a preferred meridian plane. If the ZX plane happens to be the meridian plane, then we have $\varphi = 0$; while for the ZY plane, we have $\varphi = 90$ deg. These formulas are applied throughout this text and are most useful when we deal with both electric radiation vectors and equivalent magnetic radiation vectors. For the linear radiators discussed in the previous sections, we deal only with an electric radiation vector $N = N_z$ if the linear radiator is along the Z axis. We have then the absolute value $|N|^2 = N_z \cdot N_z^*$, since $N_\varphi = 0$; $N_\theta = -N_z \sin\theta$. The power directivity is then

$$D_p = 5.232 \times 10^{-4}f^2(N_z \cdot N_z^*) \sin^2\theta \qquad \text{watts per } \textit{unit solid} \text{ angle}$$

Since we have the Laplace transformations $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, and $z = r\cos\theta$, we have also the transformations

$$N_\theta = (N_x \cos\varphi + N_y \sin\varphi)\cos\theta - N_z \sin\theta;$$
$$N_\varphi = -N_x \sin\varphi + N_y \cos\varphi \quad (835a)$$

and similarly for the magnetic radiation vector components.

126. Distant Field Effects and Power Directivity (Radiation Intensity) in the Presence of Electric and Equivalent Magnetic Radiation Vectors. According to (520), we can compute the electric and magnetic field intensities at a field point from

$$
\left.
\begin{aligned}
\mathcal{E} &= -\frac{\text{curl } \Lambda_e}{\kappa} - j\omega\Lambda_m + \frac{\text{grad (div } \Lambda_m)}{j\omega\mu\kappa} \qquad \text{volts/m;} \\
H &= \frac{\text{curl } \Lambda_m}{\mu} - j\omega\Lambda_e + \frac{\text{grad (div } \Lambda_e)}{j\omega\mu\kappa} \qquad \text{amp/m;} \\
&\text{grad (div } \Lambda) = \nabla(\nabla \cdot \Lambda); \qquad \text{curl } \Lambda = \nabla \times \Lambda
\end{aligned}
\right\} \qquad (836)
$$

The second relation was used in deriving the H_φ and H_θ field components of (833). The relation simplified then to $\mu H = \text{curl } \Lambda_m$, since we had no electric retarded vector potential Λ_e, because no magnetic current density of J_m volts per square meter and therefore also no magnetic radiation vector L in meter · volts existed. The first relation would also have simplified to the form

$$
\mathcal{E}^{(\text{volts/m})} = -j\omega\Lambda_m + \frac{\text{grad (div } \Lambda_m)}{j\mu\kappa\omega} = -j\left[\omega\Lambda_m - \frac{\nabla(\nabla \cdot \Lambda_m)}{\mu\kappa\omega}\right]
$$

since $\Lambda_e = 0$ amp · sec/m. It could have been used in finding the $\mathcal{E}_\theta$ and H_φ components. It was simpler to apply the relation $\mathcal{E}/H = \sqrt{\mu/\kappa}$ ohms for finding $\mathcal{E}$ from a known H, since we are dealing with a distant field point. Now there are many (cases for instance, wave-front radiators, electromagnetic horns, and other electromagnetic diffraction problems) where it is easier to estimate an electromagnetic field[1] in the wave front than a current and charge distribution. We deal in equivalence with magnetic and electric current sheets in place of the wave front and therefore also with an equivalent electric current density of J amperes per square meter and a magnetic current density of J_m volts per square meter. These equivalent current densities then cause the retarded magnetic and electric vector potentials Λ_m and Λ_e as defined by (830) and used in (836) for computing the electric and magnetic field intensities at a distant field point. As far as (836) is concerned, the relation holds for all distances in a pure dielectric wave medium (only μ henrys per meter and κ farads per meter effective, but $\sigma = 0$ mhos/m). However, relations (830) and the relations (832) for the same quantities, but in terms of the resultant electric and magnetic radiation vectors N and L, respectively, hold only for the distant radiation region, since only the

[1] Applications are given on pages 1089–1095, 1115–1116, 1128–1136, 1141, 1157, 1276–1278, and Sec. 246.

terms that are proportional to $1/r$ are used. We are interested only in the distant radiation terms here and can, therefore, use (832) for the respective retarded vector potentials Λ_e and Λ_m and the definitions of (831) for the electric and magnetic radiation vectors N and L. Now as far as the effect of the equivalent electric current densities of J amperes per square meter is concerned, we have the same relations as the result as found in (833*a*) from (836), since Λ_e does not exist in these relations. By using both vector potentials in (836) and proceeding as in Sec. 125, we must then obtain a solution as in (833*a*), except that both radiation vectors must appear in the associated $\mathcal{E}_\theta$ and H_φ as well as in the associated $\mathcal{E}_\varphi$ and H_θ expressions. Using the procedure that led to (833) and then to (833*a*), we find for $Z_0 = \sqrt{\mu/\kappa}$ ohms and $Z_0 \cong 376.7$ ohms for air propagation, the *effective* field values

$$
\left.
\begin{aligned}
&\left.
\begin{aligned}
\mathcal{E}_\theta &= -j\frac{L_\varphi + Z_0 N_\theta}{2r\lambda}\frac{1}{\sqrt{2}} = -j\frac{(L_\varphi + Z_0 N_\theta)f}{600\sqrt{2}\,r} \\
&= -j1.18 \times 10^{-3}\frac{(L_\varphi + 376.7N_\theta)}{r} f \\
\mathcal{E}_\varphi &= +j\frac{L_\theta - Z_0 N_\varphi}{2r\lambda}\frac{1}{\sqrt{2}} = +j1.18 \\
&\quad \times 10^{-3}\frac{L_\theta - Z_0 N_\varphi}{r} f = +j1.18 \\
&\quad \times 10^{-3}\frac{L_\theta - 376.7N_\varphi}{r} f
\end{aligned}
\right\} \text{volts/m} \\
&\left.
\begin{aligned}
H_\varphi &= \frac{\mathcal{E}_\theta}{Z_0} = \frac{\mathcal{E}_\theta}{376.7} = -j1.18 \times 10^{-3}\frac{\frac{L_\varphi}{376.7} + N_\theta}{r} f; \\
H_\theta &= -\frac{\mathcal{E}_\varphi}{Z_0} = -\frac{\mathcal{E}_\varphi}{376.7} = -j1.18 \\
&\quad \times 10^{-3}\frac{\frac{L_\theta}{376.7} - N_\varphi}{r} f \\
H_\varphi^* &= +j1.18 \times 10^{-3}\frac{\frac{L_\varphi^*}{376.7} + N_\theta^*}{r} f; \\
H_\theta^* &= +j1.18 \times 10^{-3}\frac{\frac{L_\theta^*}{376.7} - N_\varphi^*}{r} f; \quad \text{complex conjugate fields}
\end{aligned}
\right\} \text{amp/m}
\end{aligned}
\right\} \quad (837)
$$

where f is in megacycles per second and r in meters.

For the Poynting vector p and the power directivity $D_p = pr^2$, also known as the "radiation intensity," we have the detailed derivation

$$
\begin{aligned}
p^{(\text{watts/m}^2)} &= \mathcal{E}_\theta \cdot H_\varphi^* + \mathcal{E}_\varphi \cdot H_\theta^* = (1.18 \times 10^{-3})^2 \frac{f^2}{r^2} \\
&\times \left[(L_\varphi + Z_0 N_\theta) \left(\frac{L_\varphi^*}{Z_0} + N_\theta^* \right) + (L_\theta - Z_0 N_\varphi) \left(\frac{L_\theta^*}{Z_0} - N_\varphi^* \right) \right] \\
&= 1.39 \times 10^{-6} \frac{f^2}{r^2} \left(\frac{L_\varphi L_\varphi^*}{Z_0} + L_\varphi N_\theta^* + N_\theta \cdot L_\varphi^* + Z_0 N_\theta \cdot N_\theta^* \right. \\
&\qquad \left. + \frac{L_\theta \cdot L_\theta^*}{Z_0} - L_\theta N_\varphi^* - N_\varphi L_\theta^* + Z_0 N_\varphi N_\varphi^* \right) \\
&= 1.39 \times 10^{-6} \frac{f^2}{r^2} \Big[\underbrace{Z_0 (N_\theta N_\theta^* + N_\varphi N_\varphi^*)}_{\text{self-effect of electric current sheet } (= p_e)} \\
&\quad + \underbrace{\frac{1}{Z_0} (L_\theta L_\theta^* + L_\varphi L_\varphi^*)}_{\text{self-effect of magnetic current sheet } (= p_m)} + \underbrace{N_\theta L_\varphi^* - N_\varphi L_\theta^*}_{\text{mutual effect of electric on magnetic current sheet } (= p_{e,m})} + \underbrace{L_\varphi N_\theta^* - L_\theta N_\varphi^*}_{\text{mutual effect of magnetic on electric current sheet } (= p_{m,e})} \Big]
\end{aligned} \tag{838}
$$

Therefore,

$$p = p_e + p_m + p_{e,m} + p_{m,e} = p_e + p_m + 2p_{e,m} \qquad \text{watts/m}^2 \tag{838a}$$

We have the power directivity

$$
\begin{aligned}
D_p = pr^2 = 1.39 \times 10^{-6} f^2 \bigg[& 376.7(|N_\theta|^2 + |N_\varphi|^2) + \frac{|L_\theta|^2 + |L_\varphi|^2}{376.7} \\
& + 2(|N_\theta L_\varphi| - |N_\varphi L_\theta|) \bigg] \qquad \text{watts}
\end{aligned} \tag{838b}
$$

where

$$|N_\theta L_\varphi| - |N_\varphi L_\theta| = \text{Re}\,(N_\theta L_\varphi^* - N_\varphi \cdot L_\theta^*) \quad \text{absolute amount}$$
$$|N|^2 = N \cdot N^* \qquad \text{absolute amount of } N^2 \text{ or of } N^{*2}$$
$$|L|^2 = L \cdot L^* \quad \text{absolute amount of } L^2 \text{ or of } L^{*2}$$

Hence, when the electric radiation vector N as defined by (831) has the form $a + jb$, we find $|N|^2 = N \cdot N^* = (a + jb)(a - jb) = a^2 + b^2$; if $N = jb$, we have $|N|^2 = (jb)(-jb) = b^2$. The total radiated power becomes

$$
\left.
\begin{aligned}
P_r^{(\text{watts})} &= \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} pr^2 \underbrace{\sin\theta\,d\theta\,d\varphi}_{\text{solid elementary angle}} = \int_0^{\pi}\int_0^{2\pi} D_{\varphi,\theta}^2 \sin\theta\,d\theta\,d\varphi \\
&= 1.39\times 10^{-6}f^2 \int_0^{\pi}\int_0^{2\pi}\Big[\underbrace{376.7(|N_\theta|^2+|N_\varphi|^2)}_{\text{self-effect}} \\
&\quad + \underbrace{\frac{|L_\theta|^2+|L_\varphi|^2}{376.7}}_{\text{self-effect}} + \underbrace{2(|N_\theta L_\varphi| - |N_\varphi L_\theta|)}_{\text{mutual effects}}\Big]\sin\theta\,d\theta\,d\varphi \\
P_r &= P_e + P_m + 2P_{e,m} \quad \text{watts;} \\
P_e &= 5.232\times 10^{-4}f^2 \int_0^{\pi}\int_0^{2\pi} dS \quad \text{watts} \\
&\qquad dS \equiv (N_\theta\cdot N_\theta^* + N_\varphi\cdot N_\varphi^*)\sin\theta\,d\theta\,d\varphi \\
P_m &= 3.68\times 10^{-9}f^2 \int_0^{\pi}\int_0^{2\pi} (L_\theta\cdot L_\theta^* + L_\varphi\cdot L_\varphi^*)\sin\theta\,d\theta\,d\varphi \\
&\qquad \text{watts} \\
P_{e,m} &= 1.39\times 10^{-6}f^2\,\mathrm{Re}\left(\int_0^{\pi}\int_0^{2\pi} dS\right) \quad \text{watts} \\
&\qquad dS \equiv (N_\theta\cdot L_\varphi^* - N_\varphi\cdot L_\theta^*)\sin\theta\,d\theta\,d\varphi
\end{aligned}
\right\}
\tag{838c}
$$

These expressions then include electric and magnetic current densities, since we deal with electric and magnetic radiation vectors. In many cases, only the electric N vectors play a part and only the P_e term remains. This term confirms (829). Since, according to (831), the resultant electric radiation vector N is in meter · amperes, for a linear radiator, we must have $N = \int_l I\epsilon^{j\beta s\cos\psi}\,dl$ meter · amperes with a value $N = N_0 = I\,dl$ at the origin. Integrals of this type have already been used in derivations. They were used, for instance, in Sec. 113, also in connection with Figs. 124, 126, and 123.

127. Notes on Directivity and Antenna Gain. For antenna arrays, as for instance, several half-wave radiators arranged in turnstile bays,[1] it is possible to force some of the high-angle radiation toward the equatorial plane and cause a circular equatorial pattern instead of the well-known figure-eight pattern that would occur for a single horizontal half-wave dipole. For such an array it seems convenient to refer the radiation gain to the maximum field value obtained from a single half-wave radiator. We can then define a field gain as well as a power gain with

[1] HUND, A., "Frequency Modulation," pp. 338–345, McGraw-Hill Book Company, Inc., New York, 1942.

respect to a half-wave radiator. The radiation power is proportional to *either* field intensity, because the Poynting vector for the *effective* field values of $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter yields, for these inphase values,

$$p = \mathcal{E}_\theta \cdot H_\varphi = \frac{\mathcal{E}_\theta^2}{376.7} = 376.7H_\varphi^2 \qquad \text{watts/m}^2$$

We note that the field gain depends upon the field ratio but the power gain depends on the square of this ratio. Thus, for 6 turnstile bays, we have a field ratio of only 2.07, but a power ratio of 4.28. For 10 bays, the field ratio is only 2.6, but the power ratio is 6.76. This means that along the equatorial service area, the 10-bay array acts 6.76 times as powerfully. It is convenient to express the field and power advantages in decibels, and we have the gain

$$G = 10 \log_{10} \frac{P}{P_s} = 20 \log_{10} \frac{\mathcal{E}}{\mathcal{E}_s} \qquad \text{db} \tag{839}$$

In these expressions, $\mathcal{E}$ represents the electric field intensity in volts per meter, as found for a system in a desired or preferred direction, whatever may be the case, and $\mathcal{E}_s$ denotes the standard reference value. The P and P_s quantities are the corresponding power values of the electromagnetic field. Hence, a 6-bay turnstile, with a field ratio $\mathcal{E}/\mathcal{E}_s = 2.07$ of the turnstile field intensity to the intensity of a single half-wave radiator, yields

$$G = 20 \log_{10} (2.07) = 20 \times 0.31597 = 6.3194\text{-db gain}$$

over the half-wave radiator. We also find from the power ratio

$$G = 10 \log_{10} (4.28) = 10 \times 0.63144 = 6.3144\text{-db gain}$$

Inasmuch as the three-dimensional impedance of $Z_0 = 376.7$ ohms, for customary radiation space, is fixed, the above relations can be interchanged and we can use the field ratio. This ratio can be readily found from field intensity measurements, while the power measurement is a rather difficult matter at VHF's and UHF's. It is usually based on field, *i.e.*, $\mathcal{E}$, measurements and some formula based on it.

We found that the power directivity D_p, such as defined, for instance, in Eqs. (835) and (838*b*), is a measure of the radiation intensity (this is *not* the same as the field intensity, since D_p is in watts and $\mathcal{E}$ in volts per meter or used as $H = \mathcal{E}/376.7$ amperes per meter. Since the power directivity D_p is defined by

$$P_r = \int_{\theta=0}^{\theta=180°} \int_{\varphi=0}^{\varphi=360°} D_p \overbrace{\sin \theta \, d\theta \, d\varphi}^{\text{solid angle}} \qquad \text{watts} \tag{840}$$

for the total radiated power P_r over the surface $4\pi r^2$ of a sphere, and $\sin\theta\, d\theta\, d\varphi$ represents an elementary solid angle, the quantity D_p in watts must be the radiation power flow per unit solid angle in the direction φ, θ. We found in (835) and (838*b*) the power directivity D_p in the φ, θ direction by multiplying the Poynting energy vector p, which is in watts per square meter by the square of any radius r of spheres that reach into the *distant* radiation region. This means that if $\mathcal{E}_e$ and H_e denote the effective inphase values in volts per meter and amperes per meter, respectively, at a distant field point $P(r, \varphi, \theta)$, we have the relations

$$\left.\begin{aligned} p &= \frac{D_p}{r^2} = \mathcal{E}_e H_e = \frac{\mathcal{E}_e^2}{376.7} \qquad \text{watts/m}^2 \\ \mathcal{E}_e &= 19.42\sqrt{p} = 19.42\frac{\sqrt{D_p}}{r} \qquad \text{volts/m} \\ H_e &= 0.0515\sqrt{p} = 0.0515\frac{\sqrt{D_p}}{r} \qquad \text{amp/m} \end{aligned}\right\} \tag{840a}$$

where r is the distance in meters to field point r, φ, θ.

We have in these expressions all that is needed, since $\mathcal{E}_e$ volts per meter can be measured or computed from the supply current feeding a UHF antenna. For instance, if we arrange a $\lambda/2$ dipole such that the indirect ground ray, which would reflect at the ground, is screened off from reaching the ground, only the direct ray will reach a field point $P(r, \varphi, \theta = 90 \text{ deg})$ in the *equatorial plane* of a vertical dipole. We have then

$$\mathcal{E}_e = j\frac{60}{r} I_e D_\theta = j\frac{60}{r} I_e \frac{\cos(90° \cos\theta)}{\sin\theta} = j\frac{60 I_e}{r}\frac{\cos(90° \cos 90°)}{\sin 90°} = j\frac{60 I_e}{r} \quad \text{volts/m}$$

where the j factor has nothing to do with the absolute field intensity value. Hence, for a 500-Mc excitation, we have a free-space wavelength of $^{300}\!/_{500} = 0.6$ m, or 60 cm, only, which is $0.6 \times 39.37 = 23.6$ in. only. We can always guess the equivalence for meters by just calling 1 m roughly 1 yd (1.094 yd exactly) or 3 ft (3.281 ft exactly). Hence, if we choose a distance of 10 such wavelengths or more, we can calibrate any HF receiver by comparing deflections on the receiver with field values computed from the above formula as long as we choose the distance of r meters in a direction where we can expect only direct radiation and where there are no reflections from buildings or mountans that are sufficiently strong to affect the direct field intensity. Suppose we take 10 such wavelengths as the distance r. Then $\mathcal{E}_e = 6I_e$ volts per meter, and we need only to vary the effective current flow of I_e amperes at the dipole gap. This is essentially also the feeder current for an aperiodic feeder (matched feeder). It may be more convenient to use a horizontal half-wave dipole. All that is needed is then to turn the receiving linear conductor parallel with this dipole to cause maximum reception in order to apply this formula.

There are several ways of defining antenna gain. We note that *even in unobstructed radiation space*, we have—even for simple linear radiators—field patterns of revolution as shown in Fig. 106 as far as either the

electric or the magnetic field intensity in volts per meter and amperes per meter, respectively, are concerned. This means that for a half-wave radiator we have maximum field effects along the equatorial plane, the radiator acting as the pole axis. For a full-wave radiator, there is zero radiation effect along this plane or along either direction of the radiator length. We have then maximum field effects along the 54-deg directions from the radiator. For the other harmonic excitations of Fig. 106, we note there are several preferred directions with optimum maximum field effects for $\theta = 36$ deg in the case of a radiator oscillating in the double-wavelength distribution. It is the maximum directions that are normally used. For this reason, it has become customary to define radiator characteristics according to such maximum field effects, since such effects have to be employed in order to cover the longest radiation range. From field intensity formulas, we note that the radiation power given by the Poynting vector is

$$p = \frac{\mathcal{E}_\theta^2}{376.7} = \frac{(60 I_e D_\theta)^2}{376.7 r^2} = 9.55 \frac{I_e^2}{r^2} D_\theta^2 \qquad \text{watts/m}^2$$

$$D_\theta = \frac{\cos (k\ 90^\circ \cos \theta)}{\sin \theta} \qquad \text{for } k = 1, 3, 5, \cdots$$

$$D_\theta = \frac{\sin (k\ 90^\circ \cos \theta)}{\sin \theta} \qquad \text{for } k = 2, 4, 6, \cdots; \qquad \mathcal{E}_\theta = \mathcal{E}_e$$

where I_e denotes the effective current in amperes at a current antinode and r is the distance to the field point in meters. The power flow along the φ, θ direction is, therefore, proportional to the square of the space directivity, which—for a single radiator in unobstructed radiation space—happens to be defined by the meridian directivity D_θ, since we have circular symmetry about the radiator. Since directive systems have the feature of concentrating the field energy into preferred directions, *i.e.*, toward such directions, it has become customary to define the radiation advantage in a preferred direction as the radiation energy that we would have if the radiating system emitted the same total power but with a Poynting radiation flow p that is the *same* in all directions. This then assumes a fictitious omnidirectional antenna (an isotropical radiator) as a reference standard just as is done in acoustical radiation problems by a sphere that radiates sound equally well in all directions. It does not matter if such a standard can actually exist. We simply compare commercial radiators with such a hypothetical radiation condition and obtain thereby a comparison of the commercial radiators. We then obtain the radiation advantage R_a as the ratio of the Poynting vector in the direction of maximum intensity to the average Poynting vector. Therefore,

$$R_a = \frac{p_{max}}{p_{avg}} = \frac{(D^2_{\varphi,\theta})_{max}}{(D^2_{\varphi,\theta})_{avg}}; \qquad G = 10 \log_{10} (R_a) \qquad \text{db} \qquad (841)$$

where G is the corresponding decibel gain. The application is not difficult. Suppose we have a differential linear radiator, *i.e.*, an elementary linear radiator, which acts, in effect, like a Hertzian doublet radiator. We found that its meridian directivity is $D^2_\theta = \sin^2 \theta$ and $D_\varphi = 1$, since there is a circular pattern around the radiator. To find the average value, we have to bear in mind that we deal with a solid angle, since we have a pattern of rotation. The elementary solid angle is $\sin \theta \, d\theta \, d\varphi$, and we have $\int_0^\pi D^2_\theta \sin \theta \, d\theta = \int_0^\pi \sin^3 \theta \, d\theta$ in the meridian plane. This can also be seen in Fig. 17c for an elementary strip $dS = 2\pi r^2 \sin \theta \, d\theta$, which is to be multiplied by $\sin^2 \theta$. We find then

$$(D^2_\theta)_{avg} = 2/3, \qquad \text{since} \int_0^\pi \sin^3 \theta \, d\theta = \left| -1/3 \cos \theta(\sin^2 \theta + 2)\right|_{\theta=0}^{\theta=\pi}$$
$$= -1/3 \underbrace{\cos \pi}_{-1}(\underbrace{\sin^2 \pi}_{0} + 2) - [-1/3 \underbrace{\cos \theta}_{+1}(0 + 2)] = 2/3$$

because for the average of $D^2_{(\theta,\varphi)}$, we must integrate over all solid angles and divide by 4π. The condition $(D_\theta)_{max}$ occurs for $\theta = 90$ deg and is unity, because $\sin 90° = 1$ and all other integer powers of $\sin 90°$. Therefore, $R_a = 1/(2/3) = 1.5$, yielding a gain of $G = 10 \log_{10} (1.5) = 10 \times 0.1761 = 1.761$ db for an infinitesimal radiator over a uniform or isotropic radiator with the same power. For a half-wave radiator in unobstructed radiation space, we have likewise circular symmetry about the radiator. We can then use the meridian pattern. We find again $(D_\theta)_{max} = 1 = (D^2_\theta)_{max}$ for $\theta = 90$ deg. We have already found in this text[1] that $(D^2_\theta)_{avg} = 0.612$. Hence, $R_a = 1/0.612 = 1.634$, since $(D^2_\theta)_{max} = 1$; and for $\theta = 90$ deg, we have then a gain of $G = 10 \log_{10} (1.634) = 10 \times 0.21325 = 2.133$ db over an isotropic radiator of same power. A radiation advantage R_a of a half-wave radiator of 1.634 in the preferred directions means that a unidirectional radiator has to use 1.634 times the power in order to yield the same maximum field intensity $\mathcal{E}_{max}$ in the equatorial plane as does the half-wave radiator in this plane at the same distance. The reciprocal of the radiation advantage R_a is called by some writers also the "absolute" directivity, since $p_{avg}/p_{max} = P_s/P$. Herein P_s represents the power in watts radiated from an isotropic reference radiator, causing in all φ, θ directions and at all r, φ, θ space points the

[1] This value can be computed from (789a), which relation holds for $q = 1, 2, 3, 4, 5, \cdots$, *i.e.*, for $l = 0.5\lambda, \lambda, 1.5\lambda, 2\lambda, 2.5\lambda, \cdots$. To obtain $(D_\theta{}^2)_{avg}$, we have to divide A of the above-mentioned formula by 12.56, which is 4π. For the half-wave antenna, we have then $q = 1$ and find $A/12.56 = 7.67/12.56 = 0.612$.

same $\mathcal{E}_{max}$ value as the directional radiator to be compared with P watts at the distance r in the direction of maximum field intensity $\mathcal{E}_{max}$. For a half-wave radiator, $\mathcal{E}_{max}$ volts per meter occurs along the equatorial plane at a distance of r meters. For alignments, curtains, and space lattice structures of several radiation centers, with like elementary radiators such as $\lambda/2$ radiators, it is customary to compare the field advantage of the entire array with that of an individual radiator and use the concept of a relative directivity D_r. If the absolute directivity $1/R_a$ is referred to a $\lambda/2$ radiator, we have a gain $G = 10 \log_{10} [1/(1.634R_a)]$ decibels.

We note that for one linear radiator, with a circular pattern about the radiator axis, we have the total radiated power for any sphere of radius of r meters, with the sphere reaching into the distant radiation region,

$$P_r = 4\pi r^2 [p_\theta]_{avg} = 120 I_e^2 [D_\theta{}^2]_{avg} \qquad \text{watts} \tag{842}$$

where I_e represents the effective current in amperes at the current antinode. Now for an elementary radiator of infinitesimal length dl, we have I_e constant along the dl length. We have I_e also fixed along the *equivalent* length l_e defined by

$$l_e = \frac{1}{I_e} \int_0^{l=\lambda/2} I_e \sin\left(\frac{2\pi}{\lambda} l\right) dl \qquad \text{m}$$

for natural phasing. Hence, for a half-wave linear antenna,

$$l_e = \frac{1}{I_e} \int_0^{l=\lambda/2} I_e \sin\left(\frac{2\pi}{\lambda} l\right) dl = \left| -\frac{\frac{\lambda}{2\pi} I_e}{I_e} \cos\left(\frac{2\pi}{\lambda} l\right) \right|_{l=0}^{l=\lambda/2}$$

$$= -\frac{\lambda}{2\pi}\left[\underbrace{\cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{2}\right)}_{-1} - \underbrace{\cos\left(\frac{2\pi}{\lambda} 0\right)}_{+1}\right] = +\frac{2\lambda}{2\pi} = \frac{\lambda}{\pi} = \frac{2l}{\pi} = 0.637l \qquad \text{m}$$

The current I_e is the effective current value at a current antinode for the effective current distribution along the length $l = \lambda/2$. For a very short linear radiator (but *not* infinitesimally short), an elementary dl radiator, and a half-wave radiator, we have the meridian directivities

$$\left.\begin{aligned}
D_\theta &= \frac{2l}{\lambda} \sin\theta \frac{\cos(0.5\beta l \cos\theta)}{1 - (2l/\lambda)^2 \cos^2\theta} \quad \text{for zero current at either end and } (I_e)_{max} \text{ at the mid-point of } l \text{ m} \\
D_\theta &= \frac{\cos(90^\circ \cos\theta)}{\sin\theta} \quad \text{for } l = \frac{\lambda}{2} \\
D_\theta &= \frac{2\Delta l}{\lambda} \sin\theta \frac{\overbrace{\cos\left(\pi \frac{\Delta l}{\lambda} \cos\theta\right)}^{\cong 1}}{\underbrace{1 - (2\Delta l/\lambda)^2 \cos^2\theta}_{\cong 1}} \cong \frac{2\Delta l}{\lambda} \sin\theta \quad \text{for } l = \Delta l \text{ short compared with } \lambda \\
D_\theta &= \sin\theta \text{ for } l = dl \quad \text{(differential linear radiator)}
\end{aligned}\right\} \tag{842a}$$

Using these directivities and applying the average values of the respective D_θ^2 values to (842), we find

$$\left.\begin{aligned}
P_r &= 120I_e^2[D_\theta^2]_{avg} = 120 \times 0.612 I_e^2 = 73.4 I_e^2 \qquad \text{for } l = \frac{\lambda}{2} \\
P_r &= 120I_e^2[D_\theta^2]_{avg} = 120 \cdot \tfrac{2}{3} \cdot I_e^2 = 80 I_e^2 \qquad \text{for } l = dl \\
P_r &= 120I_e^2[D_\theta^2]_{avg} = 120\left[\left(\frac{2\Delta l}{\lambda}\right)^2 \frac{2}{3}\right] I_e^2 = 320\left(\frac{\Delta l}{\lambda}\right)^2 I_e^2 \qquad \text{for } l = \Delta l \\
[D_\theta^2]_{avg} &= 0.612 \quad \text{for } l = \frac{\lambda}{2} \text{ radiator}; \qquad [D_\theta^2]_{avg} = \frac{2}{3} \quad \text{for infinitesimal } dl \text{ radiator} \\
[D_\theta^2]_{avg} &= \left(\frac{2\Delta l}{\lambda}\right)^2 \frac{2}{3} \qquad \text{for very short } l = \Delta l \text{ radiator}
\end{aligned}\right\} \tag{842b}$$

confirming the radiation resistances $R_r = P_r/I_e^2$ of 73.4, 80, and $320(\Delta l/\lambda)^2$ ohms for the respective wave distributions. Hence, if Δl is only one-twentieth of a wavelength long, then we have a radiation resistance of

$$R_r = 320(0.05)^2 = 0.8 \text{ ohm}$$

In many cases, we need formulas that apply to $D^2_{\varphi,\theta}$ patterns, *i.e.*, to conditions for which we may not always have circular symmetry about the resultant radiation center axis of an antenna array. We then use the concept of the power directivity D_p, which represents the power flow in watts per unit solid angle in any φ, θ direction. This is evident from (840). If D_p represents the power flow per unit solid angle (Fig. 136). of any array under consideration (this may be also the power flow for wave front and other radiator form) and D_s the value for an isotropic or omnidirectional standard, we find the total radiated power for either case as

$$\left.\begin{aligned}
P_r &= \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} D_p \underbrace{\sin\theta\, d\theta\, d\varphi}_{\substack{\text{elementary}\\ \text{solid angle}}} \qquad \text{watts for the array} \\
P_r &= \int_0^{\pi}\int_0^{2\pi} D_s \sin\theta\, d\theta\, d\varphi = D_s \int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\varphi \\
&= 12.56 D_s \qquad \text{watts for the standard}
\end{aligned}\right\} \tag{843}$$

since

$$\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} = \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta = \Big|\underbrace{\varphi\Big|_0^{2\pi}}_{=2\pi} \cdot \Big| \overbrace{- \cos\theta\Big|_0^{\pi}}^{=-2}$$

$$= -2\pi(\cos\pi - \cos 0) = +4\pi = 15.56$$

Now for the reference standard, we have the same D_s value in all directions; and if we put its value equal to the maximum radiation power directivity of the actual radiating system under consideration and equal

to unity, we have $D_s = 1$ watt per unit solid angle and the total radiation of the reference standard as $P_s = 12.56$ watts. The power radiation disadvantage of the actual radiating system is then $12.56/P_r$, yielding a gain of

$$G = 10 \log_{10}\left(\frac{P_s}{P_r}\right) = 10 \log_{10}\left(\frac{12.56}{\int_0^\theta \int_{\varphi=0}^{\varphi=2\pi} D_p \sin\theta \, d\theta \, d\varphi}\right) \quad \text{db} \quad (844)$$

for the modified standard, based on the maximum power directivity of the actual radiator. This gives, of course, the same result as (841), for which we have $P_r = P_s = 1$ watt, but $(D_p)_{max}$ and D_s are different, and

$$G = 10 \log_{10} [12.56(D_p)_{max}] \quad \text{db} \quad (844a)$$

since

$$P_r = \iint D_p \sin\theta \, d\theta \, d\varphi = 1 \text{ watt};$$

$$P_s = \iint D_s \sin\theta \, d\theta \, d\varphi = 1 \text{ watt} = 4\pi D_s; \qquad D_s = \frac{1}{4\pi};$$

$$R_a = \frac{(D_p)_{max}}{D_s} = 4\pi(D_p)_{max}$$

because the radiation advantage R_a of the actual radiator is the maximum power directivity divided by the uniform fixed directivity D_s of the standard.

128. Application of the Gain Expression to Radiator Systems and Discussion on Beaming. It is often of interest to know the gain G in a direction φ, θ, which is not the direction of maximum power directivity. It may be a direction of minor maximum effect, for instance, as for the side lobes shown in Fig. 106. Formula (844a) then changes to

$$G = 10 \log_{10} [12.56(D_p)_{max}] + 10 \log_{10}\left[\frac{D'_p}{(D_p)_{max}}\right] \text{db}$$
$$= 10 \log_{10} (12.56 D'_p) \quad \text{db} \quad (844b)$$

In the same way, (844) changes to

$$G = 10 \log_{10}\left(\frac{12.56}{\int_0^\theta \int_0^{2\pi} D_p \sin\theta \, d\theta \, d\varphi}\right) + 10 \log_{10}\left[\frac{D'_p}{(D_p)_{max}}\right] \quad \text{db} \quad (844c)$$

where D'_p is the power directivity in the desired direction.

The radiated powers P and P_s of an array and a single radiator of the array may also be so proportioned at the source as to cause the same field intensity at any point in the preferred direction, and

$$G = 10 \log_{10} (P_s/P)$$

decibels is then the gain of the array in the preferred direction over the single radiator.

We have already applied (841) to the case of an elementary and a half-wave linear radiator, respectively. Let us see how we can confirm the result of a half-wave radiator by step-by-step reasoning with respect to elementary areas dS of Fig. 136 on an expanding wave-front sphere and corresponding to an elementary solid angle $\sin\theta\, d\theta\, d\varphi$. The gain in a preferred direction must then be due to the ratio of the power per unit solid angle flowing in the direction of maximum radiation to the average power per unit solid angle flowing in all directions from the symmetry radiation center of a certain radiator system. Now the unit area in the mks system of

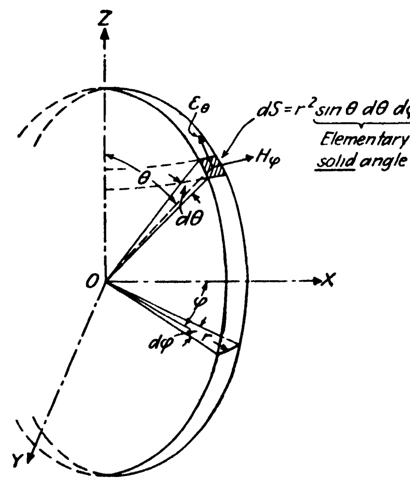


FIG. 136. Elementary solid angle and corresponding spherical surface element on a spherical wave front. O is the radiation center of the radiation system. Vertical slices of ever-increasing sphere reach into the distant radiation region. Integration from $\theta = 0°$ to $\theta = 180°$ and from $\varphi = 0°$ to $\varphi = 360°$ takes in the entire spherical surface $S = \int_\varphi \int_\theta dS = 4\pi r^2$. Such an integration over the surface S reaching through the distant radiation region can account only for the *radiated* useful power in watts.

units is 1 sq m. This is also the area to which the Poynting vector refers. If we define the unit solid angle subtended by 1-sq-m area on a sphere of radius $r = 1$ m, then the total solid angle subtended by *any* sphere must be $4\pi = 12.56$ radians. On a sphere of r meters radius due to an area of S square meters, we have then a corresponding solid angle of S/r^2. The Poynting vector p is the power in watts flowing normally across 1 sq m of the spherical surface if the radiation center of the entire radiation system coincides with the center of the sphere shown in Fig. 136. Therefore, across any spherical surface of $4\pi r^2$ square meters, we have a power transfer of pr^2 watts per unit solid angle. According to (835) and (838b), the product of the Poynting vector p and of r^2 represents the power directivity D_p, known also as the radiation intensity in any φ, θ direction. The value of D_p watts per unit solid angle then generally depends on this direction, since we *cannot* physically realize an isotropic radiator. We always have then a variable directivity factor, at least in the meridian plane of a linear radiator, *i.e.*, a D_θ factor. Generally we have a $D_{\varphi,\theta}$ factor.

If p_{max} is the value of the Poynting vector along the preferred direction (along the direction of maximum radiation effect) and P_r is the total radiated power in watts, the quantity $P_r/(4\pi)$ must be the mean, or average value of the radiated power per solid radian, since we have 4π solid radians for the entire sphere. Since $(D_p)_{max} = r^2 p_{max}$ watts per unit solid angle (along the preferred direction) and $(D_p)_{avg} = r^2 p_{avg}$ and $p = \mathcal{E}_\theta \cdot H_\varphi = \mathcal{E}_\theta^2/Z_0$ watts per square meter, where $\mathcal{E}_\theta$ volts per meter and H_φ amperes per meter represent the respective effective field values at any r, φ, θ location, we find the radiation advantage

$$R_a = \frac{(D_p)_{max}}{(D_p)_{avg}} = \frac{p_{max}}{p_{avg}} = \frac{(\mathcal{E}_\theta{}^2)_{max}/Z_0}{p_{avg}} = \frac{\mathcal{E}^2_{max}}{Z_0 p_{avg}}; \qquad Z_0 \cong 376.7 \text{ ohms for air}$$

Since, for any sphere, we have a surface of $4\pi r^2$ square meters, the average surface power density must be the total radiated power of P_r watts divided by this area, or

$$p_{avg} = \frac{P_r}{4\pi r^2} = \frac{\int\int_S p\,dS}{4\pi r^2} = \frac{\int_0^\pi \int_0^{2\pi} \frac{\mathcal{E}_\theta{}^2}{Z_0} r^2 \sin\theta\,d\theta\,d\varphi}{4\pi r^2} = \frac{\left|\varphi\right|_0^{2\pi} \cdot \int_0^\pi \frac{\mathcal{E}_\theta{}^2}{Z_0} \sin\theta\,d\theta}{4\pi}$$

$$= \frac{2\pi \int_0^\pi \frac{\mathcal{E}_\theta{}^2}{Z_0}\sin\theta\,d\theta}{4\pi} = \frac{1}{2Z_0}\int_0^\pi \mathcal{E}_\theta{}^2 \sin\theta\,d\theta = \frac{1}{2Z_0}\int^\pi KD_\theta{}^2 \sin\theta\,d\theta$$

where K is a proportionality constant depending upon the excitation of the linear radiator. For half-wavelength excitations, we have, for even and odd multiples of a half-wavelength distribution, $\sqrt{K} = j60I_e/r$, or an absolute magnitude of K equal to $3{,}600I_e^2/r^2$, where I_e denotes the effective radiator current at a current antinode. In the R_a expression, $\mathcal{E}_{max}$ stands for the maximum field intensity value in the preferred direction. Since we took $\mathcal{E}_\theta$ as the effective volts per meter in any φ, θ direction, the $\mathcal{E}_{max}$ quantity is likewise in effective volts per meter, but along the preferred direction and such that

$$\mathcal{E}_{max} = (\mathcal{E}_\theta)_{max} = \sqrt{K}\,(D_\theta)_{max}$$

For a half-wave linear radiator, we have the meridian directivity

$$D_\theta = \frac{\cos(90° \cos\theta)}{\sin\theta}, \qquad \text{and} \qquad (D_\theta)_{max} = D_{90°} = \frac{\cos(90° \cos 90°)}{\sin 90°} = 1;$$

$$\mathcal{E}^2_{max} = K(D_\theta{}^2)_{max} = K$$

Hence,

$$R_a = \frac{\mathcal{E}^2_{max}}{Z_0 p_{avg}} = \frac{K}{Z_0 \frac{1}{2Z_0}\int_0^\pi K \frac{\cos^2(90° \cos\theta)}{\sin^2\theta}\sin\theta\,d\theta} = \frac{2}{\int_0^\pi \frac{\cos^2(90° \cos\theta)}{\sin\theta}\,d\theta}$$

$$= \frac{2}{\frac{1}{2}[\log_\epsilon(2\pi\,1.7811) + \log_\epsilon(k) - \text{Ci}(2\pi k)]}$$

since for any integer half-wavelength current distribution, such as $k = 1, 2, 3, \cdots$ or $l = k\lambda/2$, we find $\int_0^\pi = \log_\epsilon(2\pi\,1.7811k) - \text{Ci}(2\pi k)$, and

$$\left.\begin{aligned} &\log(ab) = \log(a) + \log(b) \\ &\int_0^\pi = \tfrac{1}{2}[\log_\epsilon(2 \times 1.7811\pi) + \log_\epsilon(k) - \text{Ci}(2\pi k)] \\ &\qquad = \tfrac{1}{2}[2.43 + \log_\epsilon(k) - \text{Ci}(2\pi k)] \end{aligned}\right\} \quad (845)$$

and for a half-wave radiator, for which $k = 1$ and $\int_0^\pi = 2.43/2 = 1.215$, we find

$$R_a = \frac{2}{1.215} = 1.64; \qquad G = 10 \log_{10} (1.64) = 10 \times 0.2148 = 2.148 \text{ db}$$

which checks the result obtained on page 624, using formula (841), by means of the radiation advantage ratio $R_a = (D_\theta^2)_{max}/[(D_\theta^2)_{avg}]$. Figure 137 shows the solid pattern for the $\lambda/2$ radiator, based on the meridian cross-section pattern D_θ^2, causing,

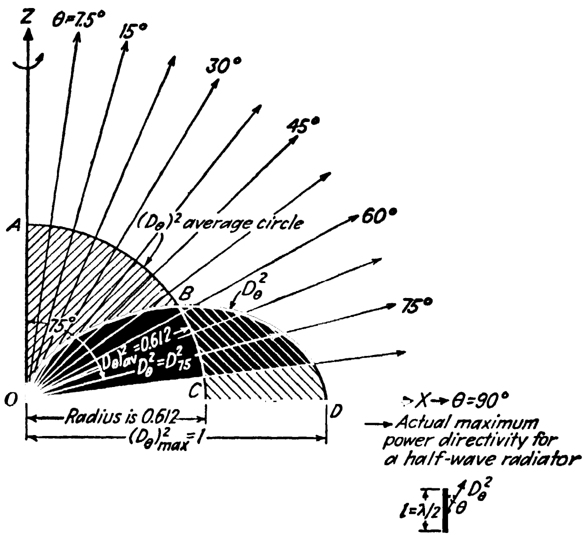


FIG. 137. Actual power directivity D_θ^2 of a half-wave radiator and equivalent power directivity $D_{\theta avg}^2$ for an isotropic radiator with the same energy content. $D_{\theta avg}^2$ is the equivalent power directivity for an omnidirectional radiator. This pattern is a surface of rotation for the space pattern and shown only for the ZX quadrant cross section. Shaded areas OAB and BCD are equal.

for the $D_{\varphi,\theta}^2$ pattern (space pattern), a surface of rotation about the linear ½λ radiator. The $(D_\theta^2)_{avg}$ pattern can also be obtained by finding the area of the D_θ^2 pattern by means of a planimeter and equating this area to a circular area πR^2, where $R = (D_\theta^2)_{avg}$.

If we now use two half-wave radiators spaced along the X axis, each radiator parallel with the Z axis, we can, according to (822), (822*a*), and (822*b*), compute the space pattern $D_{\varphi,\theta}$. We have then the comparisons for the space characteristic $D_{\varphi,\theta} = D_\theta D_g = D_\theta \cdot 1 = D_\theta$ of a single half-wave radiator.

$$\left.\begin{aligned} \mathcal{E}_P &= j60 \frac{I_e}{r} D_\theta \underbrace{D_x D_y D_z}_{= D_g} = j60 \frac{I_e}{r} D_\theta D_x && \text{for two } \frac{\lambda}{2} \text{ radiators} \\ \mathcal{E}_P &= j60 \frac{I_e}{r} D_\theta \underbrace{D_x D_y D_z}_{= D_g} = j60 \frac{I_e}{r} D_\theta && \text{for one } \frac{\lambda}{2} \text{ radiator} \end{aligned}\right\} \text{effective volts/m}$$

$$D_\theta = \frac{\cos(90^\circ \cos\theta)}{\sin\theta}; \qquad D_x = \frac{\sin(2a)}{\sin(a)};$$

$$a = 180^\circ \frac{x}{\lambda}\cos\varphi \sin\varphi + 180^\circ \frac{\tau_x}{T}; \qquad D_y = 1 = D_z$$

since the radiation centers of the two lined-up half-wave radiators are along the X axis and we have $l = \lambda/2$ directions parallel with the Z axis. We know already from Fig. 137 that the isotropic reference radiator has only an equatorial power directivity that is 0.612 if we call the corresponding equatorial directivity of the single half-wave radiator unity, yielding a radiation advantage of $1/0.612 = 1.63 = R_a$ of the half-wave radiator over the isotropic radiator in the preferred equatorial plane, which for the half-wave radiator in unobstructed radiation space happens to make the equatorial plane coincide with the preferred surface of maximum radiation effect. Hence, it would take 1.63 times the power for an isotropic radiator to give the same effect as one times the power will give for a half-wave radiator in the direction of maximum radiation. We can, therefore, find this power ratio by graphical integration as in Fig. 137 if we plot the D_θ^2 pattern in a meridian plane that coincides with the direction of maximum radiation. This assumes, however, that we have a circular polar pattern about the radiator system, as is the case for a single linear radiator. The direction of maximum radiation can be found from the $D_{\varphi,\theta}$ expression, *i.e.*, the space directivity of the field intensity, since the power space directivity is the square of it. We shall now see how this is done. Suppose we space the two half-wave radiators a distance $x = \frac{1}{4}\lambda$ apart along the X direction and phase the radiator behind the other radiator so that the half-wave radiator farthest out along the X axis has a cos ωt current function while the other radiator has a sin ωt current time function. Then we have

$$x = \frac{\lambda}{4}; \qquad \tau_x = -\frac{T}{4}; \qquad a = \frac{1}{4}\,180^\circ \cos\varphi \sin\theta - \frac{1}{4}\,180^\circ$$
$$= 45^\circ \cos\varphi \sin\theta - 45^\circ$$

$$D_x = \frac{\sin(2a)}{\sin(a)} = \frac{2\sin(a)\cos(a)}{\sin(a)} = 2\cos a$$
$$= 2\cos(45^\circ \cos\varphi \sin\theta - 45^\circ)$$

$$D_y = 1; \qquad D_z = 1$$

since there is no alignment along the Y and Z axis.

We have, at any space point $P(r, \varphi, \theta)$, an effective field intensity

$$\mathcal{E}_P^{(\text{volts/m})} = j60\frac{I_e}{r}D_\theta D_g = j60\frac{I_e}{r}D_\theta D_x$$
$$= j120\frac{I_e}{r}\underbrace{\frac{\cos(90^\circ\cos\theta)}{\sin\theta}\cos(45^\circ\cos\varphi\sin\theta - 45^\circ)}_{\text{space field directivity } (D_{\varphi,\theta})/2}$$

The radiated power in the different φ, θ directions is, therefore, proportional to $D_{\varphi,\theta}^2$. Hence, when $D_{\varphi,\theta}$ becomes a maximum, $D_{\varphi,\theta}^2$ also must be a maximum. It is not a pleasant job to find the average value of $D_{\varphi,\theta}^2$ by integration, especially for more than two radiators and in all three space directions. We note that the above expression can be greatly simplified if we make $\theta = 90$ deg, *i.e.*, go into the equatorial XY plane. We find then

$$D_{\varphi,\theta} = D_{\varphi,\theta=90} = 2\,\frac{\cos\,(90^\circ \cos 90^\circ)}{\sin 90^\circ} \cos\,(45^\circ \cos \varphi \sin 90^\circ - 45^\circ)$$
$$= 2 \cos\,(45^\circ \cos \varphi - 45^\circ) = D_\varphi$$

and have only a field directivity D_φ, which is independent of θ. Now D_φ yields a maximum when $\cos\,(45^\circ \cos \varphi - 45^\circ)$ yields a maximum, *i.e.*, if the cosine function is unity. This happens when $45^\circ \cos \varphi - 45^\circ = 0$ deg or $\cos \varphi = 1$, *i.e.*, $\varphi = 0$ deg. Therefore, $D_{\varphi=0^\circ} = 2$ and $\mathcal{E}_P = j120I_L/r$ effective volts per meter in the XY plane and along the X direction.

Figure 138 gives the equatorial D_φ^2 pattern in comparison with the pattern for only one vertical half-wave radiator. There is no longer a symmetry with respect to the Y directions, but a symmetry with respect to the X directions, so that a plot of the D^2 pattern in the XZ plane gives the additional information of the total space pattern $D_{\varphi,\theta}^2 = D_\theta^2 D_x^2$, where D_θ represents the field directivity $[\cos\,(90^\circ \cos \theta)]/\sin \theta$ holding for one half-wave radiator. The pattern in the XZ plane is given for $\varphi = 0$, and we have

$$D_{\theta,\varphi=0}^2 = 4\,\frac{\cos^2\,(90^\circ \cos \theta)}{\sin^2 \theta} \cos^2\,[45^\circ(\sin \theta - 1)]; \qquad (D_{\theta,\varphi=0}^2)_{\max} = 4$$

as against $(D_\theta)_{\max} = 1$ in the case of a single half-wave radiator. We also note that the $-X$ direction is the only direction in which we have zero field effect. We therefore can not use the graphical method for finding the average value. We must now use the method leading to formulas (844), (844*a*), (844*b*), and (844*c*). In Fig. 138, the meridian power directivity pattern in the preferred XZ plane is also shown, and Table 28 gives details with respect to these patterns. When arrays are made up of linear radiators of the same electrical lengths, such as half-wave radiators, then it is convenient to express the beaming effect of the array in terms of one individual half-wave radiator. We know that the single half-wave radiator has a radiation advantage of $R_a = 1.64$. From the maximum radiation effect along the $+X$ direction for one half-wave radiator and two half-wave radiators, spaced $\frac{1}{4}\lambda$ along the X direction and phased $\frac{1}{4}T$, we note that $D_{\max} = 4$ for two radiators and $D_{\max} = 1$ for one radiator. We have, therefore, along the X axis a radiation advantage $R_a = 4$ of the two-radiator system over a single half-wave radiator with a gain of $G = 10 \log_{10} (4) = 10 \times 0.6021 = 6.021$ db. The more radiators in the alignment the more such a gain is emphasized. This

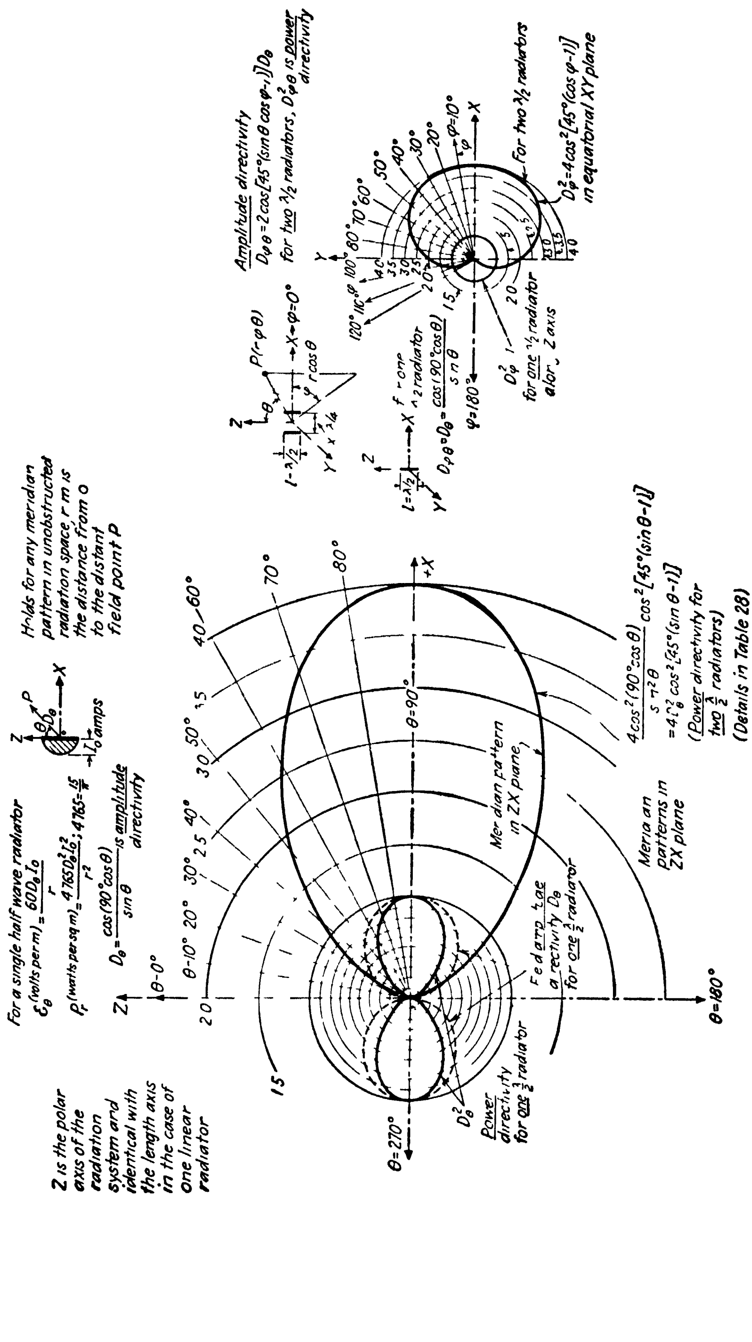


FIG. 138. Meridian, equatorial and general space relative directivity of two half-wave radiators (vertical) in comparison with a single half-wave radiator Note A spacing of $x = \lambda/4$ m refers to a 90-time-degree lead of the current phase for the front radiator (on the $+X$ axis).

can be realized from (822), (822*a*) and (822*b*), since, for the general space lattice alignment, we have the total directivity $D_{\varphi,\theta} = D_\theta D_g$, according to (822), where D_θ is the unobstructed space directivity of the individual radiator, which can also be written D_e, since it is due to only one element of equal radiators. This directivity D_e may be also a φ, θ function but is a θ function only for alignments using half-wave radiators as well as curtains made up of $\frac{1}{2}\lambda$ radiators, since we have circular symmetry about one linear radiator. According to (822), the group characteristic D_g is the product of the D_x, D_y, and D_z directivities along the respective Cartesian alignments of the space lattice structure of individual radiation centers.

TABLE 28

θ, deg	sin θ numeric	sin θ − 1 = q numeric	45°q, deg	cos (q45°) numeric	cos² (q45°) numeric	4 cos² (q45°)	D_θ^2*	$D_{\theta,\varphi}^2 = 0°$
0	0 0000	−1 0000	−45	0 7071	0 4998	1 99	0 0000	0 0000
10	0 1736	−0 8264	−37	0 7986	0 6384	2 55	0 0206	0 0525
20	0 342	−0 658	−29 6	0 8704	0 7569	3 03	0 0784	0 237
30	0 5	−0 5	−22 5	0 9239	0 854	3 41	0 1739	0 59
40	0 6428	−0 3572	−16 1	0 9608	0 9235	3 69	0 3091	1 14
50	0 766	−0 234	−10 5	0 9833	0 966	3 87	0 4802	1 86
60	0 866	−0 134	− 6 04	0 9945	0 988	3 95	0 6659	2 63
70	0 9397	−0 0603	− 2 8	0 9988	0 997	3 99	0 8317	3 32
80	0 9848	−0 0152	− 0 685	0 9999	0 998	3 999	0 9604	3 84
90	1 0000	0 0000	0 000	1 0000	1 0000	4 000	1 0000	4 0000

* The values of the D_θ^2 column are taken from Table 25

$$D_{\theta,\varphi=0°} = 4\frac{\cos^2(90° \cos\theta)}{\sin^2\theta}\cos^2[45°(\sin\theta - 1)], \qquad D_\theta = \frac{\cos(90° \cos\theta)}{\sin\theta}$$

Remarks Values for the power directivity in the XZ plane for two half-wave radiators parallel with the Z axis and lined up along the X axis with a spacing of $\frac{1}{4}\lambda$ between radiators and a $T/4$ phasing (FIG. 138 gives the characteristic in XZ plane in comparison with the characteristic in the XY plane).

According to (822*b*), these factors have the form $D = [\sin(nx)]/[\sin(x)]$, which can often be developed into a cosine product function. Suppose we have $n = 8$ radiators, then

$$D = \frac{\sin(nx)}{\sin(x)} = \frac{\sin(8x)}{\sin(x)} = \frac{2\sin(4x)\cos(4x)}{\sin(x)} = \frac{2 \times 2\sin(2x)\cos(2x)\cos(4x)}{\sin(x)}$$
$$= \frac{2 \times 2 \times 2\sin(x)\cos(x)\cos(2x)\cos(4x)}{\sin(x)} = 8\cos(x)\cos(2x)\cos(4x)$$

This shows a possibility of a maximum effect that is *eight*fold, causing 64 fold D^2 effect. For $n = 6$, we have not such a simple result at first sight, since

$$D = \frac{\sin(6x)}{\sin(x)} = \frac{2\sin(3x)\cos(3x)}{\sin(x)} = \frac{2[3\sin(x) - 4\sin^3(x)]\cos(3x)}{\sin(x)}$$
$$= 2[3 - 4\sin^2(x)]\cos(3x) \text{ etc.}$$

But we know from pages 582–583 that the group effect of n equal radiators is

$$\frac{\sin (na)}{\sin (a)} = \frac{\sin (n\delta/2)}{\sin (\delta/2)} = 1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta} + \cdots + \epsilon^{j(n-1)\delta}$$

$$\delta = 360^\circ \left(\frac{s}{\lambda} q \pm \tau f\right) \text{deg}$$

where f is in cycles per second. For $n = 6$ like radiators, which are equally spaced along a straight line, we have $(n - 1)\delta = (6 - 1)\delta = 5\delta$, yielding a group factor

$$D_g = \frac{\sin (6\delta/2)}{\sin (\delta/2)} = \frac{\sin \left[6 \times 180^\circ \left(\frac{s}{\lambda} q \pm \tau f\right)\right]}{\sin \left[180^\circ \left(\frac{s}{\lambda} q \pm \tau f\right)\right]} = 1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta}$$

$$+ \epsilon^{j4\delta} + \epsilon^{j5\delta} = 1 + \cos (\delta) + \cos (2\delta) + \cos (3\delta) + \cos (4\delta) + \cos (5\delta) + j[\quad];$$
$$[\quad] = \sin (\delta) + \sin (2\delta) + \sin (3\delta) + \sin (4\delta) + \sin (5\delta)$$

with a *possible maximum effect that is sixfold*, since each cosine term may become unity and, when added to 1, yields 6. Inasmuch as with a lattice structure, say with $n_x = 5$ along the X direction, $n_y = 6$ along the Y direction, and $n_z = 4$ along the Z directions, we have for equal radiators that are all excited in phase and equally spaced in each Cartesian direction three such group factors, namely,

$$D_g = D_x D_y D_z = (1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta})(1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta} + \epsilon^{j5\delta})(1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta})$$

$$= \Big[1 + \underbrace{\cos (\delta) + \cos (2\delta) + \cos (3\delta) + \cos (4\delta)}_{= \sum_{m=1}^{m=4} \cos (m\delta)} + j \sum_{m=1}^{m=4} \sin (m\delta)\Big] F;$$

$$F \equiv \left\{1 + \sum_{m=1}^{m=5} [\cos (m\delta) + j \sin (m\delta)]\right\} \left\{1 + \sum_{m=1}^{m=3} [\cos (m\delta) + j \sin (m\delta)]\right\}$$

Inphase currents means that in the δ expression, we have the timing $\tau = 0$ sec (no time lapse), and we have only $\delta = 360^\circ\ (s/\lambda)q$. Since for the D_x partial group directivity, $sq = x \sin \theta \cos \varphi$, for D_y, we have $sq = y \sin \theta \sin \varphi$ and, for D_z, we have $sq = z \cos \theta$, we may have a condition for which all respective δ values are either 360 deg or any integer multiple thereof. Hence, for a condition that renders

$$\frac{x}{\lambda} \sin \theta \cos \varphi = \frac{y}{\lambda} \sin \theta \sin \varphi = \frac{z}{\lambda} \cos \theta = 1, 2, 3, 4, \cdots$$

we have $\delta = 360^\circ, 2 \times 360^\circ, 3 \times 360^\circ, 4 \times 360^\circ, \cdots$, and

$$\sum_{m=1}^{m=5} \cos (m\delta) = \underbrace{\cos (360^\circ)}_{=1} + \underbrace{\cos (2 \times 360^\circ)}_{=1} + \underbrace{\cos (3 \times 360^\circ)}_{=1} + \underbrace{\cos (4 \times 360^\circ)}_{=1} + \underbrace{\cos (5 \times 360^\circ)}_{=1} = 5$$

$$\sum_{m=1}^{m=5} \sin (m\delta) = \underbrace{\sin (360°)}_{=0} + \underbrace{\sin (2 \times 360°)}_{=0} + \cdots = 0$$

$$D_x = (1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \epsilon^{j4\delta}) = (1 + 4 + j4 \times 0) = 5;$$

$$D_y = (1 + 5 + j5 \times 0) = 6$$

$$D_z = (1 + 3 + j3 \times 0) = 4; \qquad D_g = D_xD_yD_z = 5 \times 6 \times 4 = 120$$

Therefore, a 120 fold field over that of a single half-wave radiator is a possibility, and the power advantage in the preferred direction (maximum radiation) would be the square of this value. This would happen with the assumptions made. The assumptions are that each half-wave radiator is alike and has the *same* current at the current antinode in the *presence of all other current excitations* of the space lattice array of half-wave radiators and *inphase* current *excitations.* The same result can also be found, since the Poynting vector in any φ, θ direction for an alignment along the X direction, with $\lambda/2$ radiators parallel with the Z axis, is

$$p = p_{\varphi,\theta} = 9.55 \frac{I_e^2}{r^2} [D_\theta^2 D_x^2 = 9.55] \frac{I_e^2}{r^2} \frac{\cos^2 (90° \cos \theta)}{\sin^2 \theta} \frac{\sin^2 (n_x a)}{\sin^2 (a)} \quad \text{watts/m}^2$$

We note that in this expression, D_θ gives a maximum value for $\theta = 90$ deg and $(D_\theta)_{\max} = 1$, and its square is likewise unity. We find the maximum for D_x by differentiating and equating to zero. Hence,

$$\frac{d[D_x]}{da} = \frac{d\left[\dfrac{\sin (na)}{\sin (a)}\right]}{da} = \left|\frac{n_x \cos (n_x a) \sin (a) - \sin (n_x a) \cos (a)}{\sin^2 (a)}\right| = 0$$

$$n_x \cos (n_x a) \sin (a) = \sin (n_x a) \cos (a); \qquad n_x \tan (a) = \tan (n_x a)$$

$$\sin (n_x a) = \frac{n_x \cos (n_x a) \sin (a)}{\cos (a)};$$

$$D_x = \frac{\sin (n_x a)}{\sin (a)} = \frac{n_x \cos (n_x a) \sin (a)}{\cos (a) \sin (a)} = n_x \frac{\cos (n_x a)}{\cos (a)}$$

$$p_{\max} = 9.55 \frac{I_e^2}{r^2} \underbrace{[D_\theta^2]_{\max}}_{=1}[D_x^2]_{\max} = 9.55 \frac{I_e^2}{r^2} \left[n_x^2 \frac{\cos^2 (n_x a)}{\cos^2 (a)}\right]_{\max}$$

For inphase excitations with the effective current of I_e amperes at the center of each half-wave radiator, we find

$$a = 180° \left(\frac{x}{\lambda} \sin \theta \cos \varphi + \underbrace{r_x f}_{=0}\right) = 180° \left(\frac{x}{\lambda} \sin 90° \cos \varphi\right) = 180° \frac{x}{\lambda} \cos \varphi;$$

$$\left[n_x^2 \frac{\cos^2 (n_x a)}{\cos^2 (a)}\right]_{\max} = n_x^2 \frac{\cos^2 \left(n_x\, 180° \dfrac{x}{\lambda} \cos 90°\right)}{\cos^2 \left(180° \dfrac{x}{\lambda} \cos 90°\right)} = n_x^2$$

We have broadside beaming along the Y directions with a power directivity advantage that is n_x^2 fold. This advantage refers to the *maximum* effect of one linear half-wave radiator. From the derivation on pages 582–583, we know that for n equal radiators in an alignment, we have

$$D_x = \frac{\sin (na)}{\sin (a)} = \frac{\sin\left(n\frac{\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)} = 1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \cdots + \epsilon^{j(n-1)\delta};$$

$$\delta = 360°\left(\frac{s}{\lambda}\sin\theta\cos\varphi \underbrace{\pm \tau f}_{\text{timing}}\right), \qquad a = \frac{\delta}{2}$$

for the spacing s and the alignment along the X direction with a maximum effect for $\delta = 0$, since, for n equal radiators, $D_x = 1 + (n + 1) = D_{\max}$. For five radiators, we have $D_{\max} = 1 + (5 - 1) = 5$, *i.e.*, fivefold field effect with respect to the single linear radiator. The condition $\delta = 0$ occurs if

$$\frac{s}{\lambda}\sin\theta\cos\varphi \pm \tau f = 0 \qquad \text{or} \qquad \frac{s}{\lambda}\sin\theta\cos\varphi = \mp \tau f$$

Since we assumed the alignment along the X axis, we have $\sin\theta = \sin 90° = 1$, and

$$\frac{s}{\lambda}\cos\varphi = \mp \tau f$$

which, for inphase current excitations of the various radiators, causes $\tau = 0$ sec. Hence, $(s/\lambda)\cos\varphi = 0$ or $\varphi = 90$ deg, which is *broadside beaming* along the $\pm Y$ directions, since $\varphi = 270$ deg also satisfies the $(s/\lambda)\cos 270° = 0$ condition. On the other hand, if $\varphi = 0$ deg, the condition $(s/\lambda)\cos 0° = \mp\tau f$ requires that $\mp\tau f = \mp \tau/T = s/\lambda$, or beaming along the $+X$ direction requires that the spacing s in terms of the wavelength λ is either equal to $-\tau f$ or equal to $-\tau f$, since

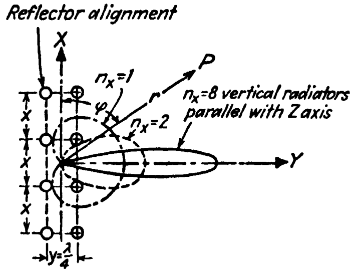
$$\left.\begin{array}{ll} \delta = 0 = 2\pi\underbrace{\left(\frac{s}{\lambda}\sin 90°\cos 0° \pm \tau f\right)}_{=0} & \text{yields } D_{\max} = 1 + (n - 1) \\ 2\pi\frac{s}{\lambda} = \beta s = \mp\omega\tau & \text{"end-firing" condition along one of the } X \text{ directions} \end{array}\right\} \qquad (846)$$

If we apply this formula to two half-wave radiators lined up along the X axis, for $s = \lambda/4$ and such a phasing that the front radiator (farthest along the $+X$ axis) is one-quarter of a period ahead, then $\tau = +T/4 = +1/(4f)$, and

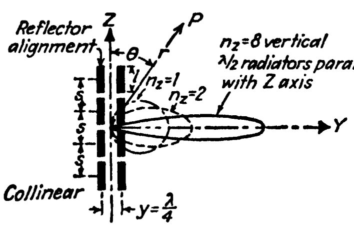
$$\beta s = \frac{2\pi}{\lambda} s = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}; \qquad \omega\tau = 2\pi f\tau = 2\pi f \frac{1}{4f} = \frac{\pi}{2};$$

$$\delta = 2\pi\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0; \qquad D_{\max} = 1 + (2 - 1) = 2$$

This is "end firing" along the $+X$ direction (axis) with double the field amplitude that one radiator would show all around its equatorial plane. We note that the "broadside firing" along the Y axis and the end firing along the X axis give the same radiation advantage over a single half-wave



P is the field point in the equatorial XY plane. Indicated alignments show only $n_x = 4$ radiators; along the Y direction, we have $n_y = 2$ radiators, and along the Z direction normal to the paper, we have $n_z = 1$ radiator.



P is the field point in the meridian ZY plane, $l = \lambda/2$, for instance. Indicated alignments parallel with the Z axis show only $n_z = 4$ radiators; along the Y direction, we have $n_y = 2$ radiators; along the X axis normal to the paper, we have $n_x = 1$ radiator. $\varphi = 90°$ plane.

FIG. 139. Beaming (unilateral) due to equal-phased front-row radiators and a backup reflector alignment.

radiator. This also holds true if the end-firing requirement $\varphi = 0$ deg or $\beta s = \mp\omega\tau$ and the broadside requirement $\varphi = 90$ deg or $\beta s \cos 90° = 0 = \omega\tau$ is *not* met, but

$$\delta = 2\pi\left(\frac{s}{\lambda}\sin 90° \cos\varphi \pm \tau f\right) = 0 \text{ yields } D_{\max} = 1 + (n - 1);$$

$$\frac{s}{\lambda}\cos\varphi = \mp\tau f \qquad (846a)$$

We then obtain a $D_{\max}$ effect in the equatorial XY plane, which is in a direction *somewhere between the* X *and* Y directions, since φ can have any values between 0 and 90 deg. Since end firing is suppressed beaming either along the $-X$ or along the $+X$ direction, respectively, but broadside firing is along both the $+Y$ and the $-Y$ directions, the general case of (846a), which occurs for angles other than $\varphi = 0$ and 90 deg, must mean a condition between end and broadside firing as far as the forward

and backward radiation is concerned. Since in the broadside case, we prevent "back" radiation only in the very opposite direction of the beam, alignments such as indicated in Fig. 139 with more than $n_x = 1$ have to be used in order to provide only forward beaming toward the Y direction. Here n_x denotes, according to (822b) and Fig. 128, the number of individual radiation centers per alignment along the X direction, while n_z is the number along the Z direction. The reflector alignment causes the broadside effect, *i.e.*, energy concentration toward the Y axis, in either case.

Since, for $n_x = n$, $\tau_x = \tau$, and $x = s$,

$$D_x = \frac{\sin(na)}{\sin(a)} = \frac{\sin\left(n\frac{\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)} = 1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \cdots + \epsilon^{j(n-1)\delta} = \frac{\epsilon^{jn\delta} - 1}{\epsilon^{j\delta} - 1}$$

and similarly for the D_y and D_z components, we must have zero field effects whenever these components vanish. For the D_x component, this happens if $\epsilon^{jn\delta} = 1$, *i.e.*, if

$$\epsilon^{jn\delta} = \cos(n\delta) + j\sin(n\delta) = 1; \qquad n\delta = 2\pi q \qquad \text{for } q = 1, 2, 3, 4, 5, \cdots$$

Therefore, in the XY plane,

$$n(\beta s\cos\varphi - \omega\tau) = \pm\, 2\pi q; \qquad \beta s\cos\varphi - \omega\tau = \pm\,\frac{2\pi q}{n}$$

where $\omega\tau$ denotes the phase lag between consecutive radiator excitations. We find

$$\frac{2\pi q}{n} = \beta s(1 - \cos\varphi) \qquad \text{for } \textit{end-fire} \text{ beaming}$$

$$\frac{2\pi q}{n} = \beta s\cos\varphi \qquad \text{for } \textit{broadside} \text{ beaming}$$

for end-fire effect

$$\frac{q\lambda}{ns} = 1 - \cos\varphi = 2\sin^2\left(\frac{\varphi}{2}\right) \text{ or } \varphi = 2\sin^{-1}\left(\sqrt{\frac{q\lambda}{2ns}}\right)$$

For broadside excitation, the angles for zero field effect are

$$\varphi = \cos^{-1}\frac{q\lambda}{ns} \qquad \text{deg}$$

Since we know the number $n = n_x$ of radiator elements with individual radiation centers along the X axis and a spacing $s/\lambda = x/\lambda$, we can, for the values $q = 1, 2, 3, \cdots$, find the angles φ with the X axis for which $D_x = 0$. We find for the width w of the main beam

$$w = 2\varphi_m = 4\sin^{-1}\left(\sqrt{\frac{\lambda}{2ns}}\right) \qquad \text{deg, for } \textit{end-fire} \text{ beaming}$$

$$w = 2\cos^{-1}\left(\frac{\lambda}{ns}\right) \qquad \text{deg, for } \textit{broadside} \text{ beaming}$$

We have seen that for half-wave radiators with the same effective current of I_e amperes at the current antinode, we have, for n equally spaced radiators of an alignment, a maximum power directivity that is n^2 times the maximum power directivity of one half-wave radiator. Inasmuch as each radiator receives only $1/n$ of the total power that one radiator would receive if it were the only radiator, the radiation advantage R_a is only n fold and we have, for the *case* of *equipartition*, a gain of $G = 10 \log_{10} (n)$ decibels with respect to that of a single half-wave radiator. Therefore, doubling the number of radiators increases the gain by $10 \log_{10} (2) = 10 \times 0.3010 = 3.01$ db. Since, for an alignment of n equal radiators, we have a field[1] directivity, for a spacing s

$$D_x = D_s = \frac{\sin (na)}{\sin (a)} = \frac{\sin (n\delta/2)}{\sin (\delta/2)} = 1 + \epsilon^{j\delta} + \epsilon^{j2\delta} + \epsilon^{j3\delta} + \cdots + \epsilon^{j(n-1)\delta}$$

$$\delta = 2\pi \left(\underbrace{\frac{s}{\lambda} \sin 90^\circ}_{\substack{\text{in equa-} \\ \text{torial } XY \\ \text{plane}}} \cos \varphi - \tau f \right) = \beta s \cos \varphi - \omega\tau; \qquad a = \frac{\delta}{2}$$

the value of the field directivity D_x (which is the group characteristic of an alignment) must vanish at $\delta = 360/n,\ 2 \times 360/n,\ 3 \times 360/n, \cdots (n - 1)360/n$ degrees if n equal radiators are in the alignment. Therefore, if $n = 6$, we have the first zero field value at $\delta = 360/6 = 60$ deg; the second zero field effect occurs for $\delta = 120$ deg followed up for $\delta = 180$, 240, and 300 deg. Therefore, between these angles, we have radiation beams also known as "lobes." As the angle φ varies from 0 deg (along $+X$ direction) to the other extreme of 180 deg (along $-X$ direction), the delay δ varies from $\beta s - \omega\tau$ to $\beta s + \omega\tau$. We have then only one maximum radiation beam, which is $360/(2n)$ degrees wide, since there are zero field effects at $\delta = 360/n$ and $-360/n$ degrees. We have also $(n - 2)$ secondary radiation beams, which decrease with increasing angle from the orientation of $(D_x)_{max}$ that occurs for $\delta = 0$ deg. This holds if the variation from $\beta s - \omega\tau$ to $\beta s + \omega\tau$ covers the variation from -180 to $+180$ deg. If the spacing s between consecutive radiators exceeds $\frac{1}{2}\lambda$ for broadside beaming or $\frac{1}{4}\lambda$ for end firing, we must expect secondary radiation beams that become more and more prominent. The secondary radiation beams (lobes) are usually undesirable for practical work.

We have already defined what is meant by a solid space angle, since $\sin \theta \, d\theta \, d\varphi$ represents an elementary solid angle. Radiation beams are

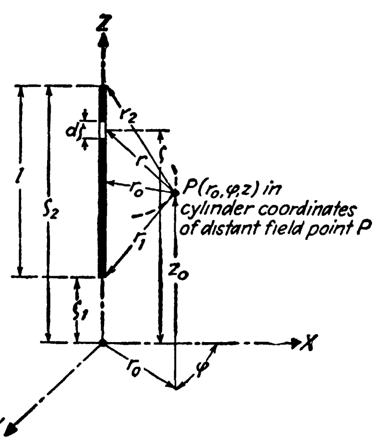
[1] We have to distinguish between the directivity D with which the electric as well as the magnetic field amplitude varies for different space locations and the directivity D^2 with which the *radiated power* acts at such space locations.

within tangential cones in some cases (for curtains, for instance, in the case of equal phasing). If radiation power is uniformly distributed in a cone of angle ψ, we have the solid angle equal to $2\pi \int_0^{\psi} \sin \psi \, d\psi = 2\pi(1 - \cos \psi)$ with a radiation advantage $R_a = 4\pi$ divided by the solid angle, or

$$R_a = \frac{4\pi}{2\pi(1 - \cos \psi)} = \frac{2}{1 - \cos \psi} = \frac{1}{(1 - \cos \psi)/2}$$

$$= \frac{1}{\sin^2 (\psi/2)}; \qquad G = 10 \log_{10} \left(\frac{1}{\sin^2 (\psi/2)} \right) \quad \text{db}$$

This is then an application of (844) and the first term of (844c).



$$r = \sqrt{r_0^2 + (z - \zeta)^2}, \quad u = r + \zeta - z; \quad \frac{du}{d\zeta} = \frac{u}{r}$$

since

$$\frac{du}{d\zeta} = \frac{d(r + \zeta - z)}{d\zeta}$$

$$= \frac{d[(r_0^2 + z^2 - 2z\zeta + \zeta^2)^{1/2} + \zeta - z]}{d\zeta}$$

$$= \tfrac{1}{2}(\underbrace{r_0^2 + z^2 - 2z\zeta + \zeta^2}_{})^{-1/2}(-2z + 2\zeta) + 1$$

$$= \frac{1}{r}$$

$$= \frac{1}{2r}(2\zeta - 2z) + 1$$

$$= \frac{\zeta - z}{r} + 1$$

$$= \frac{u - r}{r} + 1$$

$$= \frac{u - r + r}{r} = \frac{u}{r}$$

In the same way for $v = r - \zeta + z$,

we have $\frac{dv}{d\zeta} = -\frac{v}{r}$

FIG. 140. Conditions for cylindrical surfaces of actions passing through a field point $P(r_0, \varphi, z)$.

129. Notes on Integral Solutions Occurring for Linear Radiators. This has reference to a tuned linear radiator of length l with a *sinusoidal* current distribution, for which the ends of the linear radiator may be coil or condenser loaded, *i.e.*, may exhibit current[1] values. The field point is expressed in cylindrical coordinates r_0, φ, z, while any radiator element of the total length l is $d\zeta$, and the total radiator length is $l = \zeta_2 - \zeta_1$. The radiator element is a distance r from the distant field point $P(r_0, \varphi, z)$. The extremities of the linear radiator are at $z = \zeta_1$ and $z = \zeta_2$, since the radiator is parallel with the Z axis. The solid geometry is then as in Fig. 140, and the distance of r meters of a radiator element $d\zeta$ from

[1] BECHMANN, R., *Proc. IRE*, **19**, 461, March, 1931. STRATTON, J. A., "Electromagnetic Theory," Sec. 8.11, McGraw-Hill Book Company, Inc., New York, 1941.

a field point P is $\sqrt{r_0^2 - (z - \zeta)^2}$. Making the substitutions given in Fig. 140, we find $du = (u/r)\, d\zeta$ and $dv = -(v/r)\, d\zeta$. Since tuning prevails, we can assume a sinusoidal current distribution

$$I_\zeta = I_0 \sin (\beta\zeta - \psi)\epsilon^{j\omega t}$$

where ψ depends upon the loading on respective radiator ends. As mentioned, we may have a condenser or a coil loading. The electric $\mathcal{E}$ and magnetic H field associations can be found from the Hertz vector $\prod_z$ (which is in meter · volts) and

$$\mathcal{E}_z = \frac{\partial^2 \prod_z}{\partial z^2} + \beta^2 \prod_z \quad \text{volts/m;} \qquad \mathcal{E}_{r_0} = \frac{\partial^2 \prod_z}{\partial z\, \partial r_0} \quad \text{volts/m;}$$

$\mathcal{E}_{r_0}$ is negligible in distant radiation region

$$H_\varphi = -j\omega\kappa \frac{\partial \prod_z}{\partial r_0} \qquad \text{amp/m}$$

since, according to Secs. 21, 49, 50, we have, according to Eq. (315),

$$\mathcal{E}^{(\text{volts/m})} = \underbrace{\text{grad}\left(\text{div} \prod\right)}_{=\nabla\nabla\cdot\prod} - \underbrace{\mu\kappa \frac{\partial^2 \prod}{\partial t^2}}_{=-\omega^2\mu\kappa\prod} = \nabla\nabla \cdot \prod + \beta^2 \prod$$

$$H^{(\text{amp/m})} = \kappa \text{ curl}\left(\frac{\partial \prod}{\partial t}\right) = j\omega\kappa \text{ curl} \prod = j\omega\kappa\nabla \times \prod$$

The Hertz vector is along the Z direction and parallel thereto, since the thin linear radiator of length l is along the Z axis. The Hertz vector is, therefore, a time function and a function of z. For f in megacycles per second and r in meters, according to Eqs. (73) and (74), we find

$$d\prod_z = j1.435 \times 10^3 \frac{I_\zeta\, d_\zeta}{rf} \qquad \text{m} \cdot \text{volts}$$

$$\prod_z = j1.435 \times 10^3 \frac{I_0\epsilon^{j\omega t}}{f} \int_{\zeta=\zeta_1}^{\zeta=\zeta_2} \frac{\sin (\beta\zeta - \psi)}{r} \epsilon^{j\beta r}\, d\zeta \qquad \text{m} \cdot \text{volts}$$

In the above relation, we have

$$\sin (\beta\zeta - \psi) = \frac{\epsilon^{j(\beta\zeta-\psi)} - \epsilon^{-j(\beta\zeta-\psi)}}{2j} \qquad \text{according to De Moivre}$$

$$\prod_z = 7.175 \times 10^2 \frac{I_0\epsilon^{j\omega t}}{f}\left[\epsilon^{-j\psi} \int_{\zeta=\zeta_1}^{\zeta=\zeta_2} \frac{\epsilon^{j\beta(r+\zeta)}}{r}\, d\zeta - \epsilon^{j\psi} \int_{\zeta=\zeta_1}^{\zeta=\zeta_2} \frac{\epsilon^{j\beta(r-\zeta)}}{r}\, d\zeta\right]$$

It is the solution of these two integrals that concerns us in this section. With the substitutions and solutions in Fig. 140, we then find

$$\int_{\zeta_1}^{\zeta_2} = \frac{\epsilon^{j\beta(r+\zeta)}}{r}\,d\zeta = \epsilon^{j\beta z}\int_{u_1}^{u_2}\frac{\epsilon^{j\beta u}}{u}\,du \qquad \text{for } u_1 = r_1+\zeta_1-z \text{ and } u_2 = r_2+\zeta_2-z$$

$$\int_{\zeta_1}^{\zeta_2}\frac{\epsilon^{j\beta(r-\zeta)}}{r}\,d\zeta = -\epsilon^{-j\beta z}\int_{v_1}^{v_2}\frac{\epsilon^{j\beta v}}{v}\,dv \qquad \text{for } v_1 = r_1-\zeta_1+z; \quad v_2 = r_2-\zeta_2+z$$

$$\prod\nolimits_z^{(\text{m.volts})} = 717.5\,\frac{I_0\epsilon^{j\omega t}}{f^{(\text{Mc})}}\left[\epsilon^{j(\beta z-\psi)}\int_{u_1}^{u_2}\frac{\epsilon^{j\beta u}}{u}\,du + \epsilon^{-j(\beta z-\psi)}\int_{v_1}^{v_2}\frac{\epsilon^{j\beta v}}{v}\,dv\right]$$

These are well-known integrals with well-known transformations, leading to sine and cosine integrals, with tables[1] for the evaluation. We have then

$$\underbrace{\int_{u_1}^{u_2}\frac{\epsilon^{j\beta u}}{u}\,du}_{=B} = \underbrace{\int_{u_1}^{\infty}\frac{\epsilon^{j\beta u}}{u}\,du}_{=A} - \int_{2}^{\infty}\frac{\epsilon^{j\beta u}}{u}\,du$$

$$A = -\,\text{Ci}\,(\beta u_1) - j\,\text{Si}\,(\beta u_1) + j\frac{\pi}{2}; \qquad B = \Big|\text{Ci}\,(\beta u)\Big|_{u_1}^{u_2} + j\Big|\text{Si}\,(\beta u)\Big|_{u_1}^{u_2}$$

since

$$\text{Ci}\,(\beta u_1) = -\int_{u_1}^{\infty}\frac{\cos(\beta u)}{u}\,du; \qquad \text{Si}\,(\beta u_1) = \int_0^{u_1}\frac{\sin(\beta u)}{u}\,du;$$

$$\text{Si}\,(\infty) = \frac{\pi}{2}; \qquad \text{Ei}\,(jx) = (\text{Ci})\,x + j\,\text{Si}\,(x)$$

$$\epsilon^{j\beta u} = \cos(\beta u) + j\sin(\beta u)$$

We have, therefore, the integral solution

$$\left.\begin{aligned}
\int_{\zeta=\zeta_1}^{\zeta=\zeta_2}\frac{d\zeta}{r}\sin(\beta\zeta-\psi)\epsilon^{j\beta r} &= \frac{1}{2}\left[\epsilon^{j(\beta z-\psi)}\int_{u=u_1}^{u=u_2}\frac{\epsilon^{j\beta u}}{u}\,du\right.\\
&\qquad\left.+\,\epsilon^{-j(\beta z-\psi)}\int_{v=v_1}^{v=v_2}\frac{\epsilon^{j\beta v}}{v}\,dv\right]\\
&= \tfrac{1}{2}[B\epsilon^{j(\beta z-\psi)} + D\epsilon^{-j(\beta z-\psi)}]\\
B = \Big|\text{Ci}\,(\beta u) + j\,\text{Si}\,(\beta u)\Big|_{u_1}^{u_2}; &\qquad D = \Big|\text{Ci}\,(\beta v) + j\,\text{Si}\,(\beta v)\Big|_{v_1}^{v_2}
\end{aligned}\right\}\quad(847)$$

Since

$$B = \underbrace{\text{Ci}\,(\beta u_2) + j\,\text{Si}\,(\beta u_2)}_{=\text{Ei}\,(j\beta u_2)} - \underbrace{[\text{Ci}\,(\beta u_1) + j\,\text{Si}\,(\beta u_1)]}_{=\text{Ei}\,(j\beta u_1)};$$

$$D = \text{Ei}\,(j\beta v_2) - \text{Ei}\,(j\beta v_1)$$

for the function $\text{Ei}\,(jx) = \text{Ci}\,(x) + j\,\text{Si}\,(x)$, which is the integral logarithm with the purely imaginary argument jx, we have also

[1] E. Jahnke and F. Emde. There are also tables and curves in A. Hund, "Frequency Modulation," Fig. 147 and tables in the Appendix, McGraw-Hill Book Company, Inc., New York, 1942.

$$\left.\begin{aligned} B &= \text{Ei}\,(j\beta u_2) - \text{Ei}\,(j\beta u_1); \qquad D = \text{Ei}\,(j\beta v_2) - \text{Ei}\,(j\beta v_1) \\ \prod_z &= 717.5\,\frac{I_0\epsilon^{j\omega t}}{f^{(\text{Mo})}}\left\{\epsilon^{j(\beta z-\psi)}[\text{Ei}\,(j\beta u_2) - \text{Ei}\,(j\beta u_1)]\right. \\ &\quad \left. + \epsilon^{-j(\beta z-\psi)}[\text{Ei}\,(j\beta v_2) - \text{Ei}\,(j\beta v_1)]\right\} \qquad \text{m}\cdot\text{volts} \end{aligned}\right\} \tag{847a}$$

where the Ei (jx) values are computed from the Ci (x) and Si (x) values, which are taken either from tables or from curves. Moreover, the following formulas are of much use in radiation problems:

$$\text{Si}\,(x) = \int_0^x \frac{\sin u}{u}\,du = \frac{\pi}{2} - \int^\infty \frac{\sin u}{u}\,du;$$

$$\text{Ci}\,(x) = -\int_x^\infty \frac{\cos u}{u}\,du; \qquad \text{Ei}\,(x) = \int_\infty^{-x} \frac{\epsilon^{-u}}{u}\,du = \text{li}\,(\epsilon^x);$$

$$\text{li}\,(x) = \int_0^x \frac{du}{\log_\epsilon (u)}; \qquad \text{li}\,(\epsilon^x) = \text{Ci}\,(x) + j\,\text{Si}\,(x) = \text{Ei}\,(jx)$$

For sufficiently large values $|x|$ we have the approximations:

$$\text{Ei}\,(x) = \frac{\epsilon^x}{x}; \qquad \text{Ci}\,(x) = \frac{\sin x}{x}; \qquad \text{Si}\,(x) = \frac{\pi}{2} - \frac{\cos x}{x}$$

Other solutions are

$$\int_0^x \frac{1-\cos u}{u}\,du = 0.577 + \log_\epsilon (x) - \text{Ci}\,(x); \qquad \text{Si}\,(\infty) = \frac{\pi}{2}$$

$$\int_{z_1}^{z_2} \frac{\cos\,[\beta(r+z)]}{r}\,dz = \text{Ci}\,[\beta(r_2+z_2)] - \text{Ci}\,[\beta(r_1+z_1)]$$

for (848)

$$r = \sqrt{r_0^2+z^2}; \qquad r_1 = \sqrt{r_0^2+z_1^2}; \qquad r_2 = \sqrt{r_0^2+z_2^2}$$

$$\int_{z_1}^{z_2} \frac{\cos\,[\beta(r-z)]}{r}\,dz = \text{Ci}\,[\beta(r_1-z_1)] - \text{Ci}\,[\beta(r_2-z_2)]$$

$$\int_{z_1}^{z_2} \frac{\sin\,[\beta(r+z)]}{r}\,dz = \text{Si}\,[\beta(r_2+z_2)] - \text{Si}\,[\beta(r_1+z_1)]$$

$$\int_{z_1}^{z_2} \frac{\sin\,[\beta(r-z)]}{r}\,dz = \text{Si}\,[\beta(r_1-z_1)] - \text{Si}\,[\beta(r_2-z_2)]$$

$$\int_a^b = -\int_b^a; \qquad \int_a^c = \int_a^b + \int_b^c; \qquad \int_a^c - \int_a^b = \int_b^c$$

$$\int_0^\infty \frac{\cos\,(bx)}{x}\,dx = \infty; \qquad \int_0^\infty \frac{\sin\,(bx)}{x}\,dx = \frac{\pi}{2} \qquad \text{for } b > 0$$

We have also for a radiator element dz_0, where

$$r = \sqrt{\underbrace{x^2+y^2}_{=r_0^2} + (z-z_0)^2}$$

is the distance of dz_0 at $(0, 0, z_0)$ from the origin (x, y, z),

$$\left.\begin{aligned}
&\int_{z_{01}}^{z_{02}} \frac{\epsilon^{j\beta(r+z_0)}}{r}\,dz_0 = \epsilon^{j\beta z}\int_{u_1}^{u_2}\frac{\epsilon^{j\beta u}}{u}\,du = \epsilon^{j\beta z}[\mathrm{Ei}\,(j\beta u_2) - \mathrm{Ei}\,(j\beta u_1)]\\
&\int_{z_{01}}^{z_{02}} \frac{\epsilon^{j\beta(r-z_0)}}{r}\,dz_0 = -\epsilon^{-j\beta z}\int_{v_1}^{v_2}\frac{\epsilon^{j\beta u}}{u}\,du\\
&\qquad\qquad = -\epsilon^{-j\beta z}[\mathrm{Ei}\,(j\beta v_2) - \mathrm{Ei}\,(j\beta v_1)]\\
&u_1 = \sqrt{r_0^2 + (z - z_{01})^2} - (z - z_{01});\\
&\qquad\qquad u_2 = \sqrt{r_0^2 + (z - z_{02})^2} - (z - z_{02})\\
&v_1 = \sqrt{r_0^2 + (z - z_{01})^2} + (z - z_{01});\\
&\qquad\qquad v_2 = \sqrt{r_0^2 + (z - z_{02})^2} + (z - z_{02})\\
&\text{For a radiator element } dz \text{ of a radiator of length } l \text{ (Fig. 146), we have}\\
&\int_0^l \left(\frac{\epsilon^{j2\beta z} - 1}{l - z} + \frac{\epsilon^{-j\beta z} - 1}{z}\right)dz = 2\int_0^{2\beta l}\frac{\epsilon^{-ju} - 1}{u}\,du\\
&\qquad = 2 \times 0.5772 + 2\log_\epsilon (2\beta l) - 2\,\mathrm{Ci}\,(2\beta l) + j2\,\mathrm{Si}\,(2\beta l)
\end{aligned}\right\}\tag{848}$$

We have then the following transformations, which are useful in radiation problems:

(A) $$x = \int_0^\pi \frac{\cos^2\left(\frac{k\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta$$

Put $u = \cos\theta$, which has the limits ± 1, and find

$$\frac{du}{d\theta} = -\sin\theta; \qquad d\theta = -\frac{du}{\sin\theta}, \qquad \sin^2\theta = 1 - u^2;$$

$$x = \int \frac{\cos^2\left(\frac{k\pi u}{2}\right)}{\sin\theta}\left(-\frac{du}{\sin\theta}\right) = -\int \frac{\cos^2\left(\frac{k\pi u}{2}\right)}{\underbrace{\sin^2\theta}_{1-u^2}}\,du$$

But

$$\frac{1}{1 - u^2} = \frac{1}{2}\left(\frac{1}{1 + u} + \frac{1}{1 - u}\right), \qquad \cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}; \qquad \alpha = k\pi u$$

and

$$x = -\frac{1}{2}\int_1^0 \frac{1 + \cos\alpha}{1 - u^2}\,du = +\frac{1}{2}\int_0^1 \frac{1 + \cos\alpha}{1 - u^2}\,du$$

$$= \frac{1}{4}\left(\int_0^1 \frac{1 + \cos\alpha}{1 + u}\,du + \int_0^1 \frac{1 + \cos\alpha}{1 - u}\,du\right)$$

$$= \frac{1}{4}\int_{-1}^{+1} \frac{1 + \cos\alpha}{1 + u}\,du = \frac{1}{4}\int_{-1}^{+1} \frac{1 + \cos(k\pi u)}{1 + u}\,du$$

Put $1 + u = \dfrac{v}{k\pi}$

then

$$v = k\pi + k\pi u; \qquad \frac{dv}{du} = k\pi; \qquad du = \frac{dv}{k\pi}; \qquad u = \frac{v - k\pi}{k\pi}$$

$$k\pi u = v - k\pi; \qquad \cos(k\pi u) = \cos(v - k\pi) = \cos(k\pi - v) \begin{cases} = -\cos v & \text{for } k = 1, 3, 5, 7, \cdots \\ = \cos v & \text{for } k = 2, 4, 6, 8, \cdots \end{cases}$$

Proceeding with the odd half-wavelength distributions, we find

$$x = \frac{1}{4}\int_{-1}^{+1} \frac{1 + \cos(k\pi u)}{1 + u}\, du = \frac{1}{4}\int_0^{2k\pi} \frac{1 - \cos v}{1 + \frac{v - k\pi}{k\pi}} \frac{dv}{k\pi} = \frac{1}{4}\int_0^{2k\pi} \frac{1 - \cos v}{v}\, dv$$

$$= \frac{1}{4}[0.5772 + 2.303 \log_{10}(2\pi k) - \mathrm{Ci}\,(2\pi k)]; \qquad \text{where } k = 1 \text{ for } l = \frac{\lambda}{2}$$

(B) $$x = \int_{\zeta_1}^{\zeta_2} \frac{\epsilon^{j\beta(r+\zeta)}}{r}\, d\zeta \qquad \text{for cylindrical coordinates } d, \varphi, z$$

and a distance $r = \sqrt{d^2 + (z - \zeta)^2}$ (Fig. 142) from any current element $d\zeta$. Put $u = r + \zeta - z$; then

$$\frac{du}{d\zeta} = \frac{u}{r}; \qquad x = \epsilon^{j\beta z} \int_{u_1}^{u_2} \frac{\epsilon^{j\beta u}}{u}\, du$$

for

$$u_1 = r_1 + \zeta_1 - z \qquad \text{and} \qquad u_2 = r_2 + \zeta_2 - z;$$

$$\int_{u_1}^{u_2} \frac{\epsilon^{j\beta u}}{u}\, du = \int_{u_1}^{\infty} \frac{\epsilon^{j\beta u}}{u}\, du - \int_{u_2}^{\infty} \frac{\epsilon^{j\beta u}}{u}\, du;$$

But

$$\int_{u_1}^{\infty} \frac{\epsilon^{j\beta u}}{u}\, du = \int_{u_1}^{\infty} \frac{\cos(\beta u)}{u}\, du + j\int_{u_1}^{\infty} \frac{\sin(\beta u)}{u}\, du; \qquad \int_{u_1}^{\infty} \frac{\cos(\beta u)}{u}\, du = -\mathrm{Ci}\,(\beta u_1);$$

$$\int_0^{u_1} \frac{\sin(\beta u)}{u}\, du = \mathrm{Si}\,(\beta u_1); \quad \mathrm{Si}\,(\infty) = \frac{\pi}{2}; \quad \mathrm{Si}\,(x) = \int_0^{x} \frac{\sin(u)}{u}\, du = \frac{\pi}{2} - \int_x^{\infty} \frac{\sin(u)}{u}\, du$$

Hence,

$$\int_{u_1}^{\infty} \frac{\epsilon^{j\beta u}}{u}\, du = -\mathrm{Ci}\,(\beta u_1) - j\,\mathrm{Si}\,(\beta u_1) + j\frac{\pi}{2}; \qquad \int_{u_1}^{u_2} \frac{\epsilon^{j\beta u}}{u}\, du = \Big|\mathrm{Ci}\,(\beta u) + j\,\mathrm{Si}\,(\beta u)\Big|_{u_1}^{u_2}$$

(C) $$x = \int_0^{\pi} \frac{[\cos(\beta l \cos\theta) - \cos(\beta l)]^2\, d\theta}{\sin\theta}$$

Put $u = \cos\theta$, and find the solution for $\mathrm{Ci}\,(\alpha)_{\alpha\to 0} = 0.5772 + 2.303 \log_{10}(\alpha)$, with the result for

$$\mathrm{Si}\,(\alpha) = \int_0^{\alpha} \frac{\sin(\alpha)}{\alpha}\, d\alpha \qquad \text{and} \qquad \mathrm{Ci}\,(\alpha) = -\int_{\alpha}^{\infty} \frac{\cos(\alpha)}{\alpha}\, d\alpha$$

$$x = 0.5772 + 2.303 \log_{10} (2\beta l) - \text{Ci} (2\beta l) + \tfrac{1}{2} \sin (2\beta l)[\text{Si} (4\beta l) + 2 \text{ Si} (2\beta l)]$$
$$+ \tfrac{1}{2} \cos (2\beta l)[0.5772 + 2.303 \log_{10} (\beta l) + \text{Ci} (4\beta l) - 2 \text{ Ci} (2\beta l)]$$

(D)
$$x = \int_0^\pi \frac{\sin^2 \theta \sin^2 [(\beta l/2)(1 - \cos \theta)]}{(1 - \cos \theta)^2} \sin \theta \, d\theta$$

Put $u = 1 - \cos \theta$, yielding

$$\frac{du}{d\theta} = \sin \theta, \qquad x = \int_0^2 \frac{(2 - u) \sin^2 (\beta l u/2)}{u} du = \int_0^2 \frac{(2 - u)[1 - \cos^2 (\beta l u)/2]}{u} du$$
$$= \frac{1}{2} \int_0^2 \frac{2 - 2 \cos (\beta l u) - u + u \cos (\beta l u)}{u} du$$
$$= \int_0^2 \frac{1 - \cos (\beta l u)}{u} du + \frac{1}{2} \int_0^2 \cos (\beta l u) \, du - \overbrace{\frac{1}{2} \int_0^2 du}^{= -1};$$
$$\frac{1}{2} \int_0^2 \cos (\beta l u) \, du = \frac{1}{2} \left| \frac{\sin (\beta l u)}{\beta l} \right|_0^2 = \frac{\sin (2\beta l)}{2\beta l}$$

Put $u = v/\beta l$; then $dv/du = \beta l$, $du = dv/\beta l$, and

$$\int_0^2 \frac{1 - \cos (\beta l u)}{u} du = \int_0^{2\beta l} \frac{1 - \cos [\beta l v/(\beta l)]}{v/(\beta l)} \frac{dv}{\beta l} = \int_0^{2\beta l} \frac{1 - \cos v}{v} dv$$
$$= 0.5772 + 2.303 \log_{10} (2\beta l) - \text{Ci} (2\beta l)$$

Therefore,

$$x = 0.5772 + 2.303 \log_{10} (2\beta l) - \text{Ci} (2\beta l) + \frac{\sin (2\beta l)}{4\beta l} - 1$$
$$= 2.303 \log_{10} \left(12.56 \frac{l}{\lambda}\right) + \frac{\sin [12.56(l/\lambda)]}{12.56(l/\lambda)} - \text{Ci} \left(12.56 \frac{l}{\lambda}\right) - 0.4228$$

(E)
$$x = \int_{l-a}^{l+a} \sin (\beta |l - u|) \frac{\epsilon^{-j\beta|a+u|}}{|a + u|} du$$

We have in this integral an example that includes also *moduli signs.* The quantity $a + u$ is positive over the extension of the integral, permitting the omission of the moduli signs | | in the exponents as well as in the denominator. We must, however, break up the integral, because $l - u$ is positive over the range $l - a \leqq u \leqq l$ and negative over the range $l \leqq u \leqq l + a$. Therefore,

$$x = \int_{l-a}^{l} \sin [\beta(l - u)] \frac{\epsilon^{-j\beta(a+u)}}{a + u} du - \int_{l}^{l+a} \sin [\beta(l - u)] \frac{\epsilon^{-j\beta(a+u)}}{a + u} du$$
$$y = \int_{l-a}^{l} \frac{\epsilon^{-j\beta(a+2u)}}{a + u} du = \int_{l-a}^{l} \frac{\epsilon^{-j\beta a}\epsilon^{-j\beta 2u}}{a + u} du = \int_{l-a}^{l} \frac{\epsilon^{+j\beta a}\epsilon^{-j\beta a}\epsilon^{-j\beta a}\epsilon^{-j\beta 2u}}{a + u} du$$
$$= \epsilon^{j\beta a} \int_{l-a}^{l} \frac{\epsilon^{-j\beta 2a}\epsilon^{-j\beta 2u}}{a + u} du = \epsilon^{j\beta a} \int_{l-a}^{l} \frac{\epsilon^{-j2\beta(a+u)}}{a + u} du = \epsilon^{j\beta a} \int_{v=j2\beta l}^{v=j2\beta(l+a)} \frac{\epsilon^{-v}}{v} dv$$
$$\int_{jp}^{jq} \frac{\epsilon^{-v}}{v} dv = \text{Ci} (q) - j \text{ Si} (q) - \text{Ci} (p) + j \text{ Si} (p)$$

$$y = \epsilon^{j\beta a}\left\{\mathrm{Ci}\,[2\beta(l + a)] - j\,\mathrm{Si}\,[2\beta(l + a)] - \mathrm{Ci}\,(2\beta l) + j\,\mathrm{Si}\,(2\beta l)\right\}$$

$$x = \underbrace{\int_{l-a}^{l} \sin\,[\beta(l - u)]\,\frac{\epsilon^{-j\beta(a+u)}}{a + u}\,du}_{=A} - \underbrace{\int^{l+a} \sin\,[\beta(l - u)]\,\frac{\epsilon^{-j\beta(a+u)}}{a + u}\,du}_{=B} = A - B$$

$$\sin\,(\alpha) = \frac{\epsilon^{j\alpha} - \epsilon^{-j\alpha}}{2j}; \qquad A = \int_{l-a}^{l} \frac{\epsilon^{j\beta(\overbrace{l-u}^{c})} - \epsilon^{-j\beta(l-u)}}{2j}\,\frac{\epsilon^{-j\beta(\overbrace{a+u}^{d})}}{a + u}\,du$$

$$= \frac{1}{2j}\int_{l-a}^{l} \frac{\epsilon^{j\beta(c-d)} - \epsilon^{-j\beta(c+d)}}{a + u}\,du$$

$$c - d = l - u - a - u = \underbrace{l - a}_{= -a'} - 2u; \qquad c + d = l - u + a + u = l + a;$$

$$c - d = -(a' + 2u)$$

$$A = \frac{1}{2j}\left[\underbrace{\int_{l-a}^{l} \frac{\epsilon^{-j\beta(a'+2u)}}{a + u}\,du}_{\substack{=y',\text{ due to } a',\text{ instead of } a \text{ and } y \text{ and}\\ \text{solution similar to } y\\ \text{solution}}} - \underbrace{\int_{l-a}^{l} \frac{\epsilon^{j\beta(l+a)}}{a + u}\,du}_{\substack{\text{well-known}\\ \text{solution}}}\right] \cdots$$

130. The Mutual Impedance of Driven and Parasitic Linear Radiators. Radiated field power[1] can also be computed by means of the work done in driving electric currents of a known distribution *against* the electric forces produced by them. This work done then yields the *total* energy contributed to the electromagnetic field. Inasmuch as this procedure moves the surface of the power integration into the surface of an antenna, we have the comparisons

$$P^{(\text{watts})} = \iint_{S_c} (np)\,dS = P_r + jP_x \quad \text{for emf method and integration over the surface } S_c \text{ conductor}$$

$$P^{(\text{watts})} = \iint_{S} (\underset{\cdot}{n}p)\,dS = P_r \quad \text{for classic Poynting vector method with the surface } S \text{ of integration, reaching into the } \textit{distant} \text{ radiation region} \qquad (848a)$$

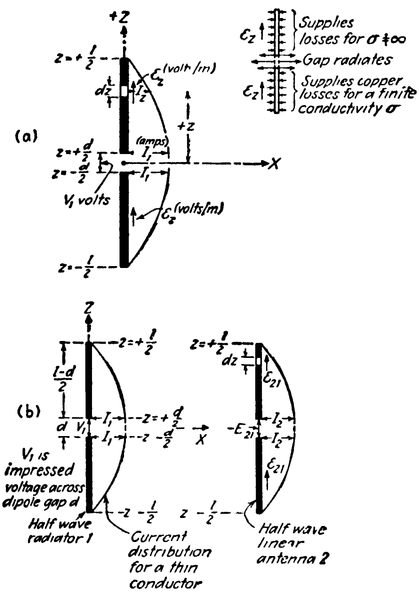
where dS denotes an element of a *completely* closed surface S_c or S, respectively, for the external unit normal vector $\underset{\cdot}{n}$. We know from

[1] Based on the induced electromotive force method due to L. Brillouin, Sur l'origine de la résistance de rayonnement, *Radio-électricité*, **3,** 147–152, April, 1922. Kliatzkin, Application to Antenna Problems, *Telegrafia i telefonia bez provodov*, **1**(40), 33 (1927). A. A. Pistolkors, *Proc. IRE*, **17,** 562, March, 1929. R. Bechmann, *ibid.*, **19,** 1471, August, 1931. P. S. Carter, *ibid.*, **20,** 1004, June, 1932. J. Labus, *Hochfrequenz und Elektroakustik*, January, 1933, p. 17. G. H. Brown, *Proc. IRE*, **25,** 78, January, 1937. S. A. Schelkunoff, *ibid.*, **27,** 660, October, 1939; **29,** 516, September, 1941.

Table 2 that the field contributions generally have real and imaginary terms (factors A, B, and C have such terms), which by no means can be neglected in the near field region and surely not at the very surface of a radiating conductor. Therefore, the integrations over the surface S_c of a radiator are more complex than over the surface S, which reaches into the distant radiation region where we have no complex field contributions, but only an inverse distance term. This accounts for the additional jP_x term in (848a), where P_x also gives the reactive power. The classic Poynting vector method is, therefore, mathematically simpler, since it includes only the radiated power term. Therefore, in mediums with no dissipation ($\sigma = 0$ mhos/m), either procedure is mathematically the same. The advantage of the emf procedure is that it is more general, according to (818a), and can be readily applied to wave propagation in dissipative mediums, such as the earth's crust and the ocean. Section 129 gives some integration expressions occurring in near field integration surfaces. The emf method must give an antenna input impedance with a real part, which is the radiation resistance (plus the small loss resistance), and an imaginary part, which is wattless field energy. This means that for one radiator, we must have a self-impedance with a self-resistance and a self-reactance. For several radiators, we must also deal with mutual effects, which are characterized by corresponding mutual impedances. We have then a means of computing the input impedances of half-wave dipoles when they are used in an alignment or some other array. It must be admitted that there are some cases where the reactive part of the impedance may be useless. Fortunately, we are always using the real part, since it accounts for the actual radiated power that we use in the distant radiation region. The reactive effects, we can "stub out," or neutralize, by choosing the radiator length somewhat shorter. We are fortunate in the case of UHF's, since we have, for instance, only half-wave individual radiators or other structures, such as quarter-wave radiator tips backed up by a metal reference sheet.

There is a radiation paradox involved as far as the Poynting vector space distribution as well as the emf method with tangential electric fields along a perfect conductor is concerned. It must be realized that the Poynting vector p in watts per square meter refers strictly to a general *integral* of electric and magnetic field relations, since we have $\int_0^{\kappa\mathcal{E}} \mathcal{E} \cdot d(\kappa\mathcal{E})$ and $\int_0^{\mu H} H \cdot d(\mu H)$. These integrals are correct but do *not necessarily* mean that electric $\mathcal{E} \cdot d(\kappa\mathcal{E})$ and magnetic $H \cdot d(\mu H)$ field volume densities are distributed throughout the field although this may be the case. Nevertheless, such assumptions lead to acceptable relations if we deal with the Poynting vector $p = p_{\varphi,\theta}$ in different space directions in order to account for the radiated power in preferred directions and the average radiation effect, so as to obtain expressions for the gain of certain antenna systems. Such acceptable relations may seem impeach-

able, *i.e.*, open to criticism, as far as the physical interpretation is concerned. It must be admitted, however, that in actual measurements, we can confirm field gains based on such assumptions. Now when examining (848*a*), we note two things. First, the Poynting vector acts normal to respective surface elements dS. As far as the classic Poynting method for expressing the *total* radiated watts is concerned, (848*a*) can be physically understood if the surface of integration is *not* the surface of a perfect conductor or even of only a good or less than good conductor. The reason for this is that for a radiator surface which is conductive (but not $\sigma = \infty$ mhos/m), the Poynting vector points *inward* and is normal to the conductor surface. It delivers then the



Effective current has a distribution value $I_z^{(\text{amp})} = I_1 \cos(\beta z)$; $\beta z = 2\pi/\lambda\, z$. *Total induced* voltage $-E_1 = -\int_{z=-l/2}^{z=+l/2} \mathcal{E}_z \times \cos(\beta z)\, dz = +V_1$ volts. *Self*-impedance $Z_{11} = -E_{11}/I_1 = +V_1/I_1$ complex ohms. The dot below the quantities indicates that we must expect a phase difference, since we did with vectors V_1 and I_1. Gap of width d for the application of the *driving* voltage $V_1^{(\text{volts})}$ is negligible (assumed). $-E_1^{(\text{volts})}$ is induced voltage across dipole gap.

Induced voltage due to radiator 1 is $E_{21} = -\int_{v=-l/2}^{z=+l/2} \mathcal{E}_{21} \cos(\beta z)\, dz$. *Mutual* impedance of radiator 1 on radiator 2 is $Z_{21} = -E_{21}/I_1$ complex ohms.

FIG. 141. Self- and mutual impedances of coupled linear radiators and explanation of the paradox of the induced emf method

joulean heat losses dissipated in the conductor. This is not an *outward* energy flow to the surface. In such a case, we have also a more or less small tangential electric field component along the surface of the conductor. On the other hand, when we deal with a hypothetically perfect conductor, there can be no tangential $\mathcal{E}$ component We have then to assume an action that brings about the outward radiation and yet does not conflict with the existence of a so-called "tangential" field distribution along the conducting surface. This leads then to the concept that the integration along the radiator surface gives only the rate of increase of heat, which energy is dissipated in the conductor, while the actual radiated power must come, for instance, in the case of a dipole, from the integration over the surface enclosing the dipole gap. This is followed up after the discussion of self- and mutual antenna impedances. Moreover, on page 655, we have an expression for the divergence of the Poynting vector p and find from (113) that the average time rate of change of the stored energy $j\omega(\kappa\mathcal{E}_e^2 - \mu H_e^2)$ vanishes and div $p = -\mathcal{E}_e \cdot J_e$ watts per cubic meter, where $\mathcal{E}_e$ is effective volts per meter and J_e effective amperes per square meter. If we deal

with a thin linear radiator, the current density J_e becomes the effective current I_e flowing in a current element. Therefore, the total power flow in watts must be equal to $-\int_\tau \mathcal{E} \cdot J^* \, d\tau$ watts, which escapes over the surface of the volume τ, expressed in cubic meters. By $\mathcal{E} \cdot J^*$ is meant the *real part of the power*, since J^* is the conjugate of J amperes per square meter. We can, therefore, also write for (848*a*)

$$P = -\text{Re} \int\int\int_\tau \mathcal{E}_e J_e^* \, d\tau + \underbrace{jP_x}_{\text{reactive}} \quad \text{watts};$$

$$-\text{Re} \int\int\int_\tau \mathcal{E}_e J_e^* \, d\tau = \text{radiated watts} = +\int\int_S (n \cdot p) \, dS \qquad (848b)$$

compared with the classic integration of the *normal* component of the Poynting vector p over a *closed* surface S including all radiation sources. This relation also shows the integration of the power expended per unit volume over the current distribution (over the region τ occupied by the radiators).

The self-impedance of a radiator depends upon the distribution of the current along the radiator. This impedance is different when in the presence of other radiators, since the mutual effect of the radiators changes the current distribution. Figure 141*a* shows a half-wave dipole with the corresponding self-impedance Z_{11}, which is the ratio of the impressed voltage V_1 and the entrance current value I_1 at the dipole gap. Figure 141*b* shows two such radiators with the mutual impedance Z_{12} of radiator 1 on radiator 2. This impedance is the negative induced voltage E_{12} across the dipole gap divided by the entrance current I_1 at the gap of radiator 1. It is assumed that we have a sinusoidal current distribution in either case. This outcome is based on the line integral of the electric force, since the voltage induced in any dz radiator element owing to the entrance current I_1 must be $\mathcal{E}_z \, dz$ volts, where $\mathcal{E}_z$ is the tangential field. By the law of reciprocity, this means that a current I_1 in the dz element causes an induced voltage of $\mathcal{E}_z \, dz$ volts at the dipole gap. If we call this induced voltage dE_{11}, we have, for the actual current $I_1 \cos(\beta z)$ in the dz element,

$$\frac{\mathcal{E}_z \, dz}{I_1} = \frac{-dE_{11}}{I_1 \cos(\beta z)}; \qquad -dE_{11} = -\mathcal{E}_z \cos(\beta z) \, dz \quad \text{volts};$$

$$-E_{11} = -\int_{-l/2}^{+l/2} \mathcal{E}_z \cos(\beta z) \, dz \quad \text{volts}$$

The induced voltage of $-E_{11}$ volts at the dipole gap must be compensated by the driving voltage of V_1 volts impressed across the dipole gap; therefore, $+V_1 = -E_{11}$, and we have the relation for the self-impedance Z_{11}, experienced at the dipole gap, as given in Fig. 141*a*. For Fig. 141*b*, we deal with two such radiators, and the field $\mathcal{E}_{21}$ of radiator 1 causes a tangential field along radiator 2, resulting in an induced voltage $-E_{21}$

$= - \int_{-l/2}^{+l/2} \mathcal{E}_{21} \cos(\beta z)\, dz$ volts across the gap of radiator 2. This voltage is then due to the entrance current I_1, and Z_{21} in Fig. 141*b* is the expression for the mutual impedance of the half-wave dipole 1 on the half-wave dipole 2. For the same reason, the current flow I_2 at the gap entrance of radiator 2 causes a mutual impedance at the gap of radiator 1

$$Z_{12} = -\frac{E_{12}}{I_2} = \frac{-\int_{-l/2}^{+l/2} \mathcal{E}_{12} \cos(\beta z)\, dz}{I_2} \quad \text{ohms}$$

Therefore, when we have two such antennas, which are driven, the voltage impressed at the dipole gap has to overcome its self-impedance as well as the mutual impedance due to the other radiator, and

$$\left.\begin{array}{l} V_1 = I_1Z_{11} + I_2Z_{12}; \quad V_2 = I_2Z_{22} + I_1Z_{21}; \quad Z_{21} = Z_{12} \\ \qquad \text{for radiator 1 driven by impressed voltage} \\ \qquad V_1\text{, and radiator 2 driven by voltage } V_2 \\ V_1 = I_1Z_{11} + I_2Z_{12}; \quad 0 = I_2Z_{22} + I_1Z_{21}; \quad Z_{21} = Z_{12} \\ \qquad \text{for radiator 1 driven by } V_1\text{, and a} \\ \qquad \textit{parasitic} \text{ radiator 2 (}\textit{not}\text{ driven)} \end{array}\right\} \quad (849)$$

It is understood that Z_1 ohms denotes the self-impedance of radiator 1 in the presence of radiator 2 and Z_2 the self-impedance of radiator 2 in the presence of radiator 1, since the current distribution is changed by the mutual effects. For n linear radiators and impressed voltages V_1, V_2, V_3, V_4, . . . , V_n, we have

$$\left.\begin{array}{l} V_1 = I_1Z_{11} + I_2Z_{12} + I_3Z_{13} + \cdots + I_nZ_{1n} \\ V_2 = I_1Z_{12} + I_2Z_{22} + I_3Z_{23} + \cdots + I_nZ_{2n} \\ V_3 = I_1Z_{13} + I_2Z_{23} + I_3Z_{33} + \cdots + I_nZ_{3n} \\ V_4 = ; \quad V_5 = ; \cdots ; \\ \qquad V_n = I_1Z_{1n} + I_2Z_{2n} + I_3Z_{3n} + \cdots + I_nZ_{nn} \end{array}\right\} \quad (849a)$$

where V_1, V_2, V_3, $\cdots$, V_n are the applied driving voltages at the current antinodes (dipole gaps in the case of half-wave dipoles); Z_1, Z_2, Z_3, $\cdots$, Z_n are the self-values in the *presence of all radiators;* Z_{12}, Z_{13}, Z_{23}, $\cdots$ are the *mutual* impedance values; Z_{11}, Z_{22}, Z_{33}, $\cdots$ Z_{nn} are the self-impedances of radiator 1, 2, 3, $\cdots$, n, for only *one* radiator present. For a driven $\lambda/2$ linear radiator with a parasitic radiator, we find for the *antinode* currents

$$I_1 = \frac{Z_{11}}{Z_{11}^2 - Z_{12}^2} V_1 \text{ amp}; \qquad I_2 = -\frac{Z_{12}}{Z_{22}} I_1 \text{ amp}$$

V_1 is driving voltage across dipole gap of radiator 1 (849*b*)

The joulean heat losses within the linear radiators can, in many cases, be neglected, and the power radiated from the nth radiator of an array is $V_n I_n \cos \varphi$, where φ is the time phase angle of the complex impedance operator 2 with the real part R, since I^2R is the radiated power. The formulas for the self- and mutual impedances can also be computed for any current distributions such as $I_0F(z)$ and linear radiators other than a half wavelength long. A half wavelength is, however, used mostly in VHF and UHF work.

131. Mutual Impedance Effect on the Self-resistance and Self-reactance of a Linear Radiator. Let us take two half-wave radiators that are parallel with the Z axis and center driven (half-wave dipoles) with impressed voltages V_1 and V_2 volts. Let us further take the condition for which the entrance currents I_1 and I_2 effective amperes at respective dipole gaps are out of phase and such that $I_2 = kI_1\underline{/\psi}$. We have, therefore, out-of-phase and unequal input currents. For the radiation centers of the $\frac{1}{2}\lambda$ radiators along the X axis,

$$\mathcal{E}_\theta^{(\text{volts/m})} = j\frac{120I_1}{r} D_\theta \epsilon^{-j\beta r}[1 + k\epsilon^{j(\psi - m)}] \qquad m \text{ is phase angle of mutual impedance}$$

$$m = \beta s \sin\theta \cos\varphi; \qquad D_\theta = \frac{\cos(90^\circ \cos\theta)}{\sin\theta}$$

where s is the spacing in meters between radiator 1 and 2, $V_1 = I_1Z_{11} + I_2Z_m$ and $V_2 = I_1Z_m + I_2Z_{22}$ are impressed dipole volts, $Z_{11} = R_{11} + jX_{11}$ is the self-impedance of radiator 1 with radiator 2 absent, $Z_{22} = R_{22} + jX_{22}$ is the self-impedance of radiator 2 with radiator 1 absent, $Z_m = R_m + jX_m$ is the mutual impedance from 1 on 2 and 2 on 1 with the absolute amount $M = \sqrt{R_m^2 + X_m^2}$, and X_m can be positive or negative.

Hence,

$$V_1 = I_1(R_{11} + jX_{11}) + kMI_1\epsilon^{j(\psi - m)} = I_1[R_{11} + jX_{11} + kM\epsilon^{j(\psi - m)}]$$

$$V_2 = I_1M\epsilon^{-j(\psi + m)} + kI_1(R_{22} + jX_{22}) = I_2\left[R_{22} + jX_{22} + \frac{M}{k}\epsilon^{-j(\psi + m)}\right]$$

$$\frac{V_1}{I_1} = Z_1 \text{ (changed self-impedance)} = R_{11} + jX_{11} + kM\cos(\psi - m) + jkM\sin(\psi - m)$$

$$= \underbrace{R_{11} + kM\cos(\psi - m)}_{\text{changed self-resistance } R_1} + j\underbrace{[X_{11} + kM\sin(\psi - m)]}_{\text{changed self-reactance } X_1}$$

$$\frac{V_2}{I_2} = Z_2 = \underbrace{R_{22} + \frac{M\cos(\psi + m)}{k}}_{R_2} + j\underbrace{\left[X_{22} - \frac{M\sin(\psi + m)}{k}\right]}_{= X_2}$$

where R_2 is changed self-resistance and X_2 is changed self-reactance. For only inphase ($\psi = 0$ deg) and antiphase ($\psi = 180$ deg) excitations and equal currents ($I_1 = I_2$ effective amperes), we have $R_1 = R_2$ ohms, but still different from the R_{11} and R_{22} values, holding only for one radiator without the presence of the other radiator.

For the radiator 2 parasitic, m is the angle of the mutual impedance $Z_m = Z_{12} = Z_{21}$, $\alpha = \tan^{-1}(X_{22}/R_{22})$, and ψ is the angle by which I_2 leads I_1, *i.e.*, $\psi = 180° + m - \alpha$. We find, for the driving voltage V_1 at the dipole gap of radiator 1,

$$V_1 = I_1Z_{11} + I_2Z_m \qquad \text{for driven } \frac{\lambda}{2} \text{ radiator 1}$$

$$0 = I_1Z_m + I_2Z_{22} \qquad \text{for parasitic } \frac{\lambda}{2} \text{ radiator 2}$$

$$I_2 = -\frac{I_1Z_m}{Z_{22}} = \frac{-MI_1}{M_{22}}\epsilon^{j(m-\alpha)} = +MI_1\epsilon^{j\psi}$$

since $Z_m = \sqrt{R_m^2 + X_m^2}\,\epsilon^{jm} = M\epsilon^{jm}$, $Z_{22} = R_{22} + jX_{22} = \sqrt{R_{22}^2 + X_{22}^2}\,\epsilon^{j\alpha} = M_{22}\epsilon^{j\alpha}$,

$$\mathcal{E}_\theta^{(\text{volts/m})} = j\frac{60}{r}D_\theta\epsilon^{-j\beta r}(I_1 + I_2\epsilon^{-j\beta s \sin\theta\cos\varphi})$$

$$= j\frac{60}{r}D_\theta I_1\epsilon^{-j\beta r}\left[1 + \frac{M}{M_{22}}\epsilon^{-j(\psi - \beta s \sin\theta\cos\varphi)}\right]$$

$$V_1^{(\text{volts})} = I_1\left(Z_{11} - \frac{Z_m^2}{Z_{22}}\right) = I_1\left[R_{11} + jX_{11} - \frac{M^2}{M_{22}}\epsilon^{j(2m-\alpha)}\right]$$

$$= I_1\left\{\underbrace{R_{11} - \frac{M^2}{M_{22}}\cos(2m - \alpha)}_{\substack{\text{changed self-resistance } R_1 \\ \text{of driven radiator 1}}} + j\underbrace{\left[X_{11} - \frac{M^2}{M_{22}}\sin(2m - \alpha)\right]}_{\substack{\text{changed self-reactance } X_1 \\ \text{of driven radiator 1}}}\right\}$$

The driven half-wave radiator 1 must then have such a self-reactance X_{11} *in the absence* of the parasitic radiator 2 that $X_{11} = (M^2/M_{22}) \sin(2m - \alpha)$. This can be *readily done by changing the length* of the driven radiator 1 somewhat. As already mentioned, the reactance effect can always be neutralized by tuning. The main purpose is to secure as good a radiation resistance as possible.

132. Remarks on the Physical Interpretation of the EMF Method. On page 650 of Sec. 130, we discussed some details with respect to the radiation paradox. We found that a linear current filament parallel to the Z axis yields, for the effective field values $\mathcal{E}_z(z)$ volts per meter and the effective current distribution of $I(z)$ amperes,

$$P = -\int_{-l/2}^{+l/2} \mathcal{E}_z(z)I^*(z)\,dz \qquad \text{watts (complex)}$$

We obtain not only the radiated power in watts but also the average

reactive interchange of the energy between the source and the field *next* to the antenna. The radiation paradox enters, since the electric field distribution $\mathcal{E}_z(z)$ along the total length of l meters of the linear radiator is tangential to the current filament. This filament is a conductor and ideally a perfect conductor. According to the boundary conditions, only a small $\mathcal{E}_z(z)$ component can exist along good conductors and none for an ideal conductor. The small electric field component transfers field power *into* and *not out of* the conductor surface; therefore, it *cannot* mean radiation into the outer space. We also found that for a half-wave dipole as in Fig. 141*a*, the respective collinear current filaments extend from $z = \frac{1}{2}d$ to $z = \frac{1}{2}l$ and $z = -\frac{1}{2}d$ to $z = -\frac{1}{2}l$. We have the complex power

$$P = -\int_{-d/2}^{+d/2} \underbrace{\mathcal{E}_z(z)}_{\text{volts/m}}\, \underbrace{I^*(z)}_{\text{amp}}\, \underbrace{dz}_{\text{m}} \quad \text{complex watts} = -I_1^* \int_{d/2}^{+d/2} \mathcal{E}_z(z)\, dz$$

$$= +I_1^* V_1 \quad \text{watts} \qquad (850)$$

since the impressed voltage V_1 *must overcome the induced voltage* $\int \mathcal{E}_z(z)\, dz$ at the dipole gap. This relation does not conflict, since the integration is made only over the surface enclosing the gap volume. This volume is air and can support a tangential $\mathcal{E}_z(z)$ component along the length of d meters. Any small tangential $\mathcal{E}_z(z)$ components along the conductor surface can exist only for finite conductivities of the conductor filaments. These components are then in antiphase with respect to the field acting along the gap air surface and cause an inward power flow instead of outward radiation. This concept then places the radiated power into the gap space. This is also the space where the entrance effective current I_1 delivers the HF power to the radiating system. According to this concept, the free ends would not contribute to the radiation, at least not in a direct way. What happens is that at respective free ends, current reflections must occur in order to secure zero current effect at these ends by means of a reflected antiphase current wave. This current wave together with the outward-moving current disturbance toward respective ends ($+l/2$ and $-l/2$) causes the stationary current distribution $I(z)$ and what happens actually in the dipole gap. Therefore, the conditions in this gap are not only due to the voltage V_1 that drives the supply effective value of I_1 amperes at the gap ends $\pm\frac{1}{2}d$ into the respective conductors but also due to the $I(z)$ distribution along the conductors. This distribution is, however, not only due to what happens at the gap ends but also due to what happens at the free ends $\pm\frac{1}{2}l$. Hence, the Poynting radiation stream outward over the gap surface is affected by either end of either conductor pole of the dipole. With such a physical interpretation, we are also in agreement that the attenuation along the

conductors has nothing to do with the radiation resistance but only with the effective HF resistance of respective conductors and the medium losses in the region into which the conductors are submerged. This interpretation is in accord with the fact that the power supply is in the gap space and the radiated power must come from this supply. It is also in accord that at the free ends, we deal with a condition of discontinuity.

As far as the approximate method for finding the radiated power is concerned, we assume a nearly sinusoidal current distribution along a thin conductor, and the electric field $\mathcal{E}_z(z)$ is computed from $I(z)$ by means of the retarded *magnetic* vector potential Λ. From the assumed *sinusoidal* distribution, the computed $\mathcal{E}_z(z)$ yields a value that is distributed over the radiator, *i.e.*, not only over the gap surface. The integral over the radiator will then be about the same as predicted by the gap value with a difference that is due to the difference between the assumed sinusoidal current distribution and the actual current distribution. The approximation is then based on the condition that for the $I(z)$ distribution sinusoidal with z, the $\mathcal{E}_z$ value differs from zero, and we have a distributed radiation impedance along the radiator with a real part $-\mathcal{E}_z(z)/I(z)$, which is the distributed radiation resistance. However, this distributed quantity will approach zero value as $I(z)$ approaches the actual distribution value.

133. Formulas for the Self- and Mutual Parameters of Coupled Linear Radiators. Formula (848a) gives the radiated power for one conductor. It was found from the Poynting vector $p = \text{Re}\,(\mathcal{E}_e \cdot H_e^*)$ watts per square meter if we deal with the effective field values, since by Green's theorem

$$P = \iiint_\tau \text{div}\,(p)\,d\tau = \iint_S (\eta p)\,dS \qquad \text{watts}$$

where the unit η vector expresses that the electromagnetic energy flux is to be taken normal outward to respective surface elements dS. If we have n conductors, the integration is simply to be extended over all n linear radiators of volume elements $d\tau$, since together they cause the resultant field. Since the total power flow in watts is equal to $-\iiint_\tau \mathcal{E}_e \cdot J_e^*\,d\tau$ (this power escapes over the surface of τ cubic meters of volume), we find from (848a)

$$P = \text{Re}\left(-\sum_{s=1}^{s=n}\iiint_{\tau_s}\sum_{r=1}^{r=n}\mathcal{E}_{sr}J_s^*\,d\tau_s\right) \qquad \text{watts}$$

$$= \text{Re}\left(-\sum^{s=n}\sum^{r=n}\int_{\zeta_{1s}}^{\zeta_{2s}}\mathcal{E}_{sr}I_s^*\,d\zeta_s\right)$$

where I_s^* is the complex conjugate value of the current distribution on the sth conductor and $\mathcal{E}_{sr}$ is the z component of the electric field at the sth conductor, due to the rth current. The volume integral has the corresponding current density J_s at the symmetry point of the volume element $d\tau$. It reduces to a line integral and to the form given if the linear radiators are all parallel with the Z axis. The coordinate system is as shown in Figs. 140 and 142 and the integrations must be extended over the total length of each of the parallel conductors (and strictly also over the end surfaces). The expression for the radiated power P, therefore, represents the rate at which the induced voltages do work on respective currents of respective radiators. The expressions are written in terms of the effective values; otherwise a factor $\frac{1}{2}$ would have to be used in front of the P expression. The electric field caused by the sth radiator is proportional to the current amplitude of this radiator. The instantaneous values of the current and the electric field are of the form

$$i_t = I_{os}(\zeta_s)\epsilon^{j\omega t} \quad \text{for current;} \qquad I_{or}U_{sr}\epsilon^{j\omega t} \quad \text{for electric field} \tag{851}$$

where U_{sr} is the coefficient introduced by R. Bechmann[1] and (ζ_s) takes care of the current distribution along the sth radiator, which is parallel with the Z axis. We find

$$P = \frac{1}{2}\,\mathrm{Re}\sum_{s=1}^{s=n}\sum_{r=1}^{r=n} I_{or}I_{os}^*Z_{sr} \quad \text{watts;}$$

$$\text{transfer impedance } Z_{sr} = \sqrt{R_{sr}^2 + X_{sr}^2}\,\Big/\tan^{-1}\left(\frac{X_{sr}}{R_{sr}}\right) \tag{851a}$$

This relation expresses the same thing as (849a) and gives, in addition, a means of deriving formulas for the transfer impedances with a *real* part, accounting for the corresponding radiation resistances with the following details:

The summation holds for $s = n$ and $r = n$, which means that we include all radiator current distributions as well as all induced fields along the n radiators. The transfer impedance is given by the general expression

$$Z_{sr} = -\int_{\zeta_{1s}}^{\zeta_{s}} U_{sr}(\zeta_s)\,d\zeta_s \quad \text{ohms} \tag{851b}$$

Hence, Z_{ss} represents the transfer impedance of the sth conductor with all other conductors removed. Therefore, its real part R_{ss} is the self-radiation resistance of the sth radiator, and jX_{ss} is the associated self-reactance. By the general *transfer* impedance Z_{sr}, we mean the mutual effect of the r radiator on the s radiator with a real component R_{sr} and a reactive component jX_{sr}. The electric field can then be computed, as is shown in Sec. 129, from the $\prod$ vector (Hertz vector), since this vector

[1] *Loc. cit.*

caused by the sth radiator is proportional to the current distribution of this radiator and the electric field $\mathcal{E}_{sr}$ is given by

$$\mathcal{E}_{sr} = \frac{\partial^2 \Pi_r}{\partial \zeta_s^2} + \beta^2 \Pi_r \qquad \text{because of } \frac{d^2 I_s^*}{d\zeta_s^2} + \beta^2 I_s^* = 0; \qquad \beta = \frac{2\Pi}{\lambda};$$

in the case of (ζ_s) is taken as a sinusoidal current distribution. Hence,

$$P = \frac{1}{2}\,\mathrm{Re}\left[\sum_{s=1}^{s=n}\sum_{r=1}^{r=n}\left(\Pi_r \frac{\partial I_s^*}{\partial \zeta_s} - I_s^* \frac{\partial \Pi_r}{\partial \zeta_s}\right)_{\zeta_{1s}}^{\zeta_{2s}}\right] \text{ watts}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{\sum_{s=1}^{s=n}\sum_{r=1}^{r=n} \beta I_{os}^*[(-1)^{k_s}\Pi_r(\zeta_{2s}) - \Pi_r(\zeta_{1s})]\right\} \qquad \text{for } \frac{\lambda}{2} \text{ distributions}$$

The first result applies to cases for which the current *at the radiator ends is not zero* (coil or capacity loaded at the ends ζ_{1s} and ζ_{2s}), while the second expression applies to customary VHF and UHF cases, where we have half-wave radiators ($k_s = 1$), since $k_s = 1, 2, 3, \cdots$ for successive excitations of this type. We come then to expressions, as dealt with in Sec. 129, that for the general case (first expression for P) yield the Hertz vector

$$\Pi_r^{(\text{in volts})} = 717.5\,\frac{I_{or}\epsilon^{j\omega t}}{f^{(\text{Mc})}}\,\{\epsilon^{j(\beta\zeta_s-\psi)}[\mathrm{Ci}\,(\beta u) + j\,\mathrm{Si}\,(\beta u)]_{u_{1r}}^{u_{2r}} + \epsilon^{-j(\beta\zeta_s-\psi)}A\}$$

$$A = [\mathrm{Ci}\,(\beta v) + j\,\mathrm{Si}\,(\beta v)]_{v_{1r}}^{v_{2r}}; \qquad u = r_{rs} + \zeta_r - \zeta_s; \qquad v = r_{rs} - \zeta_r + \zeta_s$$

Since we have respective n linear radiators parallel to the Z axis, the coordinate ζ_r refers to any point of the rth radiator and ζ_s to any point of the sth radiator. The lower point of the rth radiator is then defined by the coordinate ζ_{r1}, and ζ_{s2} means that we deal with the upper end of the sth radiator. For the same reason, I_{or} is the antinode current of the rth radiator, since I_r represents currents along the rth radiator. Since

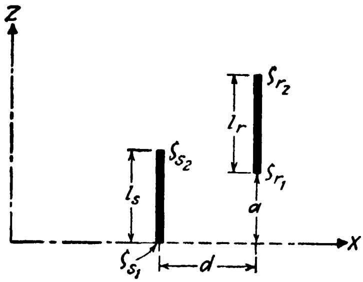
$$\mathrm{Ei}\,(jx) = \mathrm{Ci}\,x + j\,\mathrm{Si}\,x; \qquad \mathrm{Ei}\,(-jx) = \mathrm{Ci}\,x - j\,\mathrm{Si}\,x$$

we find, for two staggered linear radiators that are any integer multiple of one-half wavelength long, the Hertz vector as given in Fig. 142. In this figure, we have only the sth conductor and the rth conductor present and radiating. The length of the s radiator is then l_s meters, but such that it is equal to $\frac{1}{2}\lambda$, λ, 1.5λ, . . . The length of the r radiator is then l_r meters and may be any integer half wavelength long. On account of the staggered arrangement, the respective radiators are still parallel but shifted somewhat against each other with respect to the equatorial plane of each radiator. For this reason, we have the vertical coordinates for respective radiator ends $\zeta_{s1} = 0$, $\zeta_{s2} = l_s$, $\zeta_{r1} = a$, and $\zeta_{r2} = a + l_r$, while the relative separation in the equatorial plane is d meters. On account of the staggered arrangements, the current function of the s radiator is $\sin(\beta\zeta_s)$, but the current function of the r radiator is $\sin[\beta(\zeta_r - a)]$. According to (851) and (851*b*), the Γ_{sr} coefficient is in ohms per meter and Z_{sr}, which is the transfer impedance, is in ohms with a real component representing a radiation resistance contribution and an imaginary part referring to a reactive field contribution. We have then the expression

$$\sum Z_{sr} = -\sum_{s=1}^{s=n}\sum_{r=1}^{r=n}\int_{1s}^{2s} U'_{sr}\,d\zeta_s \qquad \text{ohms}$$

Hence, for any one association of an s radiator and an r radiator, we find the mutual transfer impedance of the radiator s on the radiator r

$$\left.\begin{aligned}
Z_{rs} &= 15\{\epsilon^{j\beta a}[\mathrm{Ei}\,(-j\beta u_{22}) + \mathrm{Ei}\,(-j\beta u_{11}) - \mathrm{Ei}\,(-j\beta u_{21}) - \mathrm{Ei}\,(-j\beta u_{12})] \\
&\quad + \epsilon^{-j\beta a}[\mathrm{Ei}\,(-j\beta v_{22}) + \mathrm{Ei}\,(-j\beta v_{11}) - \mathrm{Ei}\,(-j\beta v_{21}) - \mathrm{Ei}\,(-j\beta v_{12})]\} \quad \text{ohms} \\
\mathrm{Ei}\,(-jx) &= \mathrm{Ci}\,(x) - j\,\mathrm{Si}\,(x) \\
u_{22}, v_{22} &= \sqrt{d^2 + (a + l_r - l_s)^2} \mp (a + l_r - l_s) \\
u_{12}, v_{12} &= \sqrt{d^2 + (a + l_r)^2} \mp (a + l_r) \\
u_{21}, v_{21} &= \sqrt{d^2 + (a - l_s)^2} \mp (a - l_s); \qquad u_{11}, v_{11} = \sqrt{d^2 + a^2} \mp a
\end{aligned}\right\} \quad (851c)$$



$l_s = k_s \frac{\lambda}{2}$ for $k_s = 1, 2, 3, 4$

$l_r = k_r \frac{\lambda}{2}$ for $k_r = 1, 2, 3, 4$

We have the substitutions

$$u_1 = \sqrt{d^2 + (\zeta - \zeta_{r_1})^2} - (\zeta - \zeta_{r_1}), \quad u_2 = \sqrt{d^2 + (\zeta - \zeta_{r_2})^2} - (\zeta - \zeta_{r_2})$$

$$v_1 = \sqrt{d^2 + (\zeta - \zeta_{r_1})^2} + (\zeta - \zeta_{r_1}), \quad v_2 = \sqrt{d^2 + (\zeta - \zeta_{r_2})^2} + (\zeta - \zeta_{r_2})$$

$$\Pi_r^{(\text{m volts})} = 717.5\,\frac{I_0\,\epsilon^{j\omega t}}{f_{(\mathrm{Mc})}}\,\{\epsilon^{j\beta(\zeta-a)}[\mathrm{Ei}\,(j\beta u_2) - \mathrm{Ei}\,(j\beta u_1)] + \epsilon^{-j\beta(\zeta-a)}[\mathrm{Ei}\,(j\beta v_2) - \mathrm{Ei}\,(j\beta v)]\}$$

FIG. 142. Relations for staggered linear radiators in the plane of the radiators.

Formula (851c) is then the expression for the mutual transfer impedance for the staggered linear radiator array of Fig. 142 in the case of equal effective currents in amperes at the current antinode of each radiator. This is quite a general expression, since it can take care of two parallel linear radiators, for which $a = 0$. It can also take care of the case of collinear radiators, for which $d = 0$. Hence, we can readily cover with this formula the useful alignments with half-wave radiators in parallel and with half-wave radiators that are collinear. We can also cover the curtain of half-wave radiators, since it is simply the combination of parallel and collinear radiators. The application is not difficult. Suppose we deal with two half-wave radiators that are parallel and not staggered. Then, $a = 0$, $l_s = l_r = l = \frac{1}{2}\lambda$ meters and d meters for the spacing between the parallel radiators. We find then

$$u_{22} = u_{11} = v_{11} = v_{22} = d \quad \text{m}; \qquad u_{12} = v_{12} = \sqrt{d^2 + l^2} - l \quad \text{m};$$
$$v_{12} = u_{21} = \sqrt{d^2 + l^2} + l \qquad \text{m}$$

and the formula

$$Z_{rs} = 30\{2\ \mathrm{Ei}\ (-j\beta d) - \mathrm{Ei}\ [-j\beta(\sqrt{d^2 + l^2} + l)] - \mathrm{Ei}\ [-j\beta(\sqrt{d^2 + l^2} - l)]\} = R_{rs} - jX_{rs} \quad \text{complex ohms}$$

$$= \underbrace{60\ \mathrm{Ci}\left(\frac{2\pi}{\lambda} d\right) - 30\ \mathrm{Ci}\left[\frac{2\pi}{\lambda}(\sqrt{d^2 + l^2} + l)\right] - 30\ \mathrm{Ci}\left[\frac{2\pi}{\lambda}(\sqrt{d^2 + l^2} - l)\right]}_{= R_{rs} \text{ resistive ohms}}$$

$$-j\underbrace{\left\{60\ \mathrm{Si}\left(\frac{2\pi}{\lambda} d\right) - 30\ \mathrm{Si}\left[\frac{2\pi}{\lambda}(\sqrt{d^2 + l^2} + l)\right] - 30\ \mathrm{Si}\left[\frac{2\pi}{\lambda}(\sqrt{d^2 + l^2} - l)\right]\right.}_{= X_{rs} \text{ reactive ohms}} \quad (851d)$$

The details of this formula are

$$Z_{rs} = 15[\mathrm{Ei}\ (-j\beta u_{22}) + \mathrm{Ei}\ (-j\beta u_{22}) - \mathrm{Ei}\ (-j\beta u_{21}) - \mathrm{Ei}\ (-j\beta u_{12}) + \mathrm{Ei}\ (-j\beta u_{22}) + \mathrm{Ei}\ (-j\beta u_{22}) - \mathrm{Ei}\ (-j\beta u_{12}) - \mathrm{Ei}\ (-j\beta u_{21})]$$

$$= 15[4\ \mathrm{Ei}\ (-j\beta u_{22}) - 2\ \mathrm{Ei}\ (-j\beta u_{21}) - 2\ \mathrm{Ei}\ (-j\beta u_{12})]$$

$$= 30\{2\ \mathrm{Ei}\ (-j\underbrace{\beta d}_{A}) - \mathrm{Ei}\ [-j\underbrace{\beta(\sqrt{d^2 + l^2} + l)}_{B}] - \mathrm{Ei}\ [-j\underbrace{\beta(\sqrt{d^2 + l^2} - l)}_{C}]\}$$

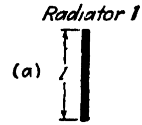
$$= 30[2\ \mathrm{Ci}\ (A) - j2\ \mathrm{Si}\ (A) - \mathrm{Ci}\ (B) + j\ \mathrm{Si}\ (B) - \mathrm{Ci}\ (C) + j\ \mathrm{Si}\ (C)]$$

$$= \underbrace{60\ \mathrm{Ci}\ (A) - 30\ \mathrm{Ci}\ (B) - 30\ \mathrm{Ci}\ (C)}_{R_{rs} \text{ ohms}} - j\underbrace{[60\ \mathrm{Si}\ (A) - 30\ \mathrm{Si}\ (B) - 30\ \mathrm{Si}\ (C)]}_{X_{rs} \text{ ohms}}$$

This represents then the mutual impedance from the s radiator on the r radiator. Since we have taken two equal radiators of length $l = \frac{1}{2}\lambda$ meters, we have $Z_{sr} = Z_{rs} = Z_m$ complex ohms, because the mutual transfer impedance of r on s must be the same as from s on r. We still have to account for the self-impedances Z_{ss} and Z_{rr} complex ohms, since $Z_s = Z_{ss} + Z_m = Z_r$, where Z_s denotes the total transfer impedance of radiator s and $Z_r = Z_{rr} + Z_m$ the total transfer impedance of radiator r. The total transfer impedance of both radiators acting as a whole is then either $2Z_r$ or $2Z_s$ ohms, since the self-impedances Z_{rr} and Z_{ss} of either radiator, without the other radiator present, must be alike. This self-impedance is found from the Z_{rs} expression, by using the spacing d *approaching* zero and putting $a = 0$. The mutual impedance Z_{rs} for two collinear half-wave radiators is found by preserving the stagger shift a, where a must be larger than the length l of each radiator, and taking the limit for d *approaching* zero value. This has to be done, since Ci (0) is infinite and we should otherwise get the indeterminate form $\infty - \infty$. We have then the formulas tabulated in Fig. 143 together with the dimensional data. Suppose we have a single half-wave radiator. Then $l = \frac{1}{2}\lambda$, $\beta l = (2\pi/\lambda)(\frac{1}{2}\lambda) = \pi$, and

$$Z_{11} = 30[0.5772 + 2.303 \underbrace{\log_{10}(6.28)}_{0.798} - C_i(6.28) + j\,\underbrace{S_i(6.28)}_{1.43}]$$
$$= 73.4 + j42.9 \text{ ohms}$$

showing a radiation resistance of $R_{11} = 73.4$ ohms and an inductive reactance of $X_{11} = 42.9$ ohms which can be compensated by choosing

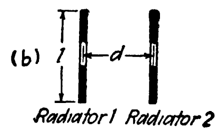


One linear radiator

$Z_{self} = Z_{11} = 30[0.5772 + 2.302 \log_{10}(2\beta l) - C_i(2\beta l) + S_i(2\beta l)] = R_{11} + jX_{11}$ ohms

$\log_{10}$, C_i and S_i functions found from tables or from curves

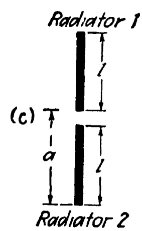
$\beta = \frac{2\pi}{\lambda}$, $Z_s = 73.4 + j42.9$ ohms for $l = \frac{\lambda}{2}$



Two parallel radiators

$Z_{12} = 60\,C_i(A) - 30\,C_i(B) - 30\,C_i(C) - j[60\,S_i(A) - 30\,S_i(B) - 30\,S_i(C)] = R_m - jX_m$ ohms

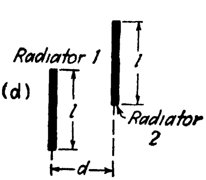
$A = \beta d$, $B = \beta(\sqrt{d^2 + l^2} + l)$, $C = \beta(\sqrt{d^2 + l^2} - l)$



Two collinear radiators $a > l$

$Z_m = Z_1 \ldots \cos(D)[-2\,C_i(2D) + C_i(E) + C_i(F) - 2.302 \log_{10}(G)]$
$+ \ldots \sin(D)[2\,S_i(2D) - S_i(E) - S_i(F)]$
$- j \ldots \cos(D)[2\,S_i(2D) - S_i(E) - S_i(F)]$
$+ j \ldots \sin(D)[2\,C_i(2D) - C_i(E) - C_i(F) - 2.302 \log_{10}(G)]$ complex ohms

$D = \beta a$, $E = 2\beta(a - l)$, $F = 2\beta(a + l)$, $G = \frac{a^2 - l^2}{a^2}$



Two staggered parallel radiators

$Z_m = \ldots 15 e^{D}\{-2 E_i(H) + E_i(K) + E_i(L)\}$
$-15 e^{-D}\{-2 E_i(M) + E_i(N) + E_i(O)\}$ complex ohms

$D = j\beta a$, $H = j\beta(\sqrt{d^2 + a^2} - a)$, $K = -j\beta[\sqrt{d^2 + (a - l)^2} - (a - l)]$

$L = -j\beta[\sqrt{d^2 + (a + l)^2} - (a + l)]$, $M = j\beta(\sqrt{d^2 + a^2} + a)$

$O = -j\beta[\sqrt{d^2 + (a + l)^2} + (a + l)]$, $N = j\beta[\sqrt{d^2 + (a - l)^2} + (a - l)]$

FIG 143 Expressions for the impedance of one linear radiator and coupled linear radiators of small cross-sectional dimensions and sinusoidal current distributions

the linear conductor somewhat shorter than a half wavelength Table 29 gives the computed radiation resistance values for different half-wave

TABLE 29*

l	0.5λ	1λ	1.5λ	2λ	2.5λ	3λ	3.5λ	4λ	4.5λ	5λ
R_{11}	73.4	93.15	106.47	115.4	120.55	126.25	131.25	134.65	138.45	141.45

* R_{11} in ohms is self radiation resistance R_s

distributions of a single linear radiator This shows that the radiation resistance is not increasing proportionally to l/λ but much slower. The formulas of Fig 143 hold only for *integer* multiples of half-wave distribu-

tions, while the general formula, for which at least one end of the linear radiator exhibits a current value, is

$$R_{11} = 30\left[2.303 \log_{10}\left(\frac{2\pi k l}{\lambda}\right) - \sin^2\left(\frac{2\pi l}{\lambda}\right) + \mathrm{Ci}\left(\frac{4\pi l}{\lambda}\right)\right] \quad \text{ohms}$$

for the factor $k = 3.56$. The radiation resistance then fluctuates as in Fig. 144 into a saturation trend. For a dipole with a distribution as in

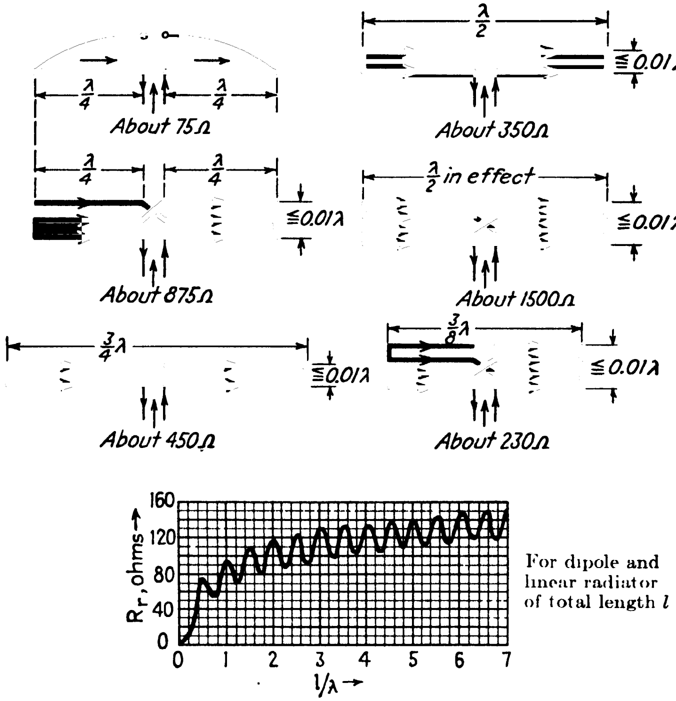


FIG. 144. Radiation resistance of special dipoles and of a linear radiator of length l in unobstructed radiation space.

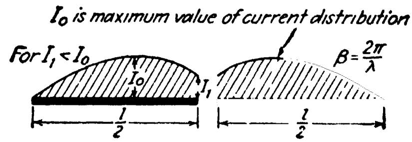
Fig. 145, we have a radiation resistance as given in this figure. There are many other numerical applications throughout this text of formulas of the type given in this section.

134. Elementary Theory for the Self- and Mutual Impedances of Linear Radiators Excited in Half-wave Current Distributions. The self- and mutual impedances are given by the relations

$$Z_{11}^{(\text{ohms})} = -\frac{\int_0^l \mathcal{E}_{11} \sin(\beta z)\, dz}{I_1} = Z_s; \quad Z_{12}^{(\text{ohms})} = -\frac{\int_0^l \mathcal{E}_{12} \sin(\beta z)\, dz}{I_1} = Z_m$$

since the self-impedance Z_{11} is equal to the induced voltage $-E_{11} = -\int_0^l \mathcal{E}_{11} \sin(\beta z)\, dz$, due to the current I_1 of an isolated radiator of length

l and the mutual impedance Z_{12} is equal to the total voltage $-E_{12} = -\int_0^l \mathcal{E}_{12} \sin (\beta z)\, dz$ induced in radiator 2 due to the current I_1 of radiator 1. It is assumed that we deal with thin linear radiators of equal length l parallel with the Z direction and that we have a sinusoidal current distribution with I_1 at the current antinode For a half-wave dipole, I_1 would be the current value at the dipole gap and the fields $\mathcal{E}_{11} \sin (\beta z)$



R_{11} refers to maximum current value I_0

$$R_{11} = 34.6 + 138.12 \log_{10} (\beta l) - 60\, \mathrm{Ci}\, (\beta l) + 30 \sin (\beta l)[\mathrm{Si}\, (2\beta l) - 2\, \mathrm{Si}\, (\beta l)] + 30 \cos (\beta l)\left[0.5772 + 2.302 \log_{10} \left(\frac{\beta l}{2}\right) + \mathrm{Ci}\, (2\beta l) - 2\, \mathrm{Ci}\, (\beta l)\right] \text{ ohms}$$

FIG 145 Radiation resistance of a dipole different from a half-wave distribution.

and $\mathcal{E}_{12} \sin (\beta z)$ volts per meter are tangential to radiator 1 or radiator 2, respectively. The voltage impressed on radiator 1 without the presence of radiator 2 must be $+V_1$ volts in order to counteract the induced voltage $-E_{11}$ at the dipole gap. For radiator 2, with radiator 1 absent, the impressed voltage is $V_2 = -E_{22}$ volts It requires an additional voltage V_{12} to counteract the effect of the additional induced voltage $-E_{12}$ in radiator 2 if the radiator 1 is present also. Suppose we have a single linear radiator 1 of length l with any integer half-wavelength current distribution; then we can find the field intensity $\mathcal{E}_z$ in effective volts per meter either by the Hertz vector method, as given in Sec. 129, or by the method using the magnetic vector potential. We have to solve for cylindrical coordinates as in Fig. 146 for a field point $P(d, \varphi, z)$, since we are interested only in tangential field components along the radiator. For the effective antinode current I_e in amperes and the distances r_1 and r_2 in meters, we have the field component parallel with the Z axis as given in Fig. 146. For a half-wave radiator, we have $k = 1$ in this formula. Since, in the expression for the self-impedance Z_{11}, the intensity $\mathcal{E}_{11}$ is the $\mathcal{E}_z$ value when the field point P moves to the surface of the radiator, we have to put $r_1 = z$ and $r_2 = l - z$ and find

$$Z_{11} = -j30 \int_0^l \left[\frac{\epsilon^{-j\beta(l-z)}}{l - z} (-1)^k - \frac{\epsilon^{-j\beta z}}{z}\right] \sin (\beta z)\, dz$$

$$= -15 \int_0^l \left(\frac{\epsilon^{-j2\beta z} - 1}{l - z} + \frac{\epsilon^{-j2\beta z} - 1}{z}\right) dz$$

$$= -30 \int_0^{2\beta l} \frac{\epsilon^{-ju} - 1}{u}\, du = 30[0.5772 + 2.303 \log_{10} (2\beta l) - \mathrm{Ci}\, (2\beta l)] + j30\, \mathrm{Si}\, (2\beta l) \quad \text{ohms}$$

The details are as follows:

$$\beta l = \frac{2\pi}{\lambda} l = \frac{2\pi}{\lambda} k \frac{\lambda}{2} = k\pi; \qquad \epsilon^{-j\beta l} = \epsilon^{-jk\pi} = \cos(k\pi) - j\sin(k\pi)$$

where $\sin(k\pi)$ is always zero for $k = 1, 2, 3, 4, 5, 6, \cdots$.

$\cos(\pi) = -1, \qquad \cos(2\pi) = +1; \qquad \cos(3\pi) = -1; \cdots, \qquad$ and

$$\epsilon^{-j\beta l} = (-1)^k = -1, +1, -1, +1, \cdots$$

$$\epsilon^{-j\beta(l-z)} = \epsilon^{-j\beta l}\epsilon^{+j\beta z} = (-1)^k \epsilon^{j\beta z}$$

$$\frac{\epsilon^{-j\beta(l-z)}}{l-z}(-1)^k = \frac{\epsilon^{j\beta z}}{l-z}(-1)^{2k} = \frac{\epsilon^{j\beta z}}{l-z}, \qquad \sin(\beta z) = \frac{\epsilon^{j\beta z} - \epsilon^{-j\beta z}}{2j}$$

$$\left[\frac{\epsilon^{-j\beta(l-z)}}{l-z}(-1)^k - \frac{\epsilon^{-j\beta z}}{z}\right]\sin(\beta z) = \left(\frac{\epsilon^{j\beta z}}{l-z} - \frac{\epsilon^{-j\beta z}}{z}\right)\frac{\epsilon^{j\beta z} - \epsilon^{-j\beta z}}{2}$$

$$= \frac{\epsilon^{j2\beta z} - 1}{j2(l-z)} - \frac{1 - \epsilon^{-j2\beta z}}{j2z}$$

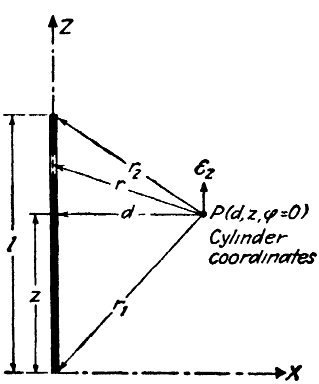
Therefore,

$$Z_{11} = -15\int_0^l \left(\frac{\epsilon^{j2\beta z} - 1}{l-z} + \frac{\epsilon^{-j2\beta z} - 1}{z}\right)dz = -30\int_0^{2\beta l} \frac{\epsilon^{-ju} - 1}{u}\,du$$

$$= -30\int_0^{2\beta l} \frac{\cos(u) - j\sin(u) - 1}{u}\,du = +30\int_0^{2\beta l} \frac{1 - \cos(u) + j\sin(u)}{u}\,du$$

$$= \underbrace{30\int_0^{2\beta l} \frac{1-\cos(u)}{u}\,du}_{30[0.577 + \log_\epsilon(2\beta l) - \mathrm{Ci}(2\beta l)]} + \underbrace{j30\int_0^{2\beta l} \frac{\sin(u)}{u}\,du}_{j30\,\mathrm{Si}(2\beta l)}$$

according to the formulas in Sec. 129.



$$\mathcal{E}_z^{(\text{volts/m})} = j30I_e\left[\frac{\epsilon^{-j\beta r_2}}{r_2}(-1)^k - \frac{\epsilon^{-j\beta r_1}}{r_1}\right]$$

$$l = k\frac{\lambda}{2},\ k = 1, 2, 3, 4$$

I_e is effective amp at antinode, and $\mathcal{E}_z$ effective volts/m

FIG. 146. Expression for the electric field parallel with the linear radiator l.

The values for the Ci x and Si x functions are taken from tables or curves, as given in Fig. 147. We note that the Ci x curve starts from $-\infty$ value, decreases its negative value with increasing argument x, intersects the zero reference axis, and oscillates about this axis into zero value (vanishingly small value). Therefore, for many applications, the

Ci x value cannot be significant. It will also be noted that for the argument of 2π, we have essentially zero value for Ci (2π). The Si x function starts with zero value, builds up gradually with increasing argument x,

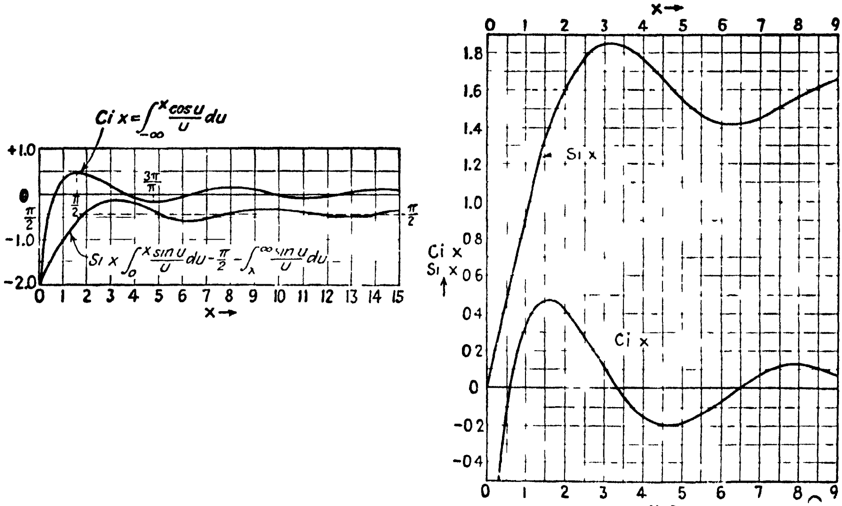


FIG 147. Integral sine and integral cosine curves For large arguments Ci x approaches zero value and Si x approaches a value 1.57

and oscillates about a $\pi/2$ value gradually into such a value. This can also be understood from the definitions of the cosine integral and the sine integral functions, since

$$\text{Ci}\,x = \int_{-\infty}^{x} \frac{\cos (u)}{u}\,du; \qquad \text{Si}\,x = \int_0^x \frac{\sin (u)}{u}\,du = \frac{\pi}{2} - \int_x^\infty \frac{\sin (u)}{u}\,du$$

$$\text{Si}\,x = x - \frac{1}{3}\frac{x^3}{2\cdot 3} + \frac{1}{5}\frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \frac{1}{7}\frac{x^7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \cdots$$

for the limits $-17 \leqq x \leqq 17$

$$\text{Ci}\,x = 0.5772 + \frac{1}{4}\log_e (x^4) - \frac{1}{2}\frac{x^2}{2} + \frac{1}{4}\frac{x^4}{2\cdot 3\cdot 4}$$

$$- \frac{1}{6}\frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots$$

for the limits $-17 \leqq x \leqq +17$

$$\text{Si}\,x = \frac{\pi}{2} - \frac{\cos x}{x}\left(1 - \frac{2}{x^2} + \frac{2\cdot 3\cdot 4}{x^4} - \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{x^6} + \cdots\right)$$

$$- \frac{\sin x}{x}\left(\frac{1}{x} - \frac{2\cdot 3}{x^3} + \frac{2\cdot 3\cdot 4\cdot 5}{x^5} - \cdots\right)$$

for *real* value $x > 17$

$$\text{Ci}\,x = \frac{\sin x}{x}\left(1 - \frac{2}{x^2} + \frac{2\cdot 3\cdot 4}{x^4} - \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{x^6} + \cdots\right)$$

$$- \frac{\cos x}{x}\left(\frac{1}{x} - \frac{2\cdot 3}{x^3} + \frac{2\cdot 3\cdot 4\cdot 5}{x^5} - \cdots\right)$$

for *real* values $x > 17$.

Therefore, for large values of the argument x, we have essentially

$$\text{Si}\, x = \frac{\pi}{2} - \frac{\cos x}{x}; \qquad \text{Ci}\, x = \frac{\sin x}{x}$$

The integrations for all cases of these functions are reduced to simple computations by means of tables or curves. It will be found that Fig 147 is most useful.

Inasmuch as we have found that the self-impedance Z_s of a linear radiator contains a real (resistive) and an imaginary (reactive) component, the tangential electric field component $\mathcal{E}_s$, for an assumed sinusoidal current distribution along the isolated radiator, must make a phase difference with the I_z current, since only the component $\mathcal{E}_s \cos \varphi_s$ is in antiphase with the I_z current, where the subscript z signifies that the radiator is parallel with the Z direction. The field intensity and the current are taken as effective values, which vary sinusoidally along the radiator. The radiated power of a radiator element dz is, therefore, $dP_s = -\mathcal{E}_s I_z \cos \varphi_z$ watts. If we have a like radiator with same excitation, parallel to the radiator mentioned above, the radiated power of the first radiator will be affected by the mutual impedance Z_m of the second radiator on the first radiator and cause a radiation power change of $dP_m = -\mathcal{E}_m I_z \cos \varphi_m$ watts in a radiator element. We have, for the total radiated power of both radiators of length l each, for an effective current distribution I_z along the radiators,

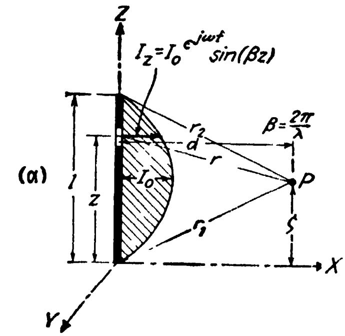
$$P_r^{(\text{watts})} = \underbrace{-\underbrace{\int_0^l \mathcal{E}_s I_z \cos \varphi_s \, dz}_{\text{self watts } P_s \text{ of radiator 1 for radiator 2 absent}} - \underbrace{\int_0^l \mathcal{E}_m I_z \cos \varphi_m \, dz}_{\text{mutual watts } P_m \text{ due to presence of radiator 2}}}_{\text{radiator power of radiator 1}}$$

$$\underbrace{-\underbrace{\int_0^l \mathcal{E}_s I_z \cos \varphi_s \, dz}_{P_s \text{ of radiator 2 for radiator 1 absent}} - \underbrace{\int_0^l \mathcal{E}_m I_z \cos \varphi_m \, dz}_{P_m \text{ mutual effect of radiator 1 on radiator 2}}}_{\text{radiated power of radiator 2}}$$

This means that the total radiated power of both like radiators is $2P_s + 2P_m$ watts and the total impedance is $2(Z_s + Z_m)$ ohms. The real component of the total impedance must be the total radiation resistance $R_r = 2(R_s + R_m)$ ohms.

135. Numerical Application of the Self- and Mutual Impedance Formulas of Linear Radiators. The formula for P_r in the last section is basic for the applications. Inasmuch as in short-wave work we employ mostly half-wave radiators, the essential relations are given in Fig. 148 with respect to such cases. The derivation of the expression for the

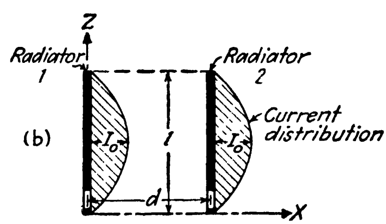
instantaneous electric field at point P is the same as given at other places of this text, except that we can no longer assume r_1 and r_2 equal. The field point is now close to the radiator and for the final application at



$r = \sqrt{d^2 + (z - \zeta)^2}$; $r_1 = \sqrt{d^2 + \zeta^2}$; $r_2 = \sqrt{d^2 + (l - \zeta)^2}$

$\mathcal{E}_P = j30I_0 \left[\frac{\epsilon^{j(\omega t - \beta r_2)}}{r_2} \cos(\beta l) - \frac{\epsilon^{j(\omega t - \beta r_1)}}{r_1} \right]$ instantaneous volts/m

for any integer $\lambda/2$ distribution along the radiator length l in meters. For P on the surface of the radiator, $d = 0$ (a thin radiator is assumed), and we have $r_1 = \zeta$ and $r_2 = l - \zeta$. Since $l = \lambda/2$, we have $2\beta l = 2 \times (2\pi/\lambda)\lambda/2 = 2\pi = 6.28$.
$R_s = R_{11} = 17.316 + 69.06 \log_{10}(6.28) - 30 \text{ Ci } (6.28) = 73.1$ ohms
$X_s = X_{11} = 30 \text{ Si } (6.28) = 42.9$ ohms



Mutual power from one radiator on the other radiator is
$P_{m(\text{watts})} = 30I_e^2\{2 \text{ Ci } (\beta d) - \text{Ci } [\beta(\sqrt{d^2 + l^2} + l)] - \text{Ci } [\beta(\sqrt{d^2 + l^2} - l)]\}$
I_e is effective amp at the current antinode
Mutual radiation resistance from one radiator on the other is
$R_{12} = R_{21} = R_m = 60 \text{ Ci } (\beta d) - 30 \text{ Ci } [\beta(\sqrt{d^2 + l^2} + l)] - 30 \text{ Ci } [\beta(\sqrt{d^2 + l^2} - l)]$ active ohms
Total radiation resistance $R_{r(\text{ohms})} = 2R_{11} + 2R_m$ for *in*phase excitation
Total radiation resistance $R_{r(\text{ohms})} = 2R_{11} - 2R_m$ for *anti*phase excitation

FIG. 148. Self- and mutual radiation resistance of linear thin half-wave radiators.

the surface of a radiator. From Fig. 148a, we find the field at the surface of the half-wave radiator by putting $d = 0$, resulting in

$$r_1 = \zeta; \qquad r_2 = l - \zeta;$$

$$\mathcal{E}_P = \mathcal{E}_z = j30I_0 \left[\frac{\epsilon^{j(\omega t - \beta l + \beta\zeta)}}{l - \zeta} \cos(\beta l) - \frac{\epsilon^{j(\omega t - \beta\zeta)}}{\zeta} \right] \quad \text{instantaneous volts/m}$$

with effective components $\mathcal{E}'_e$ and $\mathcal{E}''_e$, and

$$\mathcal{E}'_e = \frac{30I_e}{l-\zeta}\underbrace{\cos\left(\beta\frac{\lambda}{2}\right)}_{=-1}/\varphi' = -\frac{30I_e}{l-\zeta}\underline{/\varphi'}; \qquad \mathcal{E}''_e = -\frac{30I_e}{\zeta}\underline{/\varphi''} \qquad \text{volts/m}$$

$$\varphi' = -\beta l + \beta\zeta + \frac{\pi}{2}; \qquad \varphi'' = -\beta\zeta + \frac{\pi}{2}$$

since the j factor means a phase factor $\epsilon^{+j90°} = \epsilon^{+j\pi/2}$, with respect to the effective current I_e amperes at the current antinode of the linear radiator. The j factor actually means that for a time function $I_0 \sin(\omega t)$ of the antinode current, we have now a $\cos(\omega t - \beta l + \beta\zeta)$ function for the $\cos(\beta l)$ term and a $\cos(\omega t - \beta\zeta)$ function for the second function in the instantaneous $\mathcal{E}_p$ relation The corresponding power factors are then

$$\cos\varphi' = \cos(90 - \beta r_2) = \sin(\beta r_2) = \sin[\beta(l-\zeta)];$$
$$\cos\varphi'' = \cos(90 - \beta r_1) = \sin(\beta\zeta)$$

We have then for the mutual power P_m effect of radiator 2 on radiator 1, for $l = \frac{1}{2}\lambda$,

$$P_m^{(\text{watts})} = -30I_0^2\int_0^l\left[\frac{\sin(\beta r_2)}{r_2}\underbrace{\cos(\beta l)}_{-1} - \frac{\sin(\beta r_1)}{r_1}\right]\underbrace{\sin(\beta z)}_{\substack{\text{current}\\ \text{distribu-}\\ \text{tion}}}dz$$
$$= R_m I_e^2 \qquad \text{for Fig. 148}b$$

$$P_m^{(\text{watts})} = -30I_0^2\int_a^{a+\lambda/2}\left[\frac{\sin(\beta r_2)}{r_2}\underbrace{\cos(\beta l)}_{-1} - \frac{\sin(\beta r_1)}{r_1}\right]\underbrace{\sin[\beta(z-a)]}_{\text{current distribution}}dz$$
$$= R_m I_e^2 \qquad \text{for Fig. 143}d$$

since the current distribution of the second half-wave radiator has to be taken relative to radiator 1. For the first expression, we have the case of Fig. 148b, *i.e.*, the case for two parallel half-wave radiators, for which $r_2 = l - z$ and $r_1 = z$, since P lies now on the surface of radiator 1 and the field as well as the corresponding power is due to radiator 2. This is also true for the second expression of P_m, relating to Fig. 143d, where we have two parallel half-wave radiators that are staggered and, therefore, a relative current distribution $\sin[\beta' z - a]$. The mutual radiation resistances are then different, since

$$\left.\begin{array}{l} R_m = 30\{2\,\text{Ci}(\beta d) - \text{Ci}[\beta(\sqrt{d^2+l^2}+l)] - \text{Ci}[\beta(\sqrt{d^2+l^2}-l)]\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for Fig. 148}b \\ R_m = \text{for the formula given in connection with Fig. 143}d\text{, and} \\ \qquad\qquad\qquad\qquad \text{Ei}(-jx) = \text{Ci}(x) - j\,\text{Si}(x) \end{array}\right\} \text{ohms}$$

We have, in Fig. 143, all the important radiation resistance components tabulated with the corresponding radiator arrangement. From these formulas, we compute first the self-radiation resistance $R_s = R_{11}$ as in Fig. 148*a* and the mutual radiation resistance R_m of radiator 2 on radiator 1 or radiator 1 and radiator 2, in the case of two parallel radiators as in Fig. 148*b*. The total radiation resistance R_r is then the sum of twice the self-radiation resistance of each half-wave radiator when acting by itself (R_{11} value taken twice) and twice the R_m value if the respective half-wave radiators are excited in time phase. For antiphase excitations, we have to take the difference $2R_{11} - 2R_m$ ohms for R_r, since at any instant the current distribution of one radiator is in phase opposition to the current distribution of the other radiator. This is, of course, a difference only as far as the formula for antiphase excitations with respect to the formula for inphase excitation is concerned, since the R_m value can be negative, depending on the separation d of respective radiators. As far as the supply of such half-wave dipoles is concerned, we can by no means overlook the total reactance, which acts in addition to the total radiation resistance R_r and is in time quadrature. The reactance can, however, be readily neutralized by small changes in the radiator length. This will, however, somewhat offset[1] the total radiation resistance R_r. The reactance can also be stubbed out (injects opposite reactive effect).

Let us follow up the case of Fig. 148*b* and take two half-wave radiators ($l = \frac{1}{2}\lambda$) and a separation of $d = \frac{1}{2}\lambda$ meters. We find then

$$\beta l = \beta d = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi; \qquad 2\beta l = 2\pi;$$

$$\sqrt{d^2 + l^2} = \sqrt{\frac{\lambda^2}{4} + \frac{\lambda^2}{4}} = \frac{\lambda}{1.414} = 0.7071\lambda$$

$$\sqrt{d^2 + l^2} + l = (0.707 + 0.5)\lambda = 1.207\lambda;$$

$$\sqrt{d^2 + l^2} - l = (0.707 - 0.5)\lambda = 0.207\lambda$$

$$\text{Ci}\,(\beta d) = \text{Ci}\,(3.14) = 0.075;$$

$$\text{Ci}\,(2\beta l) = \text{Ci}\,(6.28) = -0.023$$

$$\beta(\sqrt{d^2 + l^2} + l) = \frac{6.28 \times 1.207\lambda}{\lambda} = 7.599$$

$$\beta(\sqrt{d^2 + l^2} - l) = \frac{6.28 \times 0.207\lambda}{\lambda} = 1.3;$$

$$\text{Ci}\,[\beta(\sqrt{d^2 + l^2} + l)] = \text{Ci}\,(7.599) = 0.1196;$$

$$\text{Ci}\,[\beta(\sqrt{d^2 + l^2} - l)] = \text{Ci}\,(1.3) = 0.4457;$$

$$\log_{10}(6.28) = 0.798$$

[1] In Hund, "Frequency Modulation," pp 278–289, 295 299, and Fig. 94, this is taken up in more detail.

According to Fig. 148*a*, we find

$$R_s = R_{11} = R_{22} = 17.316 + 69.06 \times 0.798 - 30(-0.023) = 73.4 \text{ ohms}$$
$$X_s = X_{11} = X_{22} = 30 \text{ Si } (6.28) = 30 \times 1.43 = +42.9 \text{ ohms}$$

for the self-radiation resistance and reactance of either half-wave radiator, since R_{11} and X_{11} are the radiation resistance and reactance looking into a half-wave dipole if the other half-wave radiator is absent. The same holds for R_{22} and X_{22} of the second dipole. The radiation resistance of the first dipole in the presence of the second dipole is, for inphase excitation with same antinode currents of I_e amperes, $R_{11} + R_m$ and is $R_{22} + R_m$ for the other radiator in presence of the first radiator. We find the mutual radiation resistance by means of the formula of Fig. 148*b*, as

$$R_m = 30\{2 \text{ Ci } (\beta d) - \text{Ci } [\beta(\sqrt{d^2 + l^2} + l)] - \text{Ci } [\beta(\sqrt{d^2 + l^2} - l)]\}$$
$$= 30(2 \times 0.075 - 0.1196 - 0.4457) = 30(-0.416) = -12.48 \text{ ohms}$$

This means that we have, for a half-wavelength spacing, 12.48 ohms mutual resistance from one-half wave radiator on the other, with a *negative* polarity. This is *not* a *negative* resistance, *i.e.*, an incremental resistance for a negative volt against ampere slope, but a radiation resistance that "bucks" the self-radiation resistance of each linear radiator. The total radiation resistance of radiator 1 is then $R_{11} + R_m = 73.4 - 12.48 = 60.98$ ohms. This is then also the total radiation resistance $R_{22} + R_m = 60.98$ ohms of radiator 2. Therefore, both half-wave radiators act with a radiation resistance of $R_r = 2 \times 60.98 = 121.96$ ohms and a total reactance of $2 \times 42.9 = 2X_m$ ohms, where, according to Fig. 143*b* and formula (851*d*),

$$X_m = -30\{2 \text{ Si } (\beta d) - \text{Si } [\beta(\sqrt{d^2 + l^2} + l)] - \text{Si } [\beta(\sqrt{d^2 + l^2} - l)]\}$$
$$= -30[2 \text{ Si } (3.14) - \text{Si } (7.599) - \text{Si } (1.3)]$$
$$= -30(2 \times 1.8515 - 1.52 - 1.184) = -30 \times 0.999 \cong -30 \text{ reactive ohms}$$

The mutual impedance from one half-wave radiator on the other and vice versa is then $Z_m = R_m - jX_m = -12.48 - j30$ ohms. The total reactance of each radiator in the presence of the other radiator is $j42.9 - j30 = +j12.9$ ohms. The entire reactance of both radiators is twice this value, or $j25.8$ ohms. We have, therefore, a total impedance for both radiators of $121.96 + j25.8$ ohms. That the mutual radiation resistance R_m comes out with a negative polarity is evident, since a distance of $d = \frac{1}{2}\lambda$ meters means that the retarded effect of one radiator on the other radiator, after traveling through a half-wavelength distance, has

lost half of a cycle and is, therefore, for *inphase excitations, in antiphase with the field of the other radiator*, which causes bucking effect, or antipolarity. We have, therefore, for the total radiation resistance,

$$\left.\begin{aligned} R_r &= 2R_s + 2R_m = 2R_{11} + 2R_m \quad \text{for inphase excitations} \\ R_r &= 2R_s - 2R_m = 2R_{11} - 2R_m \quad \text{for antiphase excitations} \end{aligned}\right\} \tag{852}$$

where R_m has either positive or negative polarity, depending upon the distance d in terms of the operating free-space wavelength. This is evident from Table 30, which gives the values computed from the formula for R_m, given in Fig. 148*b*. We note, therefore, that for two driven inphase radiators, for each radiator we have the input impedances $Z =$

TABLE 30. MUTUAL RADIATION RESISTANCE R_m IN OHMS FROM ONE HALF-WAVE RADIATOR ON THE OTHER FOR FIG. 148*b*

d	0λ	0.5λ	1λ	1.5λ	2λ	2.5λ	3λ	3.5λ	4.5λ	5λ	5.5λ	6λ
R_m	73.4	−12.48	4.1	−1.77	1.18	−0.75	0.42	−0.33	−0.18	0.15	−0.12	0.12

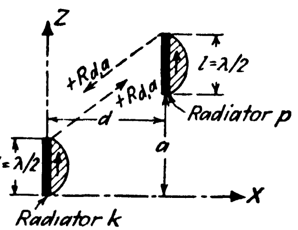
60.98 + j12.9 ohms and it takes an opposite reactance of −12.9 ohms to cancel the reactive effect. For antiphase excitations, the radiation resistance per radiator is as much as 73.4 − (−12.48) = 85.88 ohms, causing a total radiation resistance for both radiators as high as 171.76 ohms as against 121.96 ohms for the case of inphase excitations. In other words, the radiation effect can be offset by means of current phasing, by means of spacing, or by both. They are inseparable. It is a large radiation resistance that is desired, since this means actual radiation power transfer. The effective current in square and for the antinode location multiplied by the radiation resistance per radiator is the radiated power per radiator. We have already discussed that by means of equal phasing, we can readily concentrate the total radiated power into a preferred direction, call it a preferred solid angle in the case of a curtain array, for instance. Therefore, the main purpose of spacing is for the sake of beaming or energy concentration in a desired direction.

Table 30 shows how the mutual radiation effect R_m decreases with the number of half-wavelength spacings. If we had gone to a spacing of $d = 7.5\lambda$, we should have a value of only $R_m = -0.03$ ohms as against 73.4 ohms for zero distance, which is, of course, the value of the unobstructed self-resistance. We also note that R_m fluctuates between positive and negative limits owing to antiphase and inphase field effects for odd and even multiples of $\frac{1}{2}\lambda$ for the spacing d.

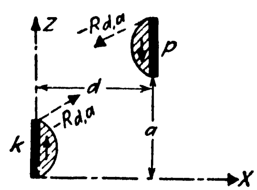
136. Numerical Applications to Half-wave Radiator Alignments, Collinear Alignments, and Curtain Arrays of Half-wave Dipoles. The

general formulas for the mutual active and reactive components of the transfer impedance are given in Eq. (851d) as well as in Fig. 143d. They apply to the staggered parallel array of two half-wave radiators. These formulas simplify to the parallel array without stagger if the offset dimension a is put equal to zero. They apply to the collinear array if

Value	Value of mutual radiation resistance R_m in ohms of one $\lambda/2$ radiator on the other $\lambda/2$ radiator														
	Value d of horizontal separation of thin radiators in terms of the wavelength λ														
a =	0	0.5λ	λ	1 5λ	2λ	2.5λ	3λ	3.5λ	4λ	4.5λ	5λ	5.5λ	6λ	6.5λ	7λ
0	73 4	12 48	4 1	−1 77	1 18	−0 75	0 42	−0 33	0 21	−0 18	0 15	−0 12	0 12	−0 1	0 06
0 5λ	26 4	−11 8	8 83	−5 75	3 76	−2 79	1 86	−1 54	1 08	−0 85	0 69	−0 57	0 51	−0 45	0 36
λ	−4 061	−0 78	3 56	−6 26	6 05	−5 67	4 51	−3 94	3 08	−2 5	2 1	−1 8	1 56	−1 18	1 14
1 5λ	1 78	0 8	−2 92	1 96	0 16	−2 4	3 24	−3 76	3 68	−3 4	3 14	−2 9	2 61	−2 31	2 06
2λ	−0 96	−1 0	1 13	0 56	2 55	2 74	−2 07	0 74	0 51	−1 3	1 82	−2 24	2 28	−2 29	2 26
2 5λ	0 58	0 15	0 42	−0 96	1 59	−0 28	−1 59	2 66	−2 49	2 0	−1 35	0 49	−0 06	−0 45	0 85
3λ	−0 43	−0 3	0 13	0 85	−0 45	−0 1	1 74	−1 03	−0 09	1 12	−1 87	1 77	−2 02	1 71	−1 32



For *in*phase excitations:
Total radiation resistance of radiator k is
$$R_k = 73.4 + R_{d,a} \text{ ohms}$$
Total radiation resistance of radiator p is
$$R_p = 73.4 + R_{d,a} = R_k \text{ ohms}$$
Radiation resistance of both radiators is
$$R_r = R_k + R_p = 2 \times 73.4 + 2R_{d,a} \text{ ohms}$$



For *anti*phase excitations:
$$R_k = 73.4 - R_{d,a} = R_p \text{ ohms}$$
$$R_r = 2 \times 73.4 - 2R_{d,a} \text{ ohms}$$

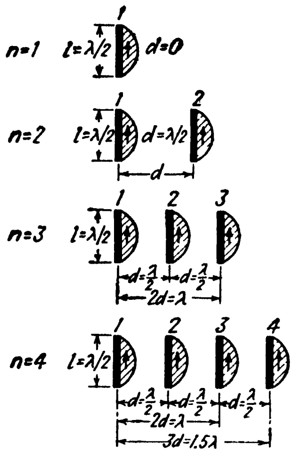
Rule: For two *in*phase radiators which are $l = \lambda/2$ long, we *add* the mutual R_m value to $R_k = 73.4$ ohms, while for two *counter*phase radiators, we subtract R_m from 73.4 ohms.

Example: For $d = 2.5\lambda$ and $a = \lambda$, the vertical 2.5λ column of the above table crosses the horizontal λ line at $R_{d,a} = R_m = -5.67$ ohms, yielding, for instance, for antiphase excitations $R_r = 2 \times 73.4 - (-2 \times 5.67)$ ohms and $R_p = R_k = 73.4 - (-5.67) = 79.07$ ohms.

FIG. 149. Computation of radiation resistance of coupled half-wave radiators.

the spacing d is put equal to zero and $a > l$, where l is the length of each linear radiator. Figure 143 gives all the formulas of concern in UHF work, and it is just a matter of evaluating these formulas as it is done in the previous section. Figure 149 gives the table of the mutual radiation resistance R_m in ohms for staggered and collinear radiators and parallel radiators without stagger. For $a = 0$, *i.e.*, the first line of this table, we have the same information as given in Table 30. It pertains then to

the array of Fig. 148*b* with spacings *d* increased in half-wavelength steps. For any spacing between, the procedure shown in Sec. 135 can be used for finding the value of R_m ohms. The column starting with 73.4 ohms, going to 26.4, to −4.064, etc., represents the cases for the R_m values of collinear alignments in half-wavelength steps. For $n = 0$, we obtain, of course, the self-radiation resistance of 73.4 ohms, since we have only

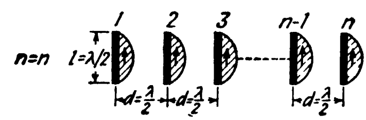


$$R_r = R_s = 73.4 \text{ ohms}$$

$$R_1 = R_2 = R_s + R_m = R_s + R_d = 73.4 + (-12.48) = 60.92 \text{ ohms}$$
$$R_r = 2R_1 = 2 \times 60.92 = 121.84 \text{ ohms}$$

$$R_1 = R_3 = R_s + R_d + R_{2d} = 73.4 + (-12.48) + 4.1 = 65.02 \text{ ohms}$$
$$R_2 = R_s + 2R_d = 73.4 + 2(-12.48) = 48.44 \text{ ohms}$$
$$R_r = R_1 + R_2 + R_3 = 2R_1 + R_2 = 2 \times 65.02 + 48.44 = 178.48 \text{ ohms}$$

$$R_1 = R_4 = R_s + R_d + R_{2d} + R_{3d} = 63.13 \text{ ohms}$$
$$R_2 = R_3 = R_s + 2R_d + R_{2d} = 52.45 \text{ ohms}$$
$$R_r = R_1 + R_2 + R_3 + R_4 = 2R_1 + 2R_2 = 4R_s + 6R_d + 4R_{2d} + 2R_{3d} = 231.16 \text{ ohms}$$



General formula for inphase exciters.

$$R_r = nR_s + 2(n-1)R_d + 2(n-2)R_{2d} + 2(n-3)R_{3d} + \cdots + 2R_{(n-1)d} \text{ ohms}$$

$R_s = 73.4\Omega$; $R_d = -12.48\Omega$; $R_{2d} = 4.1\Omega$; $R_{3d} = -1.77\Omega$
$R_{4d} = 1.18\Omega$; $R_{5d} = -0.75\Omega$; $R_{6d} = 0.42\Omega$; $R_{7d} = -0.33\Omega$
for $d = \lambda/2$ and $l = \lambda/2$, according to the table of Fig. 149.

Note how the consecutive mutual radiation resistances decrease in magnitude, owing to the fact that the mutual coupling becomes smaller and smaller as we proceed from R_d to R_{2d} to R_{3d} to R_{4d}, etc.

FIG. 150. Radiation resistances for identical inphase excitations.

one half-wave radiator. The applications to be given now and the other instructions in Fig. 149 will prove the usefulness of the general table in Fig. 149.

In Sec. 135, we found, for two half-wave radiators spaced a half wavelength apart, a total radiation resistance of 121.96 ohms for equal *inphase* excitations, while it is 171.76 ohms for equal *antiphase* current excitations. Therefore, for 10 amp flowing in each radiator at the dipole gap, we have a total radiated power of $100 \times 121.96 = 12.196$ kw for inphase excitations and 17.176 kw for antiphase excitations. If there were no mutual effect, the total radiated power would be (73.4 + 73.4)100

= 14.68 kw. Figure 150 gives the radiation resistances for inphase half-wave radiators in an alignment of parallel radiators with spacings of a half wavelength between consecutive radiators. We can then use either Table 30 or the first line of the table in Fig. 149, holding for a = 0. The inscribed formulas are then based on the following reasoning. Suppose we choose the alignment with four parallel half-wave radiators, marked 1, 2, 3, and 4. If the total radiation resistance of radiator 1 is called R_1, of radiator 2 is called R_2, the next one R_3, and the fourth one R_4, we have first for radiator 1 the radiation resistance $R_{11} = R_s = 73.4$ ohms. Since the remaining half-wave radiators 2, 3, and 4 are also in phase with radiator 1 as far as the current excitation is concerned, they cause mutual radiation resistances on radiator 1, according to the distance from this radiator. The mutual radiation resistance from radiator 2 on radiator 1 is called R_d, since the distance from radiator 1 is d meters; the mutual radiation resistance from radiator 3 on radiator 1 is called R_{2d}, since it is twice the distance to radiator 1; and R_{3d} is the mutual effect on radiator 1 from radiator 4. We have then

$$R_1 = 73.4 + R_d + R_{2d} + R_{3d} \quad \text{ohms}; \qquad R_d = -12.48 \quad \text{ohms};$$
$$R_{2d} = 4.1 \quad \text{ohms}; \quad R_{3d} = -1.77 \quad \text{ohms}$$

since, according to Table 30, we have for $d = \frac{1}{2}\lambda$ a mutual radiation resistance of $R_m = -12.48$ ohms $= R_d$. For $2d = 1\lambda$, we have $R_m = 4.1$ ohms $= R_{2d}$; and for $3d$, we have $R_m = -1.77$ ohms $= R_{3d}$. Therefore,

$$R_1 = 73.4 - 12.48 + 4.1 - 1.77 = 63.25 \text{ ohms} = R_4$$

The total radiation resistances of R_1 and R_4 are equal, since R_4 contains, besides the self-resistance of 73.4 ohms, also the R_d value due to radiator 3, the R_{2d} value due to radiator 2, and the R_{3d} value due to radiator 1. For the same reason, the total resistances of radiators 2 and 3 are equal, and we have

$$R_2 = R_3 = 73.4 + 2R_d + R_{2d} = 73.4 - 2 \times 12.48 + 4.1 = 52.54 \text{ ohms}$$

This means that the total radiation resistance of all four radiators is

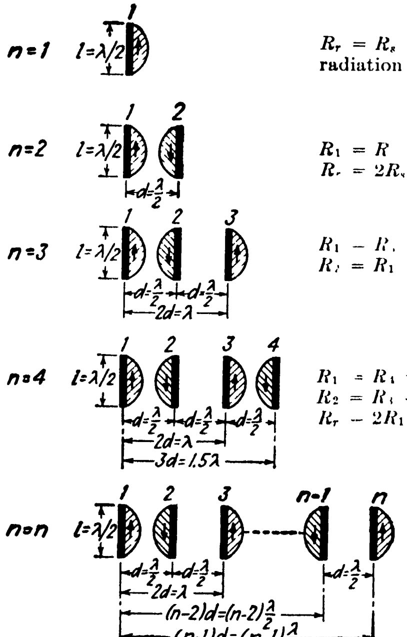
$$R_r = R_1 + R_2 + R_3 + R_4 = 2(R_1 + R_2) = 2(63.25 + 52.54) = 231.58 \text{ ohms}$$

as against $4 \times 73.4 = 293.6$ ohms if all four radiators were to radiate their power without mutual interference. This is essentially true if the four radiators are spaced far enough, since the table of Fig. 149 shows that for a distance $d = 7\lambda$, we have only a value of $R_d = 0.06$ ohms and still less for R_{2d} and R_{3d}, etc. All other computations in Fig. 150 are

now obvious. It is convenient to compute the total radiation resistance R_r of n radiators from the formula

$$R_r = nR_s + 2(n-1)R_d + 2(n-2)R_{2d} + 2(n-3)R_{3d} + \cdots + 2R_{(n-1)d} \quad \text{ohms}$$

This relation gives an idea of the total radiated power. It is, however, more important to know the $R_1, R_2, R_3, \ldots R_n$ values, since we have to



$R_r = R_s = 73.4$ ohms $= R_1$; $R_1, R_2, R_3, \ldots, R_n$ are radiation resistances of radiators $1, 2, 3, \ldots, n$

$R_1 = R_s - R_d = 73.4 - (-12.48) = 85.88$ ohms $= R_2$
$R_r = 2R_s - 2R_d = 2 \times 73.4 - 2(-12.48) = 171.76$ ohms

$R_1 = R_3 = R_s - R_d + R_{2d} = 89.8\Omega$
$R_2 = R_s - 2R_d = 98.1\Omega$; $R_r = 3R_s - 4R_d + 2R_{2d} = 270.12\Omega$

$R_1 = R_4 = R_s - R_d + R_{2d} - R_{3d} = 91.5\Omega$
$R_2 = R_3 = R_s - 2R_d + R_{2d} = 102.2\Omega$
$R_r = 2R_1 + 2R_2 = 4R_s - 6R_d + 4R_{2d} - 2R_{3d} = 387.2\Omega$

General formula:

$$R_r = nR_s - 2(n-1)R_d + 2(n-2)R_{2d} - 2(n-3)R_{3d} + \cdots + 2R_{(n-1)d} \text{ ohms}$$

$R_s = 73.4\Omega$; $R_d = -12.48\Omega$, $R_{2d} = 4.1\Omega$
$R_{3d} = -1.77\Omega$; $R_{4d} = 1.18\Omega$; $R_{5d} = -0.75\Omega$
$R_{6d} = 0.42\Omega$; $R_{7d} = -0.33\Omega$; $R_{8d} = 0.21\Omega$
$R_{9d} = -0.18\Omega$
etc., for $d = \lambda/2$ and $l = \lambda/2$ according to the table of Fig. 149.

FIG. 151. Radiation resistance of identical half-wave radiators with alternate antiphasing. n is number of individual radiators that make up the array.

drive the radiator and match the dipole impedances of the individual half-wave dipoles of the alignment. In the same way, we can compute the reactances $X_1, X_2, X_3, \cdots, X_n$ of the individual half-wave radiators, in order to obtain the complete impedance values $Z_1 = R_1 + X_1$, $Z_2 = R_2 + X_2$, $Z_3 = R_3 + X_3$, $\cdots$, $Z_n = R_n + X_n$, where the X values can be positive and negative, while, of course, the R values must always be positive, even though partially due to R_m effects (mutual effects), which

may have negative polarity. Negative polarity does not mean a negative resistance (though arithmetically behaving like it) but means "destructive" power interference. Figure 151 illustrates the cases for alternate in- and antiphasing. As brought out before, for antiphasing, we subtract the different R_m values instead of adding them, where again

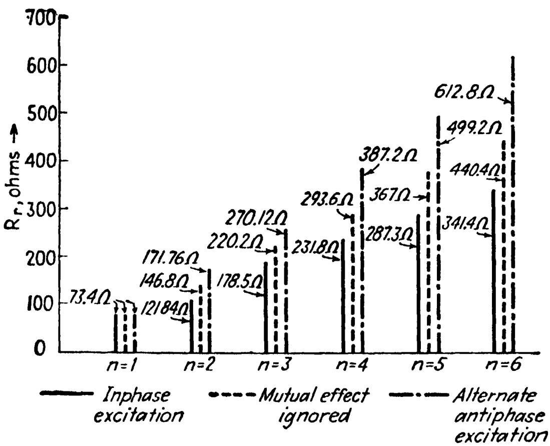
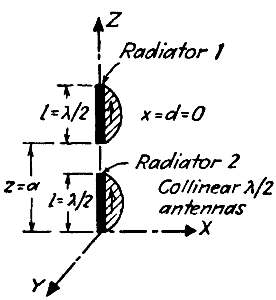


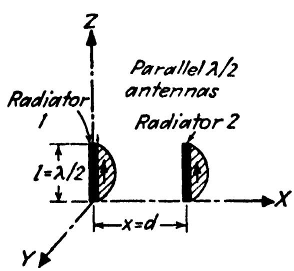
FIG. 152. Radiation resistance of half-wave radiators in parallel and spaced half of a wavelength apart for inphase and antiphase excitations and for the condition where the mutual effect is ignored. n is the number of $\lambda/2$ radiators spaced $\lambda/2$ in apart and parallel.



$$A = 2\beta\left(a - \frac{\lambda}{2}\right);\ B = 2\beta\left(a + \frac{\lambda}{2}\right);\ C = \frac{a^2}{a^2 - (\lambda^2/4)}$$

$$R_m = R_a = -15 \sin(\beta a)[\mathrm{Si}(A) - 2\,\mathrm{Si}(2\beta a) + \mathrm{Si}(B)] - 15 \cos(\beta a)[2.302 \log_{10}(C) + \mathrm{Ci}(A) - 2\,\mathrm{Ci}(2\beta a) + \mathrm{Ci}(B)] \text{ ohms}$$

For $a = \lambda/2$, the table of Fig. 149 gives
$R_a = 26.4\Omega$; $R_{2a} = -4.064\Omega$; $R_{3a} = 1.78\Omega$
$R_{4a} = -0.96\Omega$; $R_{5a} = 0.58\Omega$; $R_{6a} = -0.43\Omega$



$$R_m = R_d = -30\{2\,\mathrm{Ci}(\beta d) - \mathrm{Ci}[\beta(\sqrt{d^2 + l^2} + l)] - \mathrm{Ci}[\beta(\sqrt{d^2 + l^2} - l)]\} \text{ ohms}$$

For $d = \lambda/2$, the table of Fig. 149 gives
$R_d = -12.48\Omega$; $R_{2d} = 4.1\Omega$; $R_{3d} = -1.77\Omega$
$R_{4d} = 1.18\Omega$; $R_{5d} = -0.75\Omega$; $R_{6d} = 0.42\Omega$
$R_{7d} = -0.33\Omega$; $R_{8d} = 0.21\Omega$; $R_{9d} = -0.18\Omega$
$R_{10d} = 0.15\Omega$; $R_s = 73.4\Omega$

FIG. 153. Radiation resistance formulas and specific values for two collinear and two parallel linear radiators.

R_m may have positive or negative polarity, respectively. The computations in this figure are self-evident, since, for a four-element alignment, we have

$$R_r = 4R_s - 6R_d + 4R_{2d} - 2R_{3d} \qquad \text{ohms}$$

and for the radiator 2 as well as the radiator 3, we find the radiation resistance component of the dipole impedance $R_2 = R_s - 2R_d + R_{2d} = R_3$ ohms, where R_s is the self-resistance, $-2R_d$ is the mutual resistance

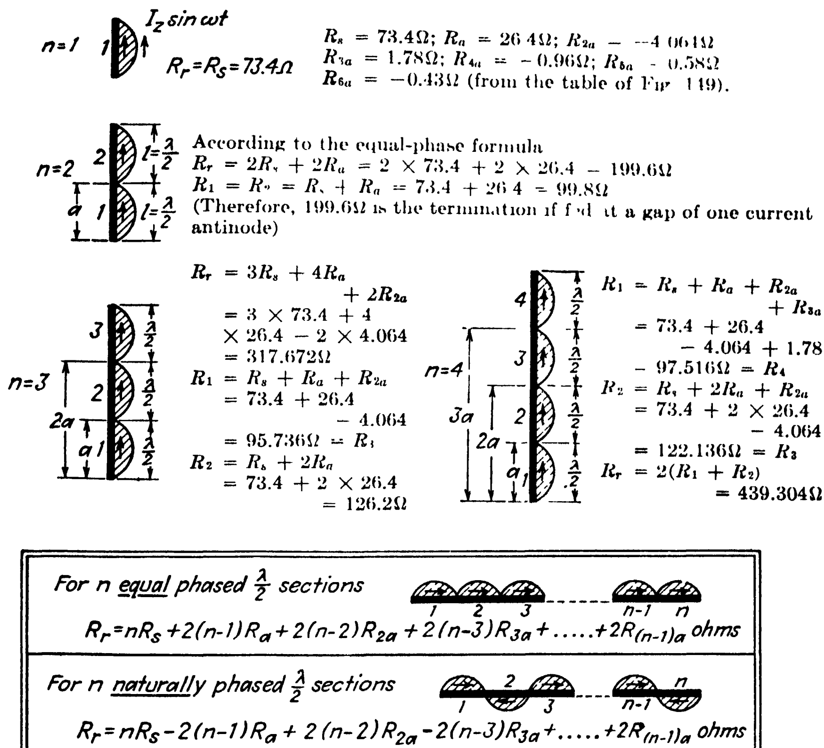


FIG. 154. Continuous collinear half-wave radiators with forced equal phasing and natural phasing.

effect of radiators 1 and 3 on radiator 2, and $+R_{2d}$ is the mutual radiation resistance effect of radiator 4 on radiator 2. The same magnitude R_2 ohms is obtained for radiator 3 so that $R_2 = R_3$ ohms. The radiation resistance expression R_r, for any alignment of alternate inphase and antiphase half-wave exciters, then includes a term that is n times the self-radiation resistance of one half-wave radiator, since we have n radiators, plus positive and negative terms taking the mutual effects into account. Figure 152 gives the relative magnitude of the entire align-

ment radiation resistance R_r of several inphase and antiphase half-wave dipoles in comparison with the values if the mutual coupling resistances R_m are ignored. Figure 153 gives the formulas for computing the mutual radiation resistance in the case of collinear and parallel half-wave radiators. For spacings d and vertical separations a, which are integer multiples of a half wavelength, the inscribed values for R_a, R_{2a}, . . . , R_d, R_{2d}, . . . are used. They are the values given in the table of Fig. 149, for which either $d = 0$ or $a = 0$. Figure 154 illustrates the application to several collinear half-wave radiators and in such a fashion that $a = \frac{1}{2}\lambda$ gives the limiting case and causes a single linear radiator with equal phasings We have now a ready means to deal with equal phasing and antiphasing (alternately), since for equal-phased radiators the mutual R_m term is to be added while it is to be subtracted for antiphasing. Alternate in- and antiphasing, for $a = \frac{1}{2}\lambda$ and $d = 0$, is equivalent to a linear conductor with "natural" phasing. The examples and illustrations of Fig. 154 are then self-explanatory. Figure 155 shows the general coupling of two half-wave radiators, where we have now $d = x$ and $a = z_0$ in order to fit in with the XYZ system of this figure. We have then

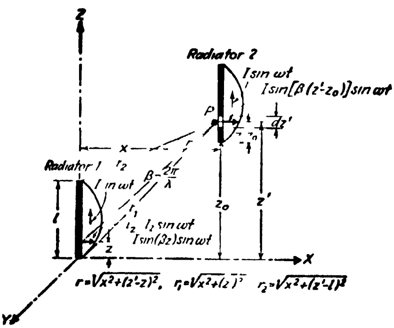


FIG. 155. Conditions for two coupled linear radiators which are staggered P is a field point near and anywhere along radiator 2

$$R_m = R_{d,a} = R_{x,z_0} = -15 \sin (\beta z_0) \left[F\left(x, z_0 - \frac{\lambda}{2}\right) - 2F(x, z_0) + F\left(x, z_0 + \frac{\lambda}{2}\right) \right] - 15 \cos (\beta z_0) \left[G\left(x, z_0 - \frac{\lambda}{2}\right) - 2G(x, z_0) + G\left(x, z_0 + \frac{\lambda}{2}\right) \right] \quad \text{ohms}$$

$$F(x, z_0) = \mathrm{Si}\, [\beta(\sqrt{x^2 + z_0^2} + x)] - \mathrm{Si}\, [\beta(\sqrt{x^2 + z_0^2} - x)];$$
$$G(x, z_0) = \mathrm{Ci}\, [\beta(\sqrt{x^2 + z_0^2} + x)] + \mathrm{Ci}\, [\beta(\sqrt{x^2 + z_0^2} - x)]$$

since

$$P_{x,z_0} = \underbrace{-I_e^2 \cdot 30 \int_{z_0}^{z_0+(\lambda/2)} \left[\frac{\sin (\beta r_2)}{r_2} + \frac{\sin (\beta r_1)}{r_1} \right] \sin [\beta(z' - z_0)]\, dz'}_{R_m \text{ mutual ohms due to radiator 1 (must be added to } R_s = 73.4 \text{ ohms of radiator 2)}}$$

mutual watts

The R_m relation degenerates then to

$$R_m = R_{d,a} = R_{0,a} = -15 \sin(\beta a) \left\{ \text{Si}\left[2\beta\left(a - \frac{\lambda}{2}\right)\right] - 2\,\text{Si}\,(2\beta a) + \text{Si}\left[2\beta\left(a + \frac{\lambda}{2}\right)\right] \right\} - 15 \cos(\beta a) \left\{ 2.303 \log_{10}\left(\frac{a^2}{a^2 - 0.25\lambda^2}\right) + \text{Ci}\left[2\beta\left(a - \frac{\lambda}{2}\right)\right] - \text{Ci}\,(2\beta a) + \text{Ci}\left[2\beta\left(a + \frac{\lambda}{2}\right)\right] \right\}$$

for two *collinear* $\frac{\lambda}{2}$ radiators $\left(d = 0;\ a \geqq \frac{\lambda}{2}\right)$

$$R_m = R_{d,a} = R_{d,0} = 30\{2\,\text{Ci}\,(\beta d) - \text{Ci}\,[\beta(\sqrt{d^2 + l^2} + l)] - \text{Ci}\,[\beta(\sqrt{d^2 + l^2} - l)]\}$$ for two *parallel radiators* $(a = 0;\ d$ any value)

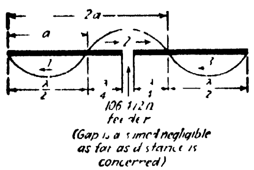
These are the expressions shown in Fig. 153 and in agreement with (851d) and the details given in connection with (851d). Formula (851c) is the general formula for the $R_m = R_{d,a}$ case (staggered parallel radiators). These expressions also confirm the expressions given by A. A. Pistolkors.[1] The general case for $R_m = R_{d,a}$ is then readily computed from the $R_{d,a}$ formula. It can be read directly from the table in Fig. 149. We find then for $d = 1.5\lambda$ and $a = 2.5\lambda$, the value $R_m = R_{d,a} = R_{1.5;2.5} = -0.96$ ohm, since the $d = 1.5\lambda$ column crosses the $a = 2.5$ line at -0.96 ohm. This is quite a small mutual effect in comparison with the self-effect of 73.4 ohms. The limiting cases of Fig. 154 ($a = \frac{1}{2}\lambda$) give broadside energy concentration for equal-phased half-wave radiators, *i.e.*, energy concentration toward the equatorial plane of each continuous linear radiator, and all the more, the more half-wave sections make up the linear radiator. The application is then the following: Suppose we have $n = 4$ half-wave radiators in a collinear alignment. Then $R_a + R_{2a} + R_{3a}$ are the mutual radiation resistance contributions on the self-resistance of 73.4 ohms of the first $\frac{1}{2}\lambda$ section on either end of the linear radiator. Since $a = \frac{1}{2}\lambda$, we find from the table in Fig. 149 the mutual contributions $R_a = 26.4$, $R_{2a} = -4.064$, and $R_{3a} = 1.78$ ohms, rendering the total radiation resistance of each half-wave end section as $R_1 = 73.4 + 26.4 - 4.064 + 1.78 = 97.516$ ohms $= R_4$. For the other two sections, we likewise obtain equal values but as shown in Fig. 154, since on each side of section 2, we have an R_a mutual resistance contribution, adding a $2R_a$ effect, plus one R_{2a} contribution to the 73.4-ohm self-effect. Now for collinear radiators as in Fig. 154, we must excite each half-wave section at the mid-gap of each section in order to prevent natural phasing. We cannot screen off, *i.e.*, suppress, antiphase

[1] PISTOLKORS, A. A., The Radiation Resistance of Beam Antennas, *Proc. IRE*, **17**, 562, March, 1929.

radiation effects, since, according to Fig. 154, we utilize all sections. The input at the mid-gap has then to match, for sections 1 and 4, a resistance of 97.516 ohms if the reactance is stubbed out (is canceled), while the respective inputs at the mid-gap of sections 2 and 3 have to match 122.156 ohms if the reactance is compensated by an opposite reactance of same magnitude. Figure 156 gives the case for *natural* phasing, for which consecutive half-wave sections are excited in antiphase. Now it is possible to energize at any one dipole gap the entire linear radiator, which is $3a$ meters long (feeder gap is neglected, just sufficient to apply the voltage), since for the limiting case we have $a = \frac{1}{2}\lambda$ meters. Let us take first the case of the gap at the symmetry center as in Fig. 156. Since we have consecutive antiphasing, we have to use the formula

$$R_r = nR_s - 2(n - 1)R_a + 2(n - 2)R_{2a} \quad \text{ohms}$$

for the total radiation resistance R_r with respect to any one mid-gap of any one half-wave section of the entire radiator length of $3a$ meters. This is the case, because the radiation resistance is, so far, expressed in terms of the effective current at a current antinode. This antinode is at the mid-point of any half-wave section, the first section beginning at either free end of the radiator. The radiation resistance can also be expressed with respect to any other current value, corresponding to another location of the linear radiator (Sec. 137). Since $a = \frac{1}{2}\lambda$, we find from the table in Fig. 149, the mutual contributions $R_a = 26.4$ ohms and $R_{2a} = -4.064$ ohms; and since $n = 3$, we have



We have the case of $n = 3$ and

$$R_r = nR_s - 2(n - 1)R_a + 2(n - 2)R_{2a}$$
$$= 3R_s - 4R_a + 2R_{2a}$$
$$= 3 \times 73.4 - 4 \times 26.4 + 2(-4.064)$$
$$= 106.472\Omega$$

since $R_s = 73.4\Omega$, $R_a = 26.4\Omega$, and $R_{2a} = -4.064\Omega$, for $a = \lambda/2$

$$R_1 = R_3 = R_s - R_a + R_{2a}$$
$$= 73.4 - 26.4 + (-4.064) = 42.936\Omega$$
$$R_2 = R_s - 2R_a = 73.4 - 2 \times 26.4 = 20.6\Omega$$

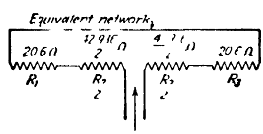
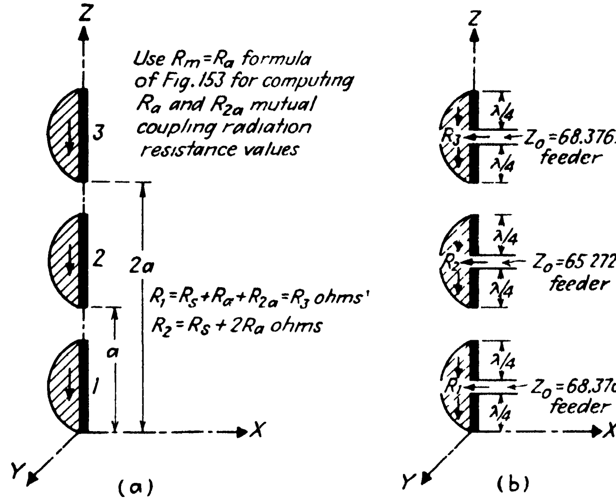


FIG. 156. Actual and equivalent network of a center-fed dipole of total length equal to $\frac{3}{2}\lambda$. The reactance is either stubbed out or canceled by means of a slightly different radiator length. $n = 3$ half-wave sections. R_1, R_2, and R_3 are the radiation resistances per $\lambda/2$ section.

$$R_r = n73.4 - 2(n - 1)R_a + 2(n - 2)R_{2a} = 106.472 \text{ ohms;}$$
$$R_2 = 73.4 - 2R_a = 73.4 - 2 \times 26.4 = 20.6 \text{ ohms}$$
$$R_1 = 73.4 - R_a + R_{2a} = 73.4 - 26.4 - 4.064 = 42.936 \text{ ohms} = R_3$$

Since R_1, R_2, and R_3 are the radiation resistances of the first, second, and third half-wave sections of the entire radiator, we must have $R_r = R_1 + R_2 + R_3 = 106.472$ ohms, with the equivalent lumped network shown

in Fig. 156, if the reactance of the gap is canceled (tuned out). It is good practice to feed at the symmetry point as shown in the figure, although we could apply the voltage to the mid-gap of section 1 or the mid-gap of section 3. In either case, we have to overcome the active resistance component of 106.472 ohms after the reactance is tuned out. If we choose the distance $a > \frac{1}{2}\lambda$, we have no longer a continuous linear radiator but the case depicted in Fig. 157. If we drive all three half-



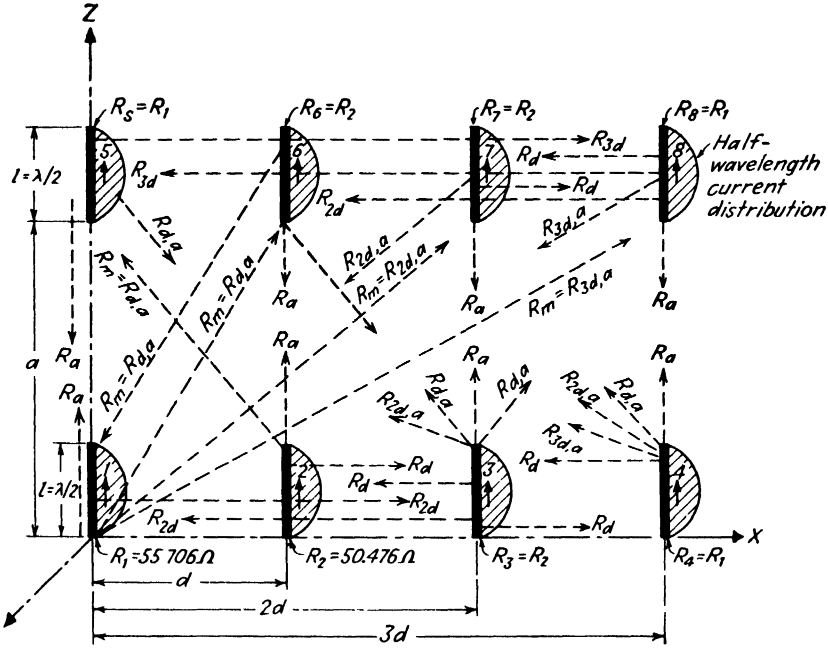
For $a = \lambda$, we find, from the table of Fig. 149, $R_m = R_a = -4.064\Omega$, $R_m = R_{2a} = -0.96\Omega$. Radiation resistance of radiator 1 is $R_1 = 73.4 - 4.064 - 0.96 = 68.376\Omega$, radiation resistance of radiator 2 is $R_2 = 73.4 - 2 \times 4.064 = 65.272\Omega$, and total radiation resistance becomes $R_r = R_1 + R_2 + R_3 = 2R_1 + R_2 = 202.024\Omega$ for $a \geq \lambda/2$.

FIG. 157. Computations for the terminating feeder lines in the case of collinear half-wave radiators.

wave radiators in phase, we have to match the active resistances of 68.376 and 65.272 ohms, respectively, for an offset of $a = \lambda$ meters. These values are fairly close, so that one and the same concentric feeder could be employed if reactance compensation is provided. Figure 158 gives the method of numerical application for the case of a half-wave radiator curtain. Since we have inphase excitations, all mutual effects have to be added to the self-effect of 73.4 ohms. We have eight half-wave radiators, where each individual radiator has the self-radiation resistance of 73.4 ohms plus R_m effects from the seven neighboring radiators. For instance, for radiator 1, we have

$$R_1 = 73.4 + \underbrace{R_d + R_{2d} + R_{3d}}_{R_m \text{ effects of 2, 3, and 4}} + \underbrace{R_a + R_{d,a} + R_{2d,a} + R_{3d,a}}_{R_m \text{ effects of 5, 6, 7, and 8}}$$
$$= 73.4 - 12.48 + 4.1 - 1.77 - 4.064 - 0.78 + 3.56 - 6.26$$
$$= 55.706 \text{ ohms}$$

On account of the symmetry of the curtain array, we find also $R_4 = R_5 = R_8 = R_1$ ohms. According to the computations in Fig. 158, we further have $R_2 = 50.476$ ohms and equal to R_3, R_6, and R_7, *i.e.*, equal to the total radiation resistance of radiators 3, 6, and 7. The radiation resistance of the entire dipole curtain is then $R_r = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_8 = 4(R_1 + R_2) = 424.728$ ohms. It would be



Example: For $d = \lambda/2$ and $a = \lambda$ the table of Fig. 149 yields
$R_d = -12.48\Omega$; $R_{2d} = 4.1\Omega$; $R_{3d} = -1.77\Omega$; $R_a = -4.064\Omega$
$R_{d,a} = -0.78\Omega$; $R_{2d,a} = 3.56\Omega$; $R_{3d,a} = -6.26\Omega$; $R_{d=0,a=0} = R_s = 73.4\Omega$

The radiation resistance R_1 of radiator is

$$R_1 = \underbrace{73.4}_{\text{its } R_s \text{ value}} + \underbrace{R_d}_{\text{due to radiator 2}} + \underbrace{R_{2d}}_{\text{due to 3}} + \underbrace{R_{3d}}_{\text{due to 4}} + \underbrace{R_a}_{\text{due to 5}} + \underbrace{R_{d,a}}_{\text{due to 6}} + \underbrace{R_{2d,a}}_{\text{due to 7}} + \underbrace{R_{3d,a}}_{\text{due to 8}} = 73.4$$

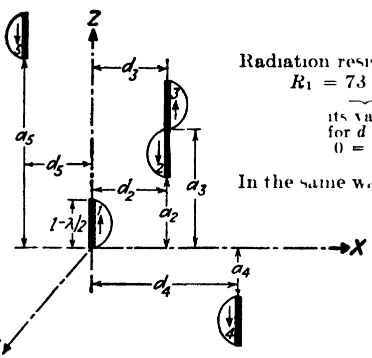
$$+ (-12.48) + 4.1 + (-1.77) + (-4.064) + (-0.78) + 3.56 + (-6.26) = 55.706\Omega$$
$$= R_4 = R_5 = R_8$$

$$R_2 = \underbrace{R_s}_{\text{due to 1}} + \underbrace{2R_d}_{\text{due to 1 and 3}} + \underbrace{R_{2d}}_{\text{due to 4}} + \underbrace{R_{d,a}}_{\text{due to 5}} + \underbrace{R_a}_{\text{due to 6}} + \underbrace{R_{d,a}}_{\text{due to 7}} + \underbrace{R_{2d,a}}_{\text{due to 8}}$$

$$= 73.4 + 2(-12.48) + 4.1 + 2(-0.78) + (-4.064) + 3.56 = 50.476\Omega$$

FIG. 158. Computations of the various radiation resistances of a curtain of half-wave dipole radiators. (Each individual half-wave radiator has, in addition to the self-radiation resistance of 73.4, resistances due to the mutual effect of all the other radiators on the particular individual radiator. The farther the other radiator is away, the smaller is the mutual radiation-resistance effect.) Total radiation resistance $R_r = 4(R_1 + R_2) = 424.728$.

difficult to realize equal inphase currents if only one half-wave radiator were driven and the remaining radiators parasitic. Fortunately, all active resistance components are close to 50 ohms, and 50-ohm concentric feeders are in order. This assumes stubbing out or tuning out the various reactances. If we have, therefore, 1-amp effective current at the current antinode, the power of 424 728 watts instead of $8 \times 73.4 = 587.2$ watts would radiate if no mutual coupling effects were present. Since the table in Fig. 149 applies to any number of integer half-wave radiators,



Radiation resistance of radiator 1 is, for instance,

$$R_1 = \underbrace{73.4}_{\text{its value for } d = 0 = a} \underbrace{- R_{d_2 a_2}}_{\text{due to 2 (minus sign since 2 is in antiphase)}} + \underbrace{R_{d_3 a_3}}_{\text{due to 3}} - \underbrace{R_{d_4 a_4}}_{\text{due to 4}} - \underbrace{R_{d_5 a_5}}_{\text{due to 5}}$$

In the same way, we find the relations for R_2, R_3, R_4, and R_5

Example If we choose $a_2 = \lambda/2$ then $a_3 = \lambda$ Choosing $a_4 = 2.5\lambda$, $a_5 = 3\lambda$, $d_2 = d_3 = \lambda/2$, $d_4 = 1.5\lambda$, and $d_5 = \lambda$, we find from the table of Fig. 149

$$R_{d_2 a_2} = R_{0.5\lambda,\,0.5\lambda} = -11.8\Omega \quad R_{d_3 a_3} = R_{0.5\lambda,\,\lambda} = -0.78\Omega$$
$$R_{d_4 a_4} = R_{1.5\lambda,\,2.5\lambda} = -0.96\Omega, \quad R_{d_5 a_5} = R_{\lambda,\,3\lambda} = 0.13\Omega$$

and

$$R_1 = 73.4 - (-11.8) + (-0.78) - (-0.96) - 0.13 = 85.25\Omega$$

For an effective current of 10 amp at all half-wave-radiator sections at the current antinode this half-wave radiator (namely 1) would contribute $10^2 \times 85.25 = 8.525$ kw field power (It *must* be realized that each half-wave radiator represents at its mid-gap a different impedance and a suitable driving voltage has to be provided to give equal antinode currents of 10 amp)

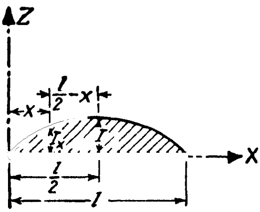
FIG. 159 The solution for an array with an irregular arrangement of inphase and antiphase half-wave radiators

we have, in Fig. 159, the conditions given in connection with this illustration, since for radiator 1 we have its own effect of 73.4 ohms plus the antiphase effect of the half-wave section 2 and the inphase effect of the half-wave section 3. Now the half-wave section 2 of the full-wave radiator, which is coupled to the half-wave radiator 1, is separated by a distance $d = \frac{1}{2}\lambda$ just as is the half-wave section 3, but 2 is offset by a distance $a_2 = \frac{1}{2}\lambda$ and radiator 3 by a distance $a_3 = \lambda$. In addition, we have to contend with half-wave radiator 5, which is in antiphase with half-wave radiator 1 and offset by a distance $a_5 = 3\lambda$ and separated a distance $d_5 = -\lambda$ with respect to half-wave radiator 1. There is also an antiphase radiator 4, which is offset by a distance $a_4 = -2.5\lambda$ and separated by a distance $d_4 = 1.5\lambda$. Using the table of Fig. 149, we find

the values given in Fig. 159. The applications to perfect obstructed wave propagation are dealt with in the following chapter.

137. Meaning of Radiation Resistance. The *radiation resistance* unlike the *customary circuit resistance* is *useful.* For *lumped* circuit constants, I^2R is the joulean heat *loss;* while for a radiating system, I^2R_r represents *useful* electromagnetic field power, which dilutes into the dis-

(a)

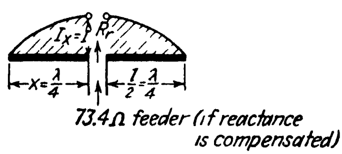


$$I_x = I \sin(\beta x) = I \cos\left[\beta\left(\frac{l}{2} - x\right)\right]$$

$$\beta = \frac{2\pi}{\lambda}$$

For $l = \lambda/2$, we have $P/I^2 = 73.4\Omega$; P is radiated power in watts

(b)



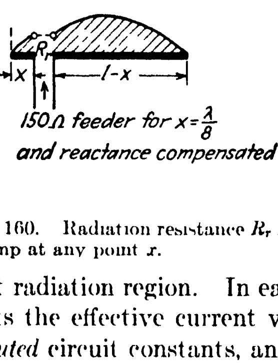
$$R_r = \frac{P}{I^2} = 73.4\Omega$$

since $x = l/2 = \lambda/4$

$$\beta x = \frac{\pi}{2};\ \sin(\beta x) = 1,\ \beta\left(\frac{l}{2} - x\right) = 0$$

$$\cos\left[\beta\left(\frac{l}{2} - x\right)\right] = 1$$

(c)



$$R_{r(\text{ohms})} = \frac{P}{I^2 \sin^2(\beta x)} = \frac{P}{I^2 \cos^2\left[\beta\left(\frac{l}{2} - x\right)\right]}$$

If $x = \lambda/8$,

$$\text{then } \beta x = \frac{2\pi}{\lambda}\frac{\lambda}{8} = \frac{\pi}{4} \text{ or } 45°$$

$$R_r = \frac{P}{I^2 \sin^2 45°} = \frac{73.4}{0.707^2} \cong 150\Omega$$

is the radiation resistance with respect to the current $I_x = I \sin(\beta x)$ entering the gap

FIG. 160. Radiation resistance R_r in ohms, radiated power P in watts, effective current I_x in amp at any point x.

tant radiation region. In each case, it is taken for granted that I represents the effective current value. In a radiating system, we have *distributed* circuit constants, and we have to take into account the location along the radiator where the reference effective current exists. Figure 160 then shows what happens. Figure 160*a* and *b* gives the conditions treated so far. These conditions refer to the effective current value at the current antinode. This is a very acceptable location for a feeder, since it takes the lowest voltage to drive power into the space. We have found that $73.4 + j42.9$ ohms is the input impedance into a half-wave dipole as long as the diameter of the radiator is small compared with its elongation and very small compared with the operating wavelength

It is true that, rigorously speaking, we cannot always assume sinusoidal current distributions, but there are times we just have to make a compromise. This compromise has proved useful in much measurement work as well as in the applied technique. Now as far as the center gap feed of a half-wave dipole is concerned, for a tuned-out reactance of $j42.9$ ohms a feeder that has a characteristic wave resistance of about 75 ohms is required. A Lecher wire feeder will not do then, since it does not seem practical to go much lower than 150 ohms. A concentric line is adaptable, since we can go to very low characteristic wave resistances. According to the theoretical condition, the radiation resistance looking into any gap location of Fig. 160 is

$$Z = \frac{73.4}{\sin^2 (\beta x)} = \frac{73.4}{\cos^2 \left[\beta \left(\frac{l}{2} - x\right)\right]}$$

for unobstructed radiation space. It is admitted that this relation assumes ideal condition, which, no doubt, is a fair approximation at the mid-point of the radiator and for points not too close to very free ends

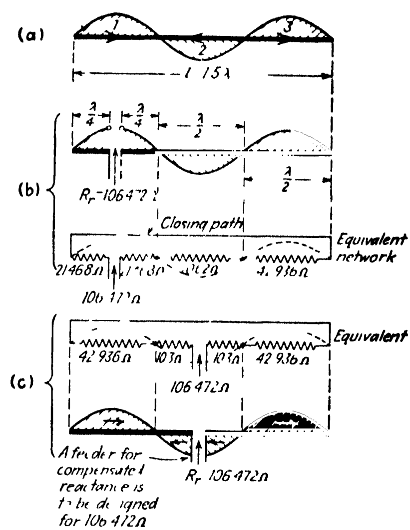


FIG. 161. Equivalent network for the radiation resistance of a 1.5λ linear radiator whose reactance is compensated.

The reason for this is that a termination at one free end would theoretically mean an infinite impedance termination but practically may mean only, say, 5,000 ohms. It is the same as with customary lumped constants and circuits. Parallel resonance requires ideally infinite input impedance but may be only 10,000 ohms. In other words, where the Q value of the system, as always, is finite, we cannot expect for the most severe condition an agreement with the formula dealing with ideal conditions. Fortunately, this is not a termination that would most likely be used. We have already mentioned in this text the case of "continuous" collinear radiators, which can be fed at any antinode location, as shown in Fig. 161. As is shown in Sec. 136, the total radiation resistance of a natural line excitation with three half-wave sections is $R_r = 106.472$ ohms, due to the radiation resistances $R_1 = 42.936 = R_3$ for sections 1 and 3 and the component $R_2 = 20.6$ ohms of the middle section 2. The total radiation resistance is then $R_r = R_1 + R_2 + R_3$ ohms. All these resistances are computed with respect to the effective

inphase current at a current antinode. Therefore, it does not matter at which current maximum location we feed the linear radiator with natural phasing. Figure 161*a* and *b* shows the equivalent and actual networks with an active resistance component of 106.472 ohms in each case. The symmetrical mid-point feed is, however, preferable for practical reasons. Figure 162 shows the case for four half-wave sections

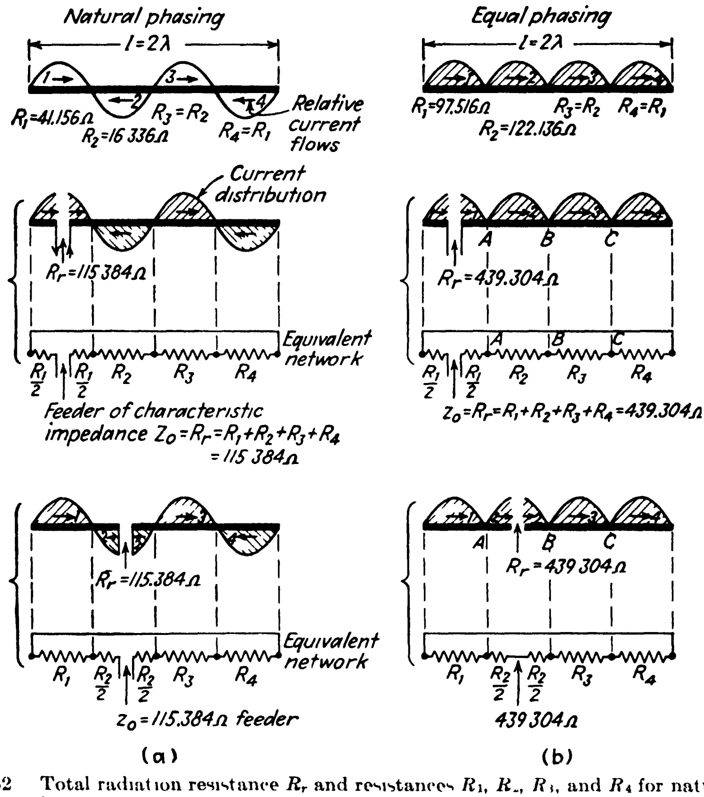


FIG. 162. Total radiation resistance R_r and resistances R_1, R_2, R_3, and R_4 for natural and forced equal phasing. (*a*) Conditions for *natural* phasing (possible with indicated feeders). (*b*) Conditions for forced *equal* phasing (not possible unless antiphase shifters are provided at *A*, *B*, and *C*).

Only the conditions in Fig. 162*a* are possible, since equal phasing would *not* occur with the drives shown in Fig. 162*b* but with quarter-wavelength phase shifters. Each half-wave section must be driven at the mid-gap by an inphase current and such that respective center gap feeder matches the resistance R_1, R_2, R_3, R_4, respectively. This can be done with the same-dimensioned feeders, having a wave impedance of about 110 ohms, since we have to match 97.516 and 122.136 ohms if the reactances are tuned out. Figure 163 gives the actual and equivalent networks for an

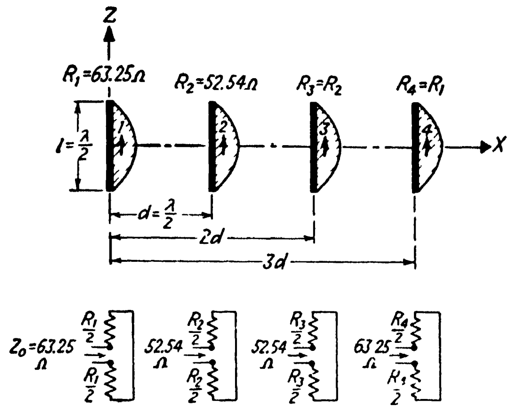
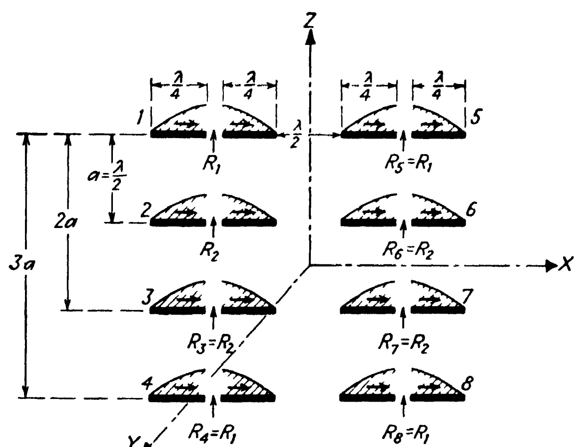


FIG. 163. Individual radiation resistances R_1, R_2, R_3, and R_4 of parallel half-wave radiators spaced half of a wavelength. Total radiation resistance of the alignment is $R_r = R_1 + R_2 + R_3 + R_4 = 2(R_1 + R_2)$ ohms. Such terminations must be satisfied for compensated half-wave-radiator reactance.



This spacing between opposing ends of the $\lambda/2$ radiators along the X direction makes $d = \lambda/2 + \lambda/4 + \lambda/4 = \lambda$ in effect. Since the excitations are inphase, the table of Fig. 149 yields

$$R_1 = \underbrace{R_{d=0,a=0}}_{\substack{=73.4\,\Omega \\ \text{self-effect} \\ \text{of radiator 1}}} + \underbrace{R_{d=0,a=\lambda/2}}_{\substack{\text{due to 2} \\ +26.4\,\Omega}} + \underbrace{R_{d=0,a=\lambda}}_{\substack{\text{due to 3} \\ -4.064\,\Omega}} + \underbrace{R_{d=0,a=1.5\lambda}}_{\substack{\text{due to 4} \\ +1.78\,\Omega}}$$

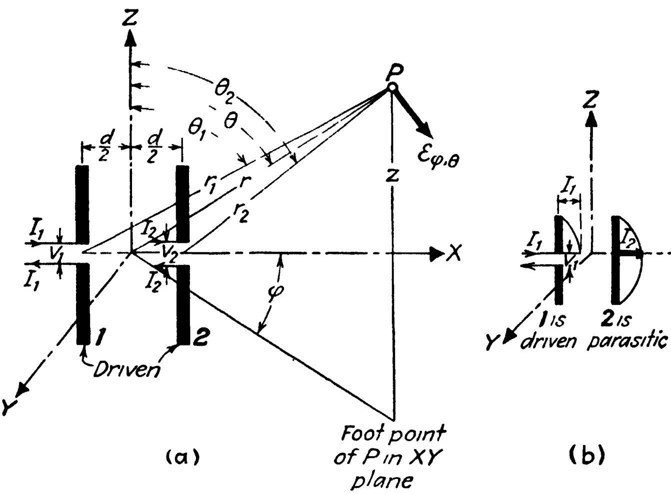
$$+ \underbrace{R_{d=\lambda,a=0}}_{\substack{\text{due to 5} \\ +4.1\,\Omega}} + \underbrace{R_{d=\lambda,a=\lambda/2}}_{\substack{\text{due to 6} \\ +8.83\,\Omega}} + \underbrace{R_{d=\lambda,a=\lambda}}_{\substack{\text{due to 7} \\ +3.56\,\Omega}} + \underbrace{R_{d=\lambda,a=1.5\lambda}}_{\substack{\text{due to 8} \\ -2.92\,\Omega}}$$

and similarly for R_2. Total radiation resistance $R_r = 4(R_1 + R_2)$ ohms.

FIG. 164. Computations for a curtain array.

alignment of four parallel radiators that are separated a distance of a half wavelength and driven in phase Also here a mean value of 52.54 and 63 25 would give the characteristic impedance for all feeders of concern As a matter of fact, a 50-ohm feeder could also be used in each case, since the mismatch would not cut down the power transfer by much. Figure 164 shows a curtain array of eight horizontal inphase half-wave dipoles with essentially again resistance terminations of the same magnitude The cases in obstructed radiation space are given in Chap VII

138. Details and Numerical Applications of Driven and Parasitic Radiators. This subject matter is partially discussed in Secs 130 and



$I_2 = kI_1\epsilon^{j\psi}$, $r_1 \gg d$, $r_2 \gg d$, $r_1 = r + \frac{d}{2}\sin\theta\cos\varphi$

$\frac{1}{r_1} \simeq \frac{1}{r} \simeq \frac{1}{r_2}$, $r_2 = r - \frac{d}{2}\sin\theta\cos\varphi$, $\theta_1 \simeq \theta \simeq \theta_2$

for P in the distant radiation region I and I_1 are currents at the current antinode which for the total length of the dipole equal to $\lambda/2$ are also equal to the feeder currents d, r, r_1, r_2 are in meters I_1 and I_2 in amp V_1 and V_2 in volts, $Z_m = Z_t$ ohms for the mutual impedance from one radiator on the other and vice versa

FIG. 165 Conditions for two driven half-wave dipoles and one driven half-wave dipole with a coupled linear half-wave dipole Note that the feeder is usually a concentric line unless the input is a higher termination than 150 ohms

131. In Fig. 165a, where both radiators are driven by dipole gap entrance currents I_1 and $I_2 = kI_1\epsilon^{j\psi}$ the total field at any elevation $(90 - \theta)$ degrees with respect to the equatorial XY plane is

$$\mathcal{E}_{\varphi,\theta} = j\frac{60I_1}{r}\epsilon^{-j\beta r_1}F_1D_{\theta_1} + j\frac{60I_2}{r}\epsilon^{-j\beta r_2}F_2D_{\theta_2}$$

volts/m; first part is due to radiator 1 and second part to radiator 2

This expression is based on the definitions for the form factor F and the meridian directivity D_θ of a linear radiator of l meters length in unobstructed space. If we take, for instance, radiator 1, we have the definitions

$$F_1 = \beta \int_{z=0}^{z=l} \frac{I_z}{I_1}\,dz; \qquad D_{\theta_1} = \frac{\beta \sin\theta \int_{z=0}^{z=l} I_z \cos(\beta z \cos\theta)\,dz}{I_1 F_1}$$

where I_1 is the effective input current at $z = 0$ for a sinusoidal current distribution toward the open end of the linear radiator and I_z denotes the effective current a distance of z meters from the input end. The form factor F_1 is referred to the current value I_1. For VHF's and UHF's, it is more convenient to refer the form factor to the *antinode* current value $I_{z_{max}}$. The form factor is then $F_1 = 1 - \cos(\beta l)$. For a half-wave dipole, we have then a gap entrance current $I_{z_{max}}$, $\beta l = (2\pi/\lambda)\frac{1}{2}\lambda = \pi$, and $F_1 = 1 - \cos(\pi) = 1 - (-1) = +2$ This is also the factor 2, which resulted in our derivation for the field intensity of a half-wave linear radiator We find then

$$\mathcal{E}_{\varphi,\theta} = j60\,\frac{I_1\epsilon^{-j\beta r}}{r}\,(F_1 D_{\theta_1}\epsilon^{-j\frac{1}{2}\beta x \sin\theta\cos\varphi} + kF_2 D_{\theta_2}\epsilon^{j\psi}\epsilon^{j\frac{1}{2}\beta x \sin\theta\cos\varphi})$$

$$= j120\,\frac{I_1 D_\theta\epsilon^{-j\beta r}}{r}\,(\epsilon^{-j\frac{1}{2}\beta x \sin\theta\cos\varphi} + k\epsilon^{j\frac{1}{2}\beta x \sin\theta\cos\varphi}\epsilon^{j\psi}) \qquad \text{volts/m}$$

The first component rotates backward by an angle $\frac{1}{2}\beta x \sin\theta\cos\varphi$, the second component is k times the first component and rotates forward by an angle $\psi + \frac{1}{2}\beta x \sin\theta\cos\varphi$. For the two half-wave radiators 1 and 2, we have $F_1 = F_2 = 2$ and the meridian directivities $D_{\theta_1} = D_{\theta_2} = D_\theta = [\cos(90^\circ \cos\theta)]/\sin\theta$. For the abbreviation $A = (j120 I_1\epsilon^{-j\beta r})/r$, we have

$$\mathcal{E}_{\varphi,\theta}^{(\text{volts/m})} = A\epsilon^{-j0.5\beta x \sin\theta\cos\varphi} + kA\epsilon^{j(\psi + 0.5\beta x \sin\theta\cos\varphi)}$$

For equal exciter currents $I_1 = I_2 = I$, *i.e.*, $k = 1$, and any relative time phase difference ψ, we find

$$\mathcal{E}_{\varphi,\theta}^{(\text{volts/m})} = A[\epsilon^{-j0.5\beta x \sin\theta\cos\varphi} + \epsilon^{j(\psi + 0.5\beta x \sin\theta\cos\varphi)}]$$

Inasmuch as the term $\frac{1}{2}\beta x \cos\varphi \sin\theta$ occurs in each component, nothing is altered if we write

$$\mathcal{E}_{\varphi,\theta}^{(\text{volts/m})} = A + kA\epsilon^{j(\psi - \beta x \sin\theta\cos\varphi)}$$ for unequal exciter currents and phase difference ψ

$$\mathcal{E}_{\varphi,\theta}^{(\text{volts/m})} = A[1 + \epsilon^{j(\psi - \beta x \sin\theta\cos\varphi)}]$$ for equal exciter currents and phase difference ψ

For inphase and equal exciter currents, we have $k = 1$, $\psi = 0$ deg, and

$$\mathcal{E}_{\varphi,\theta}^{(\text{volts/m})} = A(1 + \epsilon^{-j\beta x \sin\theta\cos\varphi}) = A[1 + \cos(\beta x \sin\theta\cos\varphi) - j\sin(\beta x \sin\theta\cos\varphi)]$$

In the equatorial XY plane, we have $\theta = 90$ deg, rendering $\sin\theta = 1 = D_\theta$. Therefore,

$$\mathcal{E}_{\varphi,\theta=90^\circ}^{(\text{volts/m})} = \mathcal{E}_\varphi = j120\,\frac{I_1\epsilon^{-j\beta r}}{r}\,\{1 + k[\cos(\psi - \beta x\cos\varphi) + j\sin(\psi - \beta x\cos\varphi)]\} \qquad \text{for } I_2 = I_1\epsilon^{j\psi};$$

$$\mathcal{E}_{\varphi,\theta=90^\circ}^{(\text{volts/m})} = \mathcal{E}_\varphi = j120\,\frac{I_1\epsilon^{-j\beta r}}{r}\,[1 + \cos(\beta x\cos\varphi) - j\sin(\beta x\cos\varphi)]$$

for inphase and equal exciter currents $I_1 = I_2$

For driven half-wave radiators with applied voltages V_1 and V_2, we generally have the impedances V_1/I_1 and V_2/I_2 complex ohms, since, according to Sec. 131 for the self-impedance Z_s and mutual impedance Z_m,

$$\left.\begin{aligned} V_1 &= I_1Z_s + I_2Z_m = I_1[R_s + jX_s + k\sqrt{R_m^2 + X_m^2}\,\epsilon^{j(m+\psi)}]; \\ V_2 &= I_2Z_s + I_1Z_m = I_2\left[R_s + jX_s + \frac{1}{k}\sqrt{R_m^2 + X_m^2}\,\epsilon^{j(m-\psi)}\right] \end{aligned}\right\}\text{volts}$$

The input impedance Z_1 at the $\lambda/2$ dipole gap of radiator 1 of Fig. 165a then becomes, for $m = \tan^{-1}(X_m/R_m)$,

$$Z_1^{(\text{ohms})} = R_s + \underbrace{k\sqrt{R_m^2 + X_m^2}\cos(m+\psi)}_{\substack{\text{deviation from } R_s = 73.4 \text{ ohms due} \\ \text{to presence of radiator 2}}} + j[X_s + \underbrace{k\sqrt{R_m^2 + X_m^2}\sin(m+\psi)}_{\substack{\text{deviation for } X_s = 42.9 \text{ ohms of} \\ \text{radiator 1 due to radiator 2}}}] \quad (853)$$

For the driven $\lambda/2$ dipole 2 of Fig. 165a, we find Z_2 from V_2/I_2 and have

$$Z_2^{(\text{ohms})} = 73.4 + \underbrace{\frac{1}{k}\sqrt{R_m^2 + X_m^2}\cos(m-\psi)}_{\substack{\text{deviation from 73.4 ohms of} \\ \text{radiator 2 due to radiator 1}}} + j\left[42.9 + \underbrace{\frac{1}{k}\sqrt{R_m^2 + X_m^2}\sin(m-\psi)}_{\substack{\text{deviation from 42.9 reactive} \\ \text{ohms of radiator 2 when acting} \\ \text{alone due to radiator 1}}}\right] \quad (853a)$$

Figure 166 gives the R_m, X_m, and Z_m curves for the mutual radiation resistance, the mutual reactance, and the mutual impedance of one half-wave radiator on another parallel half-wave radiator that is at a spacing d expressed in the free-space wavelength of $\lambda = 300/f^{(\text{Mc})}$ meters. The computations are based on the formulas of Fig. 143b and $l = \frac{1}{2}\lambda$, since we deal with half-wave dipoles. Only the case of the half-wave radiators in unobstructed radiation space is of concern in this chapter, and the values are as given on the ordinate axis For a spacing $d = \frac{1}{4}\lambda$, we find a mutual radiation resistance $R_m = 43$ ohms and a mutual reactance of $X_m = -30$ ohms, yielding the absolute value of the mutual impedance $Z_m = \sqrt{43^2 + 30^2} = 52.5$ ohms, as can also be read off directly in Fig. 166. We find then a corresponding phase angle $m = \tan^{-1} -30/43 = \tan^{-1}(-0.698) = -34.9$ deg. For equal exciter currents, we have $k = 1$, since $I_1 = I_2$; and for a relative phasing of $\psi = 50$ deg of the respective exciter currents, we find $m + \psi = -34.9 + 50 = 15.1$ deg and $m - \psi = -34.9 - 50 = -84.9$ deg. Hence, $\cos(m + \psi) = \cos(15.1$ deg$) = 0.9655$, $\sin(15.1$ deg$) = 0.2602$; $\cos(-84.9$ deg$) = \cos(84.9$

deg) = 0.0889, sin (−84.9 deg) = − sin (84.9 deg) = −0.996. Therefore,

$$R_1 = R_s + Z_m \cos(m + \psi) = 73.4 + 52.5 \times 0.9655 = 124.1 \text{ ohms}$$
$$X_1 = X_s + Z_m \sin(m + \psi) = 42.9 + 52.5 \times 0.2602 = 56.55 \text{ ohms}$$

for radiator 1

$$R_2 = 73.4 + 52.5 \cos(m - \psi) = 73.4 + 52.5 \times 0.0889 = 78.06 \text{ ohms}$$
$$X_2 = 42.9 + 52.5 \sin(m - \psi) = 42.9 - 52.5 \times 0.996 = -9.5 \text{ ohms}$$

for radiator 2

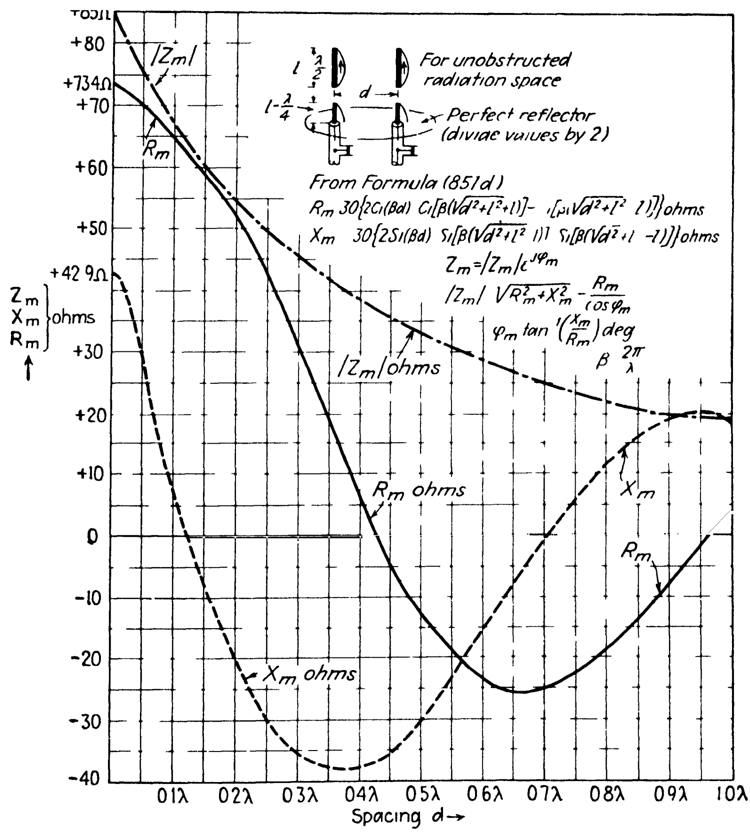
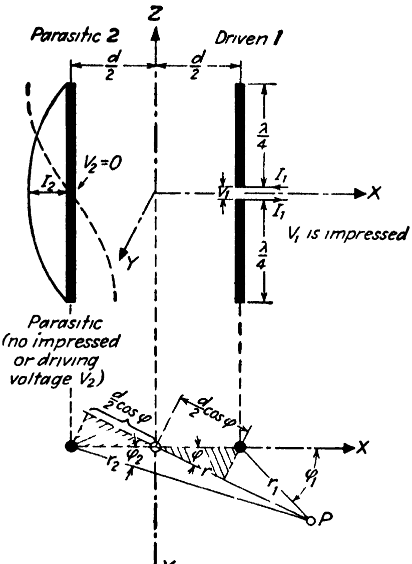


FIG 166 Mutual impedance Z_m and its mutual radiation resistance component R_m and reactance component X_m, ($Z_m = R_m + jX_m$)

There is no use adding up respective R_1 and R_2 radiation resistances and finding the value for 56.55 − 9.5 reactive ohms, since we have to feed each driven radiator separately, because we must produce a relative current phase of $\varphi = 50$ time degrees. The characteristic impedance of the feeder that supplies the power to the dipole gap of radiator 1 must then be equal to 124.1 ohms. The stub length to be used has to be such

that it causes a capacitive reactance of -56.55 ohms at the dipole gap if taken as an equivalent series compensation. It takes a feeder with a



$Z_1 = 84 + j74.4\Omega = 112 \underline{/+41.55^\circ}$ for $d = \dfrac{\lambda}{4}$

$\left.\begin{aligned} \varphi_1 \cong \varphi \cong \varphi_2 \\ \frac{1}{r_1} \cong \frac{1}{r} \cong \frac{1}{r_2} \end{aligned}\right\}$ for a distant field point P

FIG. 167. Driven and parasitic radiators. All voltages are effective volts, and all currents are effective amp; d, r, r_1, r_2, and λ are in meters

characteristic impedance of about 78 ohms for radiator 2 and an equivalent inductive reactance (series compensation) to cancel the capacitive reactance of -9.5 ohms when looking into the dipole gap of radiator 2. How this is done is shown on page 701. We note, therefore, that we have the impedances, when looking into respective dipole gaps, as

$$Z^{(\text{ohms})} = 124.1 + j56.55 \quad \text{for } \frac{\lambda}{2} \text{ dipole 1;} \qquad Z = 78.06 - j9.5 \quad \text{for } \frac{\lambda}{2} \text{ dipole 2}$$

$Z = 73.4 + j42.9$ for either dipole if we have *no mutual interference* and holds for $d = \lambda/4$ meters; $I_2 = I_1\epsilon^{j50^\circ}$ and $I_1 = I_2$. We also note that we can no longer use the same dimensions for the feeder supplying the power to dipole 1 and to dipole 2, since the radiation resistances R_1 and R_2 are altogether too different.

Figure 167 gives the case of a driven half-wave dipole and a parasitic half-wave linear radiator, for which

$$V_1 = I_1Z_s + I_2Z_m; \qquad 0 = I_2Z_s + I_1Z_m;$$

$$Z_s = \sqrt{R_s^2 + X_s^2} = \sqrt{73.4^2 + 42.9^2}\,\epsilon^{j30.35^\circ}$$

Therefore, $\psi = \tan^{-1}(42.9/73.4) = \tan^{-1}(0.585) = 30.35$ deg, and $m = \tan^{-1}(X_m/Z_m)$, which can be found from the values of Fig. 166. For the abbreviations $a = \sqrt{R_s^2 + X_s^2}$ and $b = \sqrt{R_m^2 + X_m^2}$, which are the absolute values of the impedances, we find

$$V_1 = aI_1\epsilon^{j\psi} + bI_2\epsilon^{jm};$$
$$I_2 = -\frac{I_1Z_m}{Z_s} = -\frac{bI_1\epsilon^{jm}}{a\epsilon^{j\psi}} = -\frac{bI_1}{a}\epsilon^{j(m-\psi)} = -\rho I_1\epsilon^{j(m-\psi)}.$$
$$\rho = \frac{b}{a}; \qquad V_1 = I_1[a\epsilon^{j\psi} - b\rho\epsilon^{j(m-\psi)}\epsilon^{jm}] = I_1[a\epsilon^{j\psi} - b\rho\epsilon^{j(2m-\psi)}]$$

This causes an effective input impedance

$$Z_1 = \frac{V_1}{I_1} = a\epsilon^{j\psi} - b\rho\epsilon^{j(2m-\psi)} = a(\cos\psi + j\sin\psi) - b\rho[\cos(2m-\psi) + j\sin(2m-\psi)]$$
$$= \underbrace{a\cos\psi - b\rho\cos(2m-\psi)}_{\text{effective resistance } R_1} + j[\underbrace{a\sin\psi - b\rho\sin(2m-\psi)}_{\text{effective reactance } X_1}]$$

We have, therefore, for a spacing $d = \frac{1}{4}\lambda$ meters, from Fig. 166, the values $R_m = 43$ ohms, $X_m = -30$ ohms, $b = 52.5$ ohms, $m = -34.9$ deg, $2m - \psi = -69.8 - 30.35 = -100.15$ deg, $\cos\psi = \cos 30.35° = 0.863$, $\sin\psi = 0.5052$, $\cos(2m-\psi) = \cos(-100.15°) = \cos(100.15°) = \sin(10.15°) = 0.176$, $\sin(2m-\psi) = \sin(-100.15°) = -\sin(100.15°) = -\cos(10.15°) = -0.9843$. Hence,

$$R_1 = a\cos\psi - \frac{b^2}{a}\cos(2m-\psi) = 85 \times 0.863 - \frac{52.5^2}{85}0.176 = 67.7 \text{ ohms}$$
$$X_1 = a\sin\psi - \frac{b^2}{a}\sin(2m-\psi) = 85 \times 0.5052 + \frac{52.5^2}{85}0.9843 = 74.8 \text{ ohms}$$
$$Z_1 = 67.7 + j74.8; \quad Z_1 = \sqrt{67.7^2 + 74.8^2}\,\Big/\tan^{-1}\left(\frac{74.8}{67.7}\right) = 102\epsilon^{j47.85°}$$

These are then the values experienced when looking into the dipole gap that has the applied voltage. Since the term $a\cos\psi = R_s$ and $a\sin\psi = X_s$, because $a(\cos\psi - j\sin\psi) = \sqrt{R_s^2 + X_s^2}\,\epsilon^{j\psi}$ for the dipole impedance without the presence of the parasitic half-wave radiator, we have also

$$Z_1 = \underbrace{R_s}_{73.4} - \frac{R_m^2 + X_m^2}{\sqrt{R_s^2 + X_s^2}}\cos(2m-\psi) + j\left[\underbrace{X_s}_{42.9} - \frac{R_m^2 + X_m^2}{\sqrt{R_s^2 + X_s^2}}\sin(2m-\psi)\right] \text{ ohms}; \quad Z_1 = R_1 + X_1 \text{ ohms}$$

where R_1 is the effective resistance and X_1 is the effective reactance in ohms. The driven $\lambda/2$ dipole must be tuned in such a way that the total

reactance vanishes, or

$$\frac{R_m^2 + X_m^2}{\sqrt{R_s^2 + X_s^2}} \sin (2m - \psi) = 42.9 \text{ ohms}$$

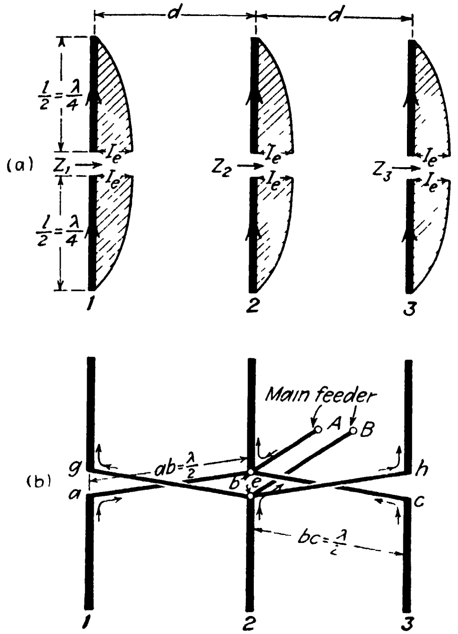
For an effective current of 1 amp ($= I_1$), the radiated power becomes

$$P_r = R_s - \frac{R_m^2 + X_m^2}{\sqrt{R_s^2 + X_s^2}} \cos (2m - \psi) = 67.7 \text{ watts}$$

instead of 73.4 watts in unobstructed radiation space. This is a power disadvantage of 67.7/73.4 = 0.925. However, the parasitic $\lambda/2$ radiator is not used for an over-all power gain, but for a gain in a *desired direction.*

139. Notes on Feeders, Quarter-wavelength Sleeves, and Such Sleeves as Partial Radiators. From the analysis in the foregoing sections (just given), it is difficult to realize a common feed, as in Fig. 168*b*, to meet the conditions for equal inphase effective currents of I_e amperes at respective dipole gaps. As far as the transposed feeder lines between radiator 1 and radiator 2 and radiator 3 and radiator 2 are concerned, we can readily secure voltage inphase conditions at respective dipole gaps if the transpositions correspond to a half-wavelength along the conductors *ab* and *cb*. The spacing *d* depends upon the free-space wavelength λ_0, which is 300 divided by the frequency in megacycles per second, while the wavelength along the transposed line is somewhat different and equal to λ meters. Even though we put this distance equal to $\frac{1}{2}\lambda$, we have, for radiators 1 and 3, an input impedance of $Z_1 = Z_3 = 73.4 + j42.9 + Z_d + Z_{2d}$ complex ohms and, for radiator 2, an impedance $Z_2 = 73.4 + j42.9 + 2Z_d$ complex ohms. Taking $d = \frac{1}{2}\lambda$ meters, we find from Fig. 166 the values $R_m = R_d = -12.48$ ohms, $X_m = X_d = -30$ ohms, $R_{2d} = +4.1$ ohms, which values could also be found from the table in Fig. 149 for the distance equal to λ meters. Inasmuch as the curves in Fig. 166 are computed from the inscribed formulas, we can also find X_{2d} from

$$\begin{aligned} X_{2d} &= -30\{2\,\text{Si}\,(2\beta d) - \text{Si}\,[\beta(\sqrt{4d^2 + l^2} + l)] - \text{Si}\,[\beta(\sqrt{4d^2 + l^2} - l)]\} \\ &= -30\left\{2\,\text{Si}\left(2\frac{2\pi}{\lambda}\frac{\lambda}{2}\right) - \text{Si}\left[\frac{2\pi}{\lambda}\left(\sqrt{4\frac{\lambda^2}{4} + \frac{\lambda^2}{4}} + \frac{\lambda}{2}\right)\right] \right. \\ &\qquad \left. - \text{Si}\left[\beta\left(\underbrace{\sqrt{\frac{5\lambda^2}{4}}}_{=\frac{2.24\lambda}{2}} - \frac{\lambda}{2}\right)\right]\right\} \\ &= -30\left[2\,\text{Si}\,(6.28) - \text{Si}\left(\frac{2\pi}{\lambda}\frac{3.24\lambda}{2}\right) - \text{Si}\left(\frac{2\pi}{\lambda}\frac{1.24\lambda}{2}\right)\right] \\ &= -30[2\,\text{Si}\,(6.28) - \text{Si}\,(10.15) - \text{Si}\,(3.88)] \\ &= -30(2 \times 1.41 - 1.66 - 1.7765) = -30 \times 0.62 = -18.6 \text{ ohms} \end{aligned}$$



$Z_1 = \sqrt{65.02^2 + 31.5^2}\,\epsilon^{j\varphi_1}$ ohms

$\varphi_1 = \tan^{-1}\left(\frac{31.5}{65.02}\right)$ deg

$Z_2 = \sqrt{48.44^2 + 17.1^2}\,\epsilon^{j\varphi_2}$ ohms

$\varphi_2 = -\tan^{-1}\left(\frac{17.1}{48.44}\right)$ deg

for $d = \frac{\lambda}{2}$

$Z_1 = Z_3 = 65.02 + j31.5\Omega$
$Z_2 = 48.44 - j17.1\Omega$

Equal I_e entrance currents at all three dipole gaps are basic for our directivity patterns as well as inphase conditions.

A, *B* are terminals of the UHF source. A main feeder can provide equal effective antinode values only at the gaps of radiators 1 and 3, but *not* at 2, since $Z_1 = Z_3$ and $Z_1 \neq Z_2$ ohms

$Z_1 = Z_3$; $Z_1 = Z_s + Z_d + Z_{2d}$ complex ohms; $Z_2 = Z_s + 2Z_d$ complex ohms; $Z_d = -12.48 - j30\Omega$; $Z_{2d} = 4.1 + j18.6\Omega$.

Since even for thin radiators with assumed sinusoidal current distributions we have

$$Z_1 = Z_3 = 72.5\epsilon^{j26.3°} \text{ ohms and } Z_2 = 51.4\epsilon^{-j19.5°} \text{ ohms}$$

we find for *V* volts impressed across the *b* and *e* terminals of the mid-gap of radiator 2 essentially also *V* volts across terminals *g* and *a* and *h* and *c* of respective radiators 1 and 3. *But* the entrance current I_2 at the mid-gap of dipole 2 is

$$I_2 = \frac{V}{51.4\epsilon^{-j19.5°}} = \frac{V}{51.4}\epsilon^{j19.5°} \text{ amp}$$

and the entrance currents I_1 and I_3 at respective mid-gaps of dipoles 1 and 3 are

$$I_1 = I_3 = \frac{V}{72.5\epsilon^{j26.3°}} = \frac{V}{72.5}\epsilon^{-j26.3°} \text{ amp}$$

which differ not only in magnitude but also in relative phase with respect to current I_2.

FIG. 168. A main feeder terminating at the dipole gap of radiator 2 will not cause equal and inphase currents at all three dipole gaps.

Table 31 tabulates the computation of mutual reactances, resistances, and impedances between two half-wave radiators, for different spacings *d* in terms of the wavelength, for ratios d/λ from 0 to 1. These results are used in Fig. 166. We find then for Fig. 168

$$R_1 = 73.4 - 12.48 + 4.1 = 65.02 \text{ ohms} = R_3$$
$$R_2 = 73.4 + 2(-12.48) = 48.44 \text{ ohms}$$
$$X_1 = 42.9 - 30 + 18.6 = 31.5 \text{ ohms} = X_3$$
$$X_2 = 42.9 - 2 \times 30 = 17.1 \text{ ohms}$$

TABLE 31.—MUTUAL EFFECTS BETWEEN HALF-WAVE RADIATORS

$\frac{d}{\lambda}$	βd	Si βd	Ci βd	$\sqrt{d^2+l^2}$	$\sqrt{d^2+l^2}+l = a$	$\sqrt{d^2+l^2}-l = b$	$A = \beta a$ numeric	$B = \beta b$ numeric	Si A	Ci A	Si B	Ci B	R_m, ohms	X_m, ohms	$\frac{X_m}{R_m} = c$	$\varphi_m = \tan^{-1} c$, deg	$\cos \varphi_m$	$\lvert Z_m\rvert = \frac{R_m}{\cos \varphi_m} = \sqrt{R_m^2 + X_m^2}$
0.1	0.628	0.6	0.02	0.5099λ	1.0099λ	0.0099λ	6.28	0.0624	1.42	−0.023	0.01	−2.2	64.11	6.9	0.1075	6.1	0.9943	64.4
0.2	1.256	1.15	0.43	0.5385λ	1.0385λ	0.0385λ	6.49	0.242	1.42	0.0	0.24	−0.9	52.8	−19.2	−0.364	−20	0.9387	55.9
0.25	1.57	1.39	0.48	0.5567λ	1.0567λ	0.0567λ	6.64	0.356	1.425	0.0015	0.4	−0.4	40.755	−28.65	−0.705	−35.2	0.8171	49.75
0.3	1.884	1.55	0.48	0.583λ	1.083λ	0.083λ	6.81	0.52	1.435	0.005	0.496	−0.17	33.75	−35.07	−1.04	−46.1	0.6934	46.65
0.35	2.198	1.67	0.38	0.6082λ	1.1082λ	0.1082λ	6.97	0.68	1.45	0.08	0.65	0.07	18.3	−37.2	−2.03	−63.8	0.4415	41.4
0.4	2.512	1.785	0.285	0.6403λ	1.1403λ	0.1403λ	7.16	0.88	1.47	0.09	0.85	0.24	7.2	−37.5	−5.2	−79.1	0.1891	38.1
0.45	2.826	1.84	0.15	0.671λ	1.171λ	0.171λ	7.36	1.075	1.49	0.11	~1	0.36	−5.1	−35.7	7	81.9	0.1409	36.2
0.5	3.14	1.85	0.07	0.707λ	1.207λ	0.207λ	7.59	1.3	1.52	0.12	1.19	0.44	−12.48	−30	2.405	67.4	0.3843	32.5
0.55	3.454	1.84	−0.02	0.742λ	1.242λ	0.242λ	7.81	1.52	1.54	0.12	1.35	0.47	−18.9	−23.7	1.25	51.4	0.6239	30.3
0.6	3.768	1.8	−0.1	0.781λ	1.281λ	0.281λ	8.08	1.76	1.58	0.118	1.49	0.46	−23.34	−15.9	0.683	34.5	0.8256	28.2
0.65	4.082	1.745	−0.16	0.819λ	1.319λ	0.319λ	8.3	2	1.6	0.112	1.6	0.42	−25.56	−8.7	0.34	18.8	0.9466	27
0.7	4.396	1.68	−0.19	0.86λ	1.36λ	0.36λ	8.55	2.26	1.62	0.1	1.7	0.35	−24.9	−1.2	0.0482	2.75	0.99885	25
0.8	5.024	1.55	−0.18	0.943λ	1.443λ	0.443λ	9.05	2.78	1.66	0.06	1.83	0.19	−18.3	11.7	−0.64	−32.7	0.8415	21.7
0.85	5.338	1.49	−0.16	0.985λ	1.485λ	0.485λ	9.35	3.04	1.66	0.02	1.85	0.11	−13.5	15.9	−1.178	−49.7	0.6468	20.9
0.9	5.652	1.45	−0.12	1.03λ	1.53λ	0.53λ	9.62	3.33	1.66	−0.01	1.848	0.03	−7.8	18.24	−2.34	−66.9	0.3923	19.85
0.95	5.966	1.42	−0.075	1.072λ	1.572λ	0.572λ	10.0	3.59	1.66	−0.045	1.83	−0.06	−1.35	20.1	−14.9	−86.15	0.067	19.5
1.00	6.28	1.41	−0.04	1.118λ	1.618λ	0.618λ	10.15	3.88	1.66	−0.055	1.78	−0.165	4.1	18.6	4.4	77.2	0.2215	19

Remarks. Based on formula (51d) and Fig. 166. d is spacing of the two parallel half-wave radiators and $l = \frac{1}{2}\lambda$ the physical length of each radiator; $\beta = \frac{2\pi}{\lambda}$;

$$R_m{}^{(\text{ohms})} = +30\left\{2\,\text{Ci}(\beta d) - \text{Ci}\left[\beta\left(\sqrt{d^2+l^2}+l\right)\right] - \text{Ci}\left[\beta\left(\sqrt{d^2+l^2}-l\right)\right]\right\}$$

$$X_m{}^{(\text{ohms})} = -30\left\{2\,\text{Si}(\beta d) - \text{Si}\left[\beta\left(\sqrt{d^2+l^2}+l\right)\right] - \text{Si}\left[\beta\left(\sqrt{d^2+l^2}-l\right)\right]\right\}$$

The cosine integral (Ci−) functions and sine integral (Si−) functions are read off on curves or from tables.

$R_m{}^{(\text{ohms})}$ mutual resistance, $X_m{}^{(\text{ohms})}$ mutual reactance, $Z_m e^{j\varphi_m}$ mutual impedance: contributions of one $\frac{\lambda}{2}$ radiator on the other $\frac{\lambda}{2}$ radiator

and the input impedances Z_1, Z_2, and $Z_3 = Z_1$ as given in Fig. 168. A common supply feeder as indicated in Fig. 168*b*, therefore, cannot provide equal input currents at respective dipole gaps of radiators 1, 2, and 3. We note that the central dipole (dipole 2) has a capacitive input reactance while the dipoles on either side exhibit an inductive gap reactance.

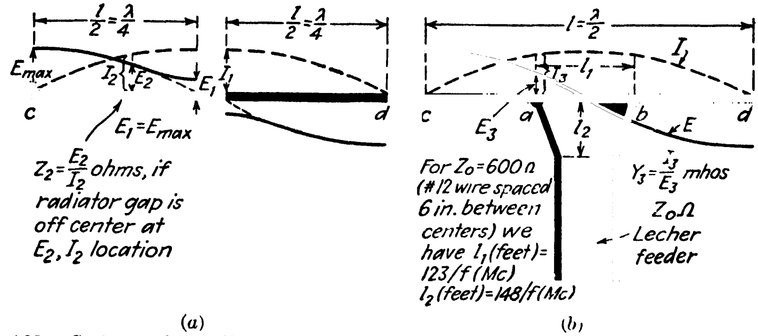
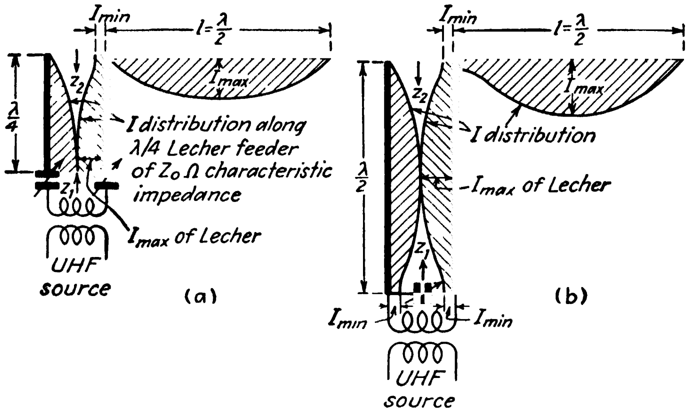


FIG. 169. Series and parallel feeding system of a half-wave radiator (*a*) Series feed (*b*) Shunt feed. λ along such radiators is about 5 per cent smaller than λ_0 in free space (λ_0 m $= 300/f^{(\text{Mc})}$ and 1 m = 39.37 in.)

From a practical point of view, it is evident that individual reactance canceling seems imperative. It is to be realized that we assumed sinusoidal distributions along the half-wave radiators and that the effect of the diameter of the radiators is also of no consequence. The effect of the diameter is analyzed in Sec. 140.

Figure 169 gives the series- and shunt-feed systems of a half-wave radiator, where again sinusoidal distributions are assumed. Inasmuch as for the series feed the feeder looks into an impedance of $73.1 + j42.9$ ohms, there must be a voltage E_1 at the driving points and we deal with pseudo nodes. As far as customary transmission-line theory is concerned, we have an input impedance of Z_0E_1/E_{max} ohms if radiation effects are ignored, where Z_0 represents the characteristic impedance of the half-wave dipole. If $0.5l$ is shorter than $\frac{1}{4}\lambda$, we should then have $-Z_0 \cot (\frac{1}{2}\beta l)$ ohms for the input impedance, which is a capacitive reactance and can be made to cancel an equal inductive reactance. This is also evident from the discussions in Secs. 74 to 84. A supply at the gap of a half-wave dipole is, according to the classic line theory, a case of series resonance, with a low input resistance experienced at the gap points, while the case of Fig. 170 calls for a high input resistance, since I_{min} is the entrance current and corresponds to the case of parallel resonance. Figure 171*a* shows that a quarter-wave section can be used for changing from a low-impedance input to a high-impedance output. Figure 171*b* illustrates the case for which we have a low impedance across points AA and across BB a higher impedance. It is, therefore, possible to find in Fig. 171*c* a position for which E_2/I_2 is equal to the input impedance of the dipole. Figure 172 then shows applications where a nonresonant feeder can be used to bring about the proper impedance transformation, where in each case a short-circuited quarter-wavelength stub and the locations p and q are made use of. Figure 173 gives the application for mid-gap drives of a half-wave

dipole. The lower the Z_0 value of the nonresonant feeder the closer the locations p and q must be next to the short-circuited end of the half-wave stub. It is this stub which acts as impedance-matching transformer. For the open-ended stub, the location is then next to the dipole gap, since the impedance E/I is lowest near this gap. For the $\frac{1}{2}\lambda$ stub, we have also the dotted position for terminating the feeder, since



$Z_0 = \sqrt{Z_1 Z_2}$ (quarter-wave impedance matching)

$Z_1 = Z_0 \dfrac{I_{\max}}{I_{\min}}$; $Z_2 = Z_0 \dfrac{I_{\min}}{I_{\max}}$

$Z_1 = Z_0 \dfrac{I_{\min}}{I_{\max}} = Z_2$

(half-wave impedance transformer is 1 : 1 matching)

FIG. 170. Lecher wire end feed of a half-wave radiator. (For the quarter-wavelength feeder the input impedance Z_1 is low and series resonance is a suitable match; for the half-wavelength feeder parallel resonance is a suitable match, since $Z_1 = Z_2$ is a high impedance.) The formulas with $I_{\min}$ and $I_{\max}$ refer to these extreme values of the Lecher feeder. Since the fields of the respective parallel-wire Lecher lines are in opposition, the external field is only very small.

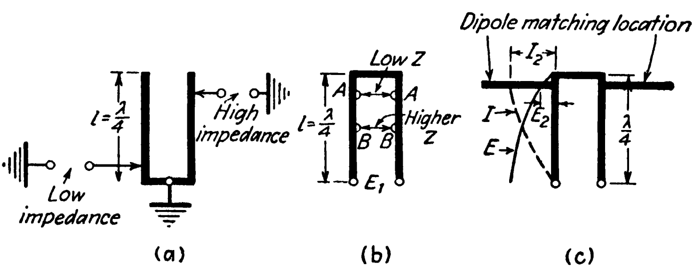


FIG. 171. Quarter-wavelength impedance converters.

we deal here with a 1 : 1 impedance transformer as far as either end of the stub is concerned. In connection with Figs. 172 and 173, we have to bear the following in mind: We have a nonresonant feeder of any suitable length and characteristic impedance Z_0 ohms. The stub has the same characteristic impedance. For Fig. 172, locations P_0 and P are the load terminations. The actual power flow occurs through the active antenna terminal P as far as the entrance current at P is concerned and the field between P and P_0. The entrance current at P is zero, according to the

distribution indicated in the figure. In reality, it is a minimum value, since the Q value of the half-wave radiator is not infinite, and we have at the P end of the half-wave radiator a I_{min} value, while there is a zero value at the other end and a I_{max} value at the mid-point of the half-wave radiator (see Fig. 170). For Fig. 173, the dipole is where the load points of respective stub lines terminate. In either Fig. 172 or 173, we have, when looking into the supply points p and q (toward the load), an input impedance Z_0 ohms if the feeder is properly matched (properly located). It is based

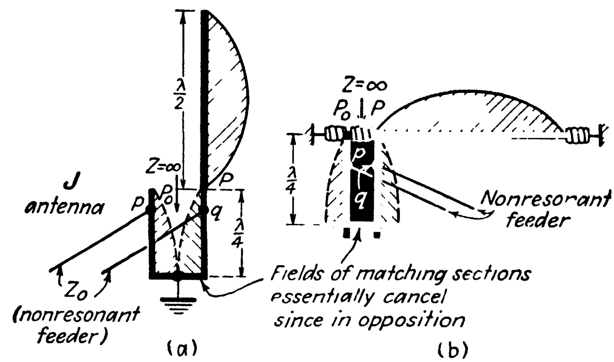


FIG. 172. Use of nonresonant feeders with matching sections.

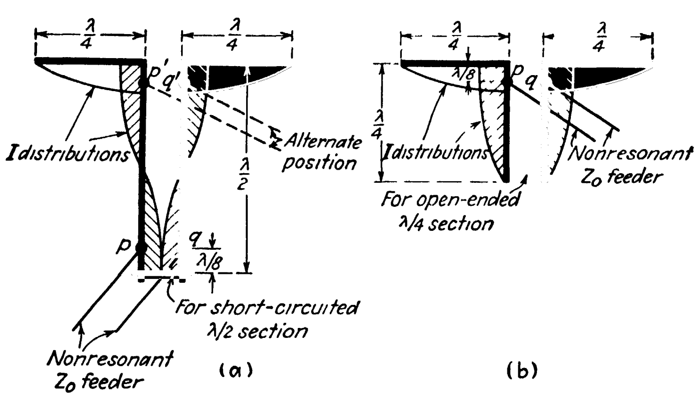


FIG. 173. Half-wavelength and quarter-wavelength sections in connection with an aperiodic feeder. For Z_0, any value from about 150 to 800Ω.

on the condition that for any line, when looking toward the load end, at a voltage maximum, we have an input *resistance* of rZ_0 ohms for $r = E_{max}/E_{min}$. At a voltage minimum, we have a pure resistance effect of Z_0/r ohms. Therefore, somewhere between the maximum and minimum position, we must have a state of affairs, where the input impedance is equal to Z_0 ohms, which is the *position marked p and q*. Figure 174 gives the arrangement for exciting two sets of ½λ radiators in time quadrature. This procedure of feeding and stubbing is based on the condition shown in Fig. 175, where we can locate positions of any phase. For instance, voltage E_a is 90 time degrees behind the input voltage, and E_b is 270 time degrees behind the input voltage. Therefore, E_c is 360 time degrees behind the E_a voltage, *i.e.*, in phase with it. Now

in Fig. 174, the antennas C and D are driven over the actual line lengths $cd = ab$, while the other two antennas A and B are driven over the actual line lengths $cf = ae$, which are longer. But $cf - cd = 2\lambda/8 = \frac{1}{4}\lambda$ as is $ae - ab = \frac{1}{4}\lambda$. This causes a lag of 90 time degrees. Moreover, a feeder section $\lambda/8$ long corresponds to an electrical length of $\beta l = 45$ deg. For an open-ended HF line, the impedance looking into the line is $Z_i = -jZ_0 \cot(\beta l)$ ohms; while for the short-circuited line, it is $Z_i = jZ_0 \tan(\beta l)$ ohms. In each case, we have for $l = \lambda/8$ meters, cot 45 deg = tan 45 deg = 1 and $Z_i = Z_0$ ohms. Figure 176 shows how a 500-ohm Lecher wire feeder trans-

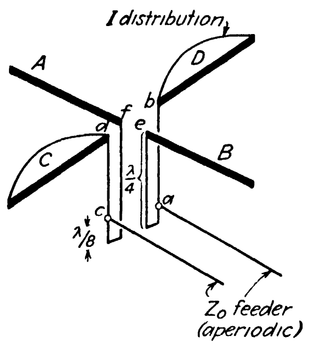


FIG. 174. Power supply to a turnstile bay by means of an aperiodic feeder. Radiators A and B have a zero current distribution when C and D have indicated maximum I distribution.

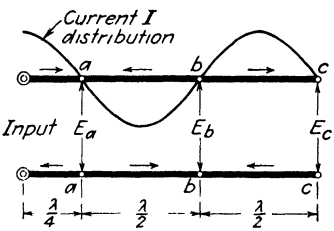


FIG. 175. The distribution along a feeder determines the relative phase with respect to the input end.

forms into a 70-ohm (strictly 73.4 ohms, or whatever other numerical value is taken as correct dipole termination) by using a 187-ohm quarter-wavelength section. By 187 ohms is meant the characteristic impedance of the $\frac{1}{4}\lambda$ section. If the Lecher wire feed had been chosen $Z_0 = 300$ ohms, the impedance transformation would require only a characteristic impedance of 145 ohms, which cannot be practically met with a Lecher line. For this reason, a *concentric* matching section is shown The reactance of the gap is canceled by a somewhat shorter dipole length, for which we also have only about 70 instead of 73.4 ohms radiation resistance. In each case, no attention was paid to the grounding of the supply and its effect on the feeder potential with respect to ground

The quarter-wave section impedance transformer can also be used by referring to the standing-wave ratio $r = E_{max}/E_{min} = I_{max}/I_{min}$ of a mismatched line with Z_0 ohms for the characteristic impedance. At a E_{max} location, we have a pure resistance of rZ_0 ohms; and at a minimum location (E_{min}), we have a pure resistance Z_0/r ohms. The line to be matched then uses a $\frac{1}{4}\lambda$ section with a characteristic impedance $Z_0' = Z_0\sqrt{r}$ ohms between the E_{max} and the remainder of the line. We can also use a $\frac{1}{4}\lambda$ section with a characteristic impedance $Z_0'' = Z_0/\sqrt{r}$ ohms between the voltage minimum and the remainder of the feeder. The arrangements of Fig. 176 are satisfactory provided the feeder terminals A and B are balanced with respect to ground. If this is not the case, we have to

use other means to bring about symmetry at respective dipole halves with regard to ground potential. This can be done by $\frac{1}{4}\lambda$ sleeves, as is shown in Fig. 177. The respective sleeves have two *entirely different* functions. With the sleeve open at the longer end, as in Fig. 177*a*, the impedance Z looking into the open end is very high (ideally ∞), since the other end is short-circuited. Therefore, the feeder current along the

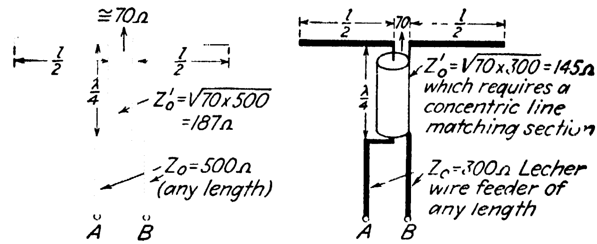
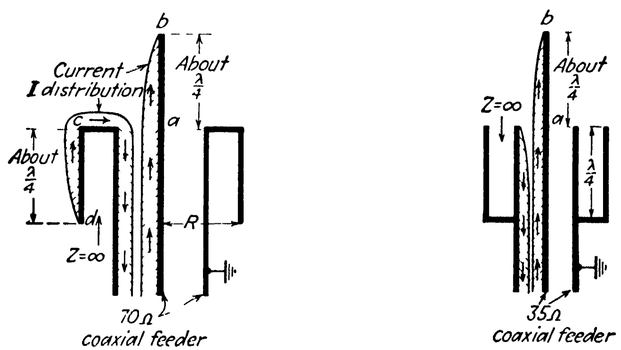


FIG. 176. Lecher wire feed of a dipole. $Z_0' = 194$ and $Z_0 = 500\Omega$ are possible values for Lecher wire feeders. $l/2$ is somewhat shorter than $\lambda/4$ in order to neutralize the reactance. This causes a somewhat smaller radiation resistance of about 70Ω.



(a) Portions *ab* and *cd* radiate

(b) Only portion *ab* radiates

FIG. 177. Quarter-wave sleeves as radiators and as insulators.

inside wall of the 70-ohm concentric will flow *readily* around the short-circuit path of the sleeve (which is part of the concentric line, bent backward) and then decrease according to the cosine law toward the lower end of the sleeve. Therefore, the extension *ab* of the continued central conductor of the concentric feeder must act as the upper half and the outside cylinder of length *cd* as the lower half of a half-wave radiator. It is admitted that the lower half has a diameter that is no longer negligibly small compared with the length *cd*. This must affect the current distribution. Overlooking this fact for the time being, we have in Fig. 177*a* a sort of offset half-wave dipole in comparison with the well-known J antenna of Fig. 178. It is called a J antenna, since the open-ended

quarter-wavelength Lecher together with the half-wave radiator extension looks like a J. The impedance looking into the J opening of Fig. 178*a* is likewise very high (ideally ∞), since the other end of the $\frac{1}{4}\lambda$ double line is short-circuited. It may be grounded at its mid-point. Therefore, the quarter-wave Lechers of Fig. 178*a* and *b* act, in effect, as insulators, even though consisting of metal. In Fig. 177*b*, the cup-

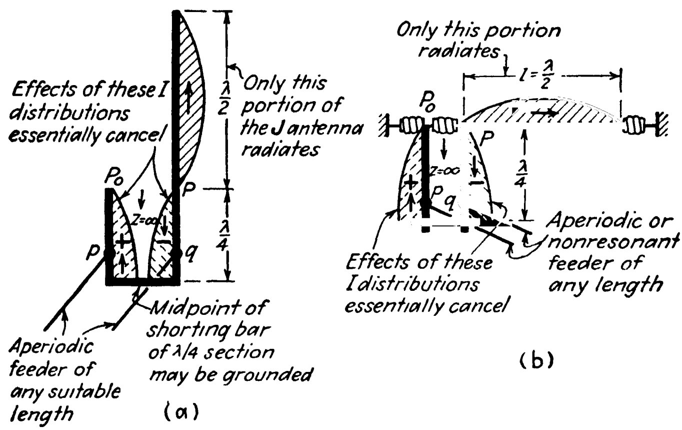


FIG. 178. The J antenna with an aperiodic feeder.

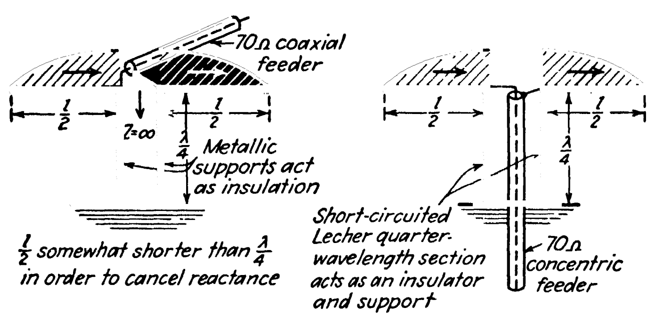


FIG. 179. Metallic insulators for half-wave radiators used as supports.

shaped sleeve acts as an insulator and *not* as a portion of the radiator as in Fig. 177*a*. Such $\frac{1}{4}\lambda$ metallic insulators are also made use of in Fig. 179. Figure 180 shows how dipole gap feeder points *a* and *b* can be kept dynamically insulated from the ground potential where the outer conductor is.

This is the $\frac{1}{2}\lambda$ dipole excitation with the insulation toward ground as compared with the $\frac{1}{4}\lambda$ single-pole excitation of Fig. 177*b*, where the $\frac{1}{4}\lambda$ sleeve with the open end at the top prevents the current from creeping down the outside wall of the $\frac{1}{4}\lambda$ sleeve. Therefore, the open $\frac{1}{4}\lambda$ sleeve of Fig. 177*b* prevents the outside surface of the sleeve from becoming a seat of a current distribution; while for Fig. 177*a*, a current

distribution is enhanced owing to the short-circuit path at the top, which makes it easy for a current to follow around the bent outside conductor of the concentric

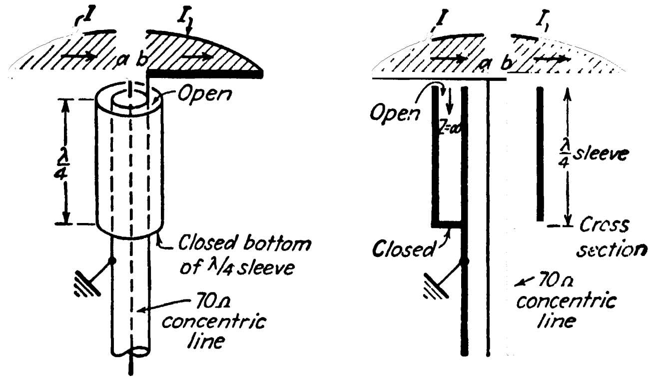


FIG. 180. Quarter-wavelength sleeve keeps points a and b dynamically insulated from ground.

feeder. Therefore, simply by inverting the $\frac{1}{4}\lambda$ concentric sleeve, we can either insulate ("cool off" the current distribution) a $\frac{1}{4}\lambda$ radiator from ground or continue ("heat up" the current distribution) to a half-wave radiator.

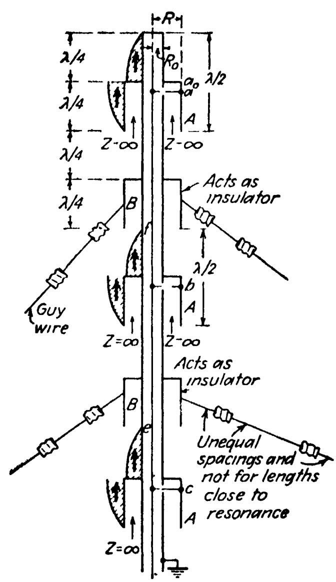


FIG 181. Alternate mounting of quarter-wave sleeves along a coaxial line can be used as spaced half-wave radiators.

An alternate mounting of quarter-wave sleeves along a concentric feeder is shown in Fig. 181. Note that only sleeves A are excited (driven) by means of respective a, b, and c connectors to the central conductor of the coaxial line. The distance between tap a and tap b is exactly 1 wavelength, as is the distance from tap b to tap c. Hence, the indicated current distributions are all in phase and are each a half wavelength. Tap a then excites sleeve A in a quarter-wavelength current distribution with a maximum current value at a_0. The short circuit a_0 to the outer conductor of the concentric line continues this action on the outside surface of the concentric line and causes a cosinoidal current decrease in the top direction of the outside surface of the coaxial feeder. The $\frac{1}{4}\lambda$ sleeves, marked B, are not excited and prevent any current distributions from creeping upward from e or f, since these sleeves act as metallic insulators. They can be used for guying the vertical

antenna system. For both cases, shown in Figs. 177a and 181, a circular horizontal pattern about the radiators exists in unobstructed radiation space.

For customary thin linear ½λ radiators, we have a meridian directivity

$$D_\theta = \frac{\cos(90° \cos\theta)}{\sin\theta} \qquad \text{for thin radiator}$$

$$D_\theta = J_0(k)\frac{\cos(90° \cos\theta)}{\sin\theta}; \qquad k = \frac{6.28 \sin\theta}{\lambda} r \qquad \text{for a radiator with radius } r$$

where the Bessel function of zeroth order proceeds as in Fig. 182. Hence, at places for which $k = 2.4$, 5.52, 8.65, 11.79, · · · , there can be no radiation effect at all

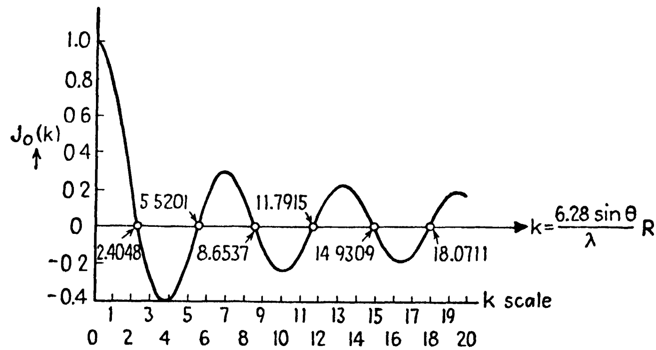


FIG. 182. Significant arguments k for vanishing Bessel factors.

The matching of shunt-fed dipoles as in Fig. 183 has been discussed at several places in literature.[1] It is assumed then that

$$\alpha l = \frac{R_r}{Z_0}\left[1 - \frac{\sin(2\beta l)}{2\beta l}\right] \quad \text{for any length of each pole of the dipole in nepers}$$

$$\alpha l = \frac{R_r}{Z_0} \qquad \text{nepers for } \frac{\lambda}{2} \text{ dipole } (l = \lambda/4 \text{ m})$$

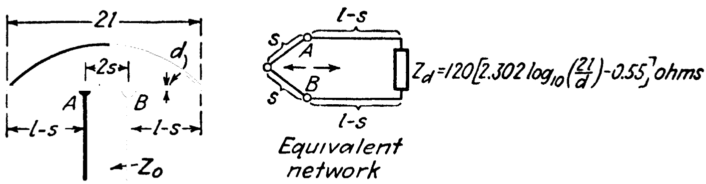
even though the attenuation α of a transmission line is, strictly taken, not affected by the radiation resistance, since radiation is a terminal effect (consult pages 491 498). By making such assumptions, due to Siegel and Labus, we have, for a horizontal half-wave dipole over a perfect reflector, the relation for the electric length βs given in Fig. 183 if we make the characteristic impedance Z_0 of the feeder equal to the computed

[1] CARTER, P. S., *Proc. IRE*, **20**, 1021, June, 1932. SIEGEL, E., and T. LABUS, *Hochfrequenz und Elektroakustik*, **43**, 166, May, 1934. KAUFMANN, H., *ibid.*, **60**, 160, December, 1942. G. GLINSKI, *Proc. IRE*, **33**, 408, June, 1945, gives also curves and error computations.

characteristic impedance Z_d of the dipole. Putting $Z_d/R_r = Z_0/73.4 = p$, we find

$$\tan(\beta s) = \frac{\sqrt{p}}{p-1} \cong \beta s \qquad \text{for values} \leqq 0.176$$

From this expression, we can compute the length s or find the ratio p if the linear radiator length $2s$ is assumed across which the feeder is connected.



For $Z_0 = Z_d$, we find $\cot(\beta s) = \frac{(Z_d^2/R_r) - Z_0}{\sqrt{(Z_d^2/R_r)\,Z_0}} = \frac{(Z_d/R_r) - 1}{\sqrt{Z_d/R_r}}$

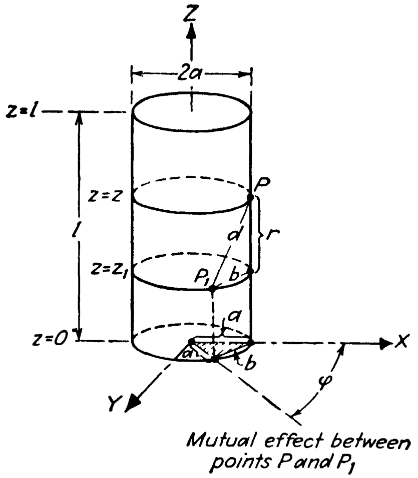
$R_r = 73.4\Omega$ and about 70Ω for reactance compensation for which l is somewhat smaller than $\lambda/4$

FIG. 183. Conditions for a matched shunt feed.

140. Essential Parts of the Theories by Oseen and Hallén, the Theory by Schelkunoff, and Expressions by King, Blake, and Harrison for Cylindrical Radiators and Radiators of Arbitrary Shape. This section has reference to procedures that also take the *transverse* dimensions of an elongated radiator into account.[1] R. King, F. G. Blake, Jr., and C. W. Harrison, Jr., used the expression derived by Hallén for the input impedance of a symmetrical cylindrical radiator for tables, curves, and special expressions for the components of the input impedance of dipoles and mutual impedances, for the case of linear radiators where the diameter of the radiator is not ignored. The Hallén integral expression is based on the assumption that the square of the ratio of the radiator length to its diameter is large compared with the natural logarithm of this ratio. The Hallén method for the entrance impedance formula is due to *suc-*

[1] SCHELKUNOFF, S. A., *Proc. IRE*, **29**, 493, September, 1941; *Jour. Appl. Phys.*, **15**, 54, January, 1944; *Proc. IRE*, **33**, 872, December, 1945. BRILLOUIN, L., *Elec. Commun.*, **22**, 11 (1944); *Quart. Appl. Math.*, **1**, 201, October, 1943. OSEEN, C. W., *Arkiv. Mat. Astron. Fysik*, **9**, Nos. 12 and 30, 1913. HALLÉN, E., *Nova Acta Regiae Soc. Sci. Upsaliensis* **11**, (4), **1**, November, 1938. BOUWKAMP, C. J., *Physica*, **9**, 609, July, 1942. GRAY, M. C., *Jour. Appl. Phys.*, **15**, 61, January, 1944. KING, R., and F. G. BLAKE, JR., *Proc. IRE*, **30**, 335, July, 1942. KING, R., and C. W. HARRISON, JR., *Jour. Appl. Phys.*, **15**, 170, February, 1944. STRATTON, J. A., and L. J. CHU, *ibid.*, **12**, 230, March, 1941. KING, R., and C. W. HARRISON, JR., *Proc. IRE*, **31**, 548, October, 1943. HARRISON, JR., C. W., *Proc. IRE*, **33**, 398, June, 1945; **34**, 204P, April, 1946. ABRAHAM, M., *Ann. Physik*, **66**, 435, 1898. MIE, G., *Ann. Physik*, 1900, p. 248. PAGE, LEIGH, and N. I. ADAMS, JR., *Phys. Rev.*, May 15, 1938, p. 819.

cessive approximations and such that the radiator *length* must be *short* compared with the wavelength. The formulas given by Marion C. Gray are *not* based on such a broad assumption as far as applications to longer radiator lengths are concerned. For this reason, her formula checks the experimental result much better at the first antiresonance. Nevertheless, the formulas due to King, Blake, and Harrison give satisfactory results for half-wave dipoles. It is such dipoles that are used at UHF's. The theory by Schelkunoff is based directly on the Maxwellian field relations and also takes care of the cross section of the dipole radiator.



$$d^2 = r^2 + 2a^2(1 - \cos\varphi)$$

$$\psi(r) = \underbrace{\frac{1}{2\pi}\int_{\varphi=0}^{\varphi=2\pi} \frac{\epsilon^{-j\beta d}}{d}\, d\varphi}$$

average of $(\epsilon^{-j\beta d})/d$ taken all around at the z_1 section when current flows and potentials at z

$$d^2 = r^2 + b^2;\; b^2 = a^2 + a^2 - 2a\,a\cos\varphi$$
$$= 2a^2(1 - \cos\varphi)$$

FIG. 184. The $\psi(r)$ function at UHF's for which we have current only on a very thin surface skin.

In the second approximation, the current distribution depends on the size of the transverse cross section as well as on the shape of the longitudinal cross section of the radiator. It applies, therefore, to wide-band dipoles. The Schelkunoff procedure is based on the conical wire as a radiator pole of a dipole (similar to the Barrow electromagnetic horns). The conditions for off-resonance can be found just as readily as the conditions for resonance. This is, therefore, a method which differs from the well-known method using a spheroid. The characteristic feature of the Schelkunoff procedure rests in the property that the relations holding for the biconical radiator can be changed to dipoles of *different* shape and applies to cases *other than cylindrical* dipoles. In the Schelkunoff method, we deal with waves along the biconical radiators or their altered shapes. This seems more in line with the thought of engineers accustomed to line equations, although such a feature seems nonessential. The end effect of the radiator is also separable from the effect of the shape in the second approximation.

The classic linear radiator theory dates back to the announcement of the Abraham theory. It is a free-oscillation theory and refers to a sinusoidal current distribution in the case of a filament radiator. For thin radiators, the radius of the radiator is negligible in comparison with its length. This subject matter is treated in Sec. 45 by means of the electric scalar potential of V volts and the magnetic vector potential of Λ volt · second per meter with the following essential results:

A thin radiator of radius a and length l in meters is oriented along the Z axis yielding, according to (254), the relations

$$V(\zeta) = \frac{1}{4\pi\kappa}\int_0^l \frac{[\rho]\,dz}{r} \text{ volts}; \qquad \Lambda(\zeta) = \frac{\mu}{4\pi}\int_0^l \frac{[I(z)]\,dz}{r} \text{ volt}\cdot\text{sec/m};$$

$$[\rho] = \rho\epsilon^{-j\beta r} \qquad [I(z)] = I\epsilon^{-j\beta r}$$

According to (255), for the distance $r = |z - z_1|$ meters between any two points located at z_1 and z along the filament radiator (Fig. 151), the respective *retarded*[1] potentials at any location z are

$$V(z) = \frac{1}{4\pi\kappa}\int_0^l \frac{[\rho(z_1)]}{r}\,dz_1 = j\frac{\epsilon^{j\omega t}}{4\pi\omega\kappa}\int_0^l \frac{\partial I(z_1)}{\partial z_1}\frac{\epsilon^{-j\beta r}}{r}\,dz_1 \qquad \text{volts}$$

$$\Lambda(z) = \frac{\mu}{4\pi}\epsilon^{j\omega t}\int_0^l \frac{I(z_1)\epsilon^{-j\beta r}}{r}\,dz_1 \qquad \text{volt sec/m}$$

$$\mathcal{E} = -\text{grad } V - \frac{\partial\Lambda}{\partial t}; \qquad \mathcal{E}(z) = -\frac{\partial V}{\partial z} - \frac{\partial\Lambda}{\partial t} \qquad \text{volts/m}$$

where the *magnetic* vector potential Λ is entirely along the Z direction, since the thin linear radiator is along it. These results cause serious confusion if examined for certain limits, since we assume a very thin wire, which would cause, in the function $F = (\epsilon^{-j\beta r})/r$, a very large and unreasonable value near the surface of the thin radiator. For a negligible radius $a = 0$, we should actually obtain an infinite value that is unreasonable. L. Brillouin[2] replaces this "confusing" function F by a function $\psi_\omega(r)$, which acts like the F function for large values of the distance r but causes only a finite, very large value for zero distance, which is $r = 0$. Since the function ψ depends upon $|z - z_1|$, we also have $\partial\psi/\partial z = -\partial\psi/\partial z_1$ and find

$$V(z) = j\frac{\epsilon^{j\omega t}}{4\pi\omega\kappa}\int_0^l \frac{\partial I}{\partial z_1}\psi(r)\,dz_1; \qquad \Lambda(z) = \frac{\mu\epsilon^{j\omega t}}{4\pi}\int_0^l \frac{\partial I}{\partial z_1}\psi(r)\,dz_1$$

$$\mathcal{E}(z) = -\frac{\partial V}{\partial z} - \frac{\partial\Lambda}{\partial t} = -j\frac{\epsilon^{j\omega t}}{4\pi\omega\kappa}\int_0^l \left[\frac{\partial I}{\partial z_1}\frac{\partial\psi}{\partial z} + \beta^2 I(z_1)\psi(r)\right]dz_1 \qquad \text{volts/m}$$

[1] It is essential that we work with retarded values, unlike in customary field theory dealing with low-frequency currents, where we can use conditions similar to the ones of the electrostatic field. This means that the instantaneous electromagnetic field cannot be applied even for points next to the radiator or along it. This is due to the requirement that, for instance, the field at point z_1, when somewhere near the upper end of a half-wave radiator, *must* be computed with respect to the field that existed at z_1 half of a period sooner if z is taken near the lower end.

[2] See p. 235; consult also *Elec. Commun.*, **14**, 15 (1944).

If we *integrate* the ψ function *by parts*, we find

$$\mathcal{E}(z) = -j\frac{\epsilon^{j\omega t}}{4\pi\omega\kappa}\left\{-\left|\psi(r)\frac{\partial I}{\partial z_1}\right|_{z_1=0}^{z_1=l} + \underbrace{\int_0^l\left[\frac{\partial^2 I}{\partial z_1^2} + \beta^2 I(z_1)\right]\psi(r)\,dz_1}_{\text{vanishes for resonance lengths of } l}\right\} \quad \text{volts/m} \qquad (854)$$

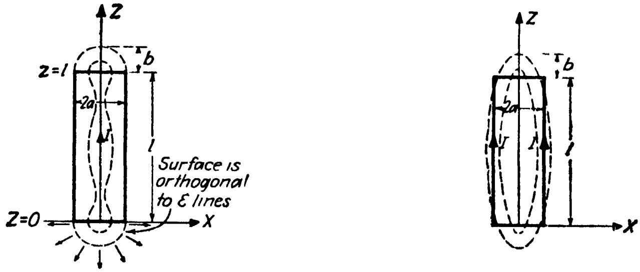
It is this outcome which is of *great importance*. If we use a sinusoidal current distribution ($I(z)$ distribution), as in Sec. 46, for a very thin linear radiator, we have for the qth harmonic excitation, $\beta = \pi q/l = 2\pi/\lambda$ and $I(z_1) = I \sin(\pi q z_1/l)$, which gives the tangential radiator field

$$\mathcal{E}_t = j\frac{I\epsilon^{j\omega t}}{4\pi\omega\kappa}\frac{\pi q}{l}[(-1)^q\psi(l-z) - \psi(z)] \qquad \text{volts/m}$$

This then gives the result used so far in connection with linear radiators, where we always take for granted a sinusoidal current distribution, and (a/l) exceedingly small. That the integral in (854) happens to cancel out for resonance does not mean that it is not important. We must realize that this integral involves the $\psi(r)$ function, which generally accounts for a much larger value. If we do not limit ourselves to a sinusoidal current function along the length l of the radiator, we have for $I(z)$ any current distribution that is to be found from the boundary condition at the surface of the radiator. Doing this, we shall find again the relation given in (854). It is, therefore, necessary that the tangential field $\mathcal{E}_t = \mathcal{E}(z)$ along the cylindrical radiator must have an external driving field $[-\mathcal{E}(z)]$ superposed on it, in order to satisfy zero electric field requirement along a perfect conductor.

Figure 184 shows how to find the $\psi(r)$ function at VHF's and UHF's, since the current exists only at the very surface of the cylindrical radiator. This simplifies matters greatly. It is also of importance, for solid cylindrical conductors, to consider the effect of the end faces. We find then the ψ function logarithmically infinite for $r = 0$, but nevertheless $\int_0^{r_0} \log(r)\,dr = r_0(\log r_0 - 1)$ is *finite*.

Figure 185 illustrates the difference between the assumptions made by Oseen and by Hallén for the field distribution model. The procedures of Oseen and Hallén are essentially the same. The indicated viewpoints of Oseen and Hallén are due to using a different ψ function definition in order to include the effects of the end surfaces also. The ψ function discussed so far takes care of the finite radius a of the linear radiator. The expression for $\psi(r)$ in Fig. 184 satisfies the requirement that for perfectly conducting radiators, the currents as well as the charges are on the surface of the radiator at z_1 for VHF's and UHF's. However, we need the field at another point z on the surface of the radiator, satisfying the boundary condition that there can be no tangential electric field com-



Oseen substitutes actual surface currents by a fictitious axial current I, and takes the charges as well as the current along the cylinder *axis* at z_1, while the field is taken at the surface of radius a at z

Hallén takes actual currents I only along the cylinder surface of radius a, and calculates the *field* on the axis (radius $a = 0$) instead of surface (radius a) of the cylinder. Currents and charges on the surface of radius a at z_1, and field is computed along the axis ($a = 0$) at z.

FIG. 185. Field distribution and shape of radiator (Oseen and Hallén models).

ponent along a perfect conductor. To meet this *intricate* condition, both Oseen and Hallén *assume* that we have a condition for ψ and a condition for I as in

$$\psi = \frac{\epsilon^{-j\beta r_1}}{r_1}; \qquad r_1 = \sqrt{(z - z_1)^2 + a^2} = \sqrt{r^2 + a^2}; \qquad I = 0$$

for $z = 0$ and $z = l$

where $z = 0$ and $z = l$ refer to the free ends of the radiator. It so happens that the error due to the I condition partially offsets the error due to the ψ condition.

As to details, in the Oseen model, the metallic surface is a cylinder with rounded caps. We have, therefore, a cylinder of radius a and a metallic body that is $l + 2b$ long, where b is of the same order of magnitude as the radius a. The rounded-end caps depend, however, on the mode of oscillation. The Hallén model resembles a long ellipsoid and is likewise $l + 2b$ long. The shapes are found from the field distribution. The Hallén method applies only to solid cylindrical radiators with rounded-end caps. The success of the Hallén procedure depends on the treatment of the above-mentioned integral relation. Schelkunoff[1] has shown that with careful approximations acceptable results are possible. For thick cylinders, Oseen's model should be easier to apply. This subject matter is treated in detail by L. Brillouin,[2] who has shown that with Oseen's and Hallén's viewpoints, we must solve the integral equation

$$\int_0^l \left[\frac{\partial^2 I}{\partial z_1^2} + \beta^2 I(z_1)\right]\psi(r)\,dz_1 - \left(\frac{\partial I}{\partial z_1}\right)_{z_1=l}\psi(l-z) + \left(\frac{\partial I}{\partial z_1}\right)_{z_1=0}\psi(z) = -j4\pi\omega\kappa\mathcal{E}_1(z)$$

where $\mathcal{E}_1$ is the driving electric field that is applied to the radiator. The method of *successive approximations* may then give the desired solution. The procedures used by Hallén and King are along another path. They are based on the behavior of the

[1] *Proc. IRE*, **33**, 872, December, 1945.

[2] See Fig. 185; consult also *Elec. Commun.*, **22**, 11 (1944).

scalar potential of V volts and the magnetic vector potential of Λ_z volt · seconds per meter on the surface of a cylindrical radiator. For the cylinder coordinates ρ, φ, z, we have then

$$\underbrace{\frac{\partial}{\partial\rho}\left(\frac{\rho\partial\Lambda_z}{\partial\rho}\right)}_{\substack{\text{for } \rho \text{ nearly } a,\\ \text{we have also}\\ \text{this term} = 0}} + \underbrace{\frac{1}{\rho}\frac{\partial^2\Lambda_z}{\partial\varphi^2}}_{\substack{=0\\ \text{for sym-}\\ \text{metrical}\\ \text{solutions}}} + \underbrace{\rho\frac{\partial^2\Lambda_z}{\partial z^2} + \beta^2\rho\Lambda_z}_{\substack{\text{remains along the}\\ \text{cylindrical surface}\\ (\rho = a)}} = 0;$$

$$\underbrace{\frac{\partial^2\Lambda_z}{\partial z^2} + \beta^2\Lambda_z = 0;}_{\substack{\text{hence, vector}\\ \text{potential is}\\ \text{a harmonic}\\ \text{function}}} \qquad \frac{\partial\Lambda_z}{\partial z} = -j\omega\mu\kappa V \qquad \text{since div } \Lambda = -\mu\kappa\frac{\partial V}{\partial t}$$

which means that the *potential distribution V is* also *sinusoidal* while the *current distribution is* not *sinusoidal*. This is due to the Oseen model and applies also to the Hallén model. We have, therefore,

$$\Lambda_z = \frac{\mu}{4\pi}\int I(z_1)\psi(r)\,dz_1 \qquad \textit{first fundamental relation} \text{ used by Hallén and King}$$

a sinusoidal function. Hence, for a driven dipole with the origin at the dipole gap, the integration extends from $-\frac{1}{2}l$ to $+\frac{1}{2}l$, and

$$\Lambda_z = -j\sqrt{\mu\kappa}\,[\underbrace{C_1\cos(\beta z)}_{\substack{\text{is continuous}\\ \text{at } z = 0}} + \tfrac{1}{2}V_0\sin(\beta|z|)] \qquad \textit{second fundamental relation}, \text{ used by Hallén and King}$$

for the impressed voltage of V_0 volts, which drives the current into the dipole gap. The constant C_1 is found from the condition $I(z) = 0$ for $z = \pm\frac{1}{2}l$. The sine term is, however, discontinuous on $\partial\Lambda_z/\partial z$, since

$$\frac{\partial\Lambda_z}{\partial z_{+0}} - \frac{\partial\Lambda_z}{\partial z_{-0}} = -j\omega\mu\kappa V_0$$

this *discontinuity accounts for* the *impressed* dipole *voltage* at $z = -0$ and $z = +0$. Inasmuch as the corresponding current distribution $I(z)$ is *not* known, we have to solve for it. We find then

$$\left.\begin{aligned}
\int_{z_1=-\frac{1}{2}l}^{z_1=\frac{1}{2}l} I(z_1)\psi(r)\,dz_1 &= \int_{-l/2}^{l/2} I(z_1)\frac{\epsilon^{-j\beta r_1}}{r_1}\,dz_1\\
&= I(z)\int_{-l/2}^{l/2}\frac{dz_1}{r_1} + \int_{-l/2}^{l/2}\frac{I(z_1)\epsilon^{-j\beta r_1} - I(z)}{r_1}\,dz_1;\\
r_1 = \sqrt{(z-z_1)^2 + a^2};&\qquad \text{for } \Omega = 2\log_\epsilon\left(\frac{l}{a}\right)\\
\text{and}\qquad&\\
\Delta = \log_\epsilon\left\{\frac{1}{4}\left[\sqrt{1+\left(\frac{a}{0.5l-z}\right)^2}+1\right]\right.&\left.\left[\sqrt{1+\left(\frac{a}{0.5l+z}\right)^2}+1\right]\right\}
\end{aligned}\right\} \qquad (855)$$

we find

$$\left.\begin{aligned}\int_{-l/2}^{l/2}\frac{dz_1}{r_1} &= \log_\epsilon\left[\frac{\sqrt{\left(\frac{l}{2}-z\right)^2+a^2}+\frac{l}{2}-z}{\sqrt{\left(\frac{l}{2}+z\right)^2+a^2}-\frac{l}{2}-z}\right]\\ &= \Omega+\log_\epsilon\left(1-\frac{z^2}{0.25l^2}\right)+\Delta\end{aligned}\right\} \tag{855}$$

where the term Δ is finite near either open end of the dipole radiator, but infinitesimal everywhere else. We find then

$$I(z) = j\underbrace{\frac{4\pi}{\Omega Z_0}\left[C_1\cos(\beta z)+\frac{V_0}{2}\sin(\beta|z|)\right]}_{=A} - \frac{1}{\Omega}\left[I(z)\log_\epsilon\left(1-\frac{z^2}{l^2/4}\right)+I(z)\Delta\right.$$
$$\left.+\int_{-l/2}^{l/2}\frac{I(z_1)\epsilon^{-j\beta r_1}-I(|z|)}{r_1}\,dz_1\right],\qquad Z_0 = 376.7\text{ ohms} \tag{855a}$$

The relations used by R. King, F. G. Blake, and C. W. Harrison[1] are therefore, based on the conditions that the potential distribution is sinusoidal and the second fundamental relation, besides the assumptions by Oseen and Hallén for the ψ and I requirements. Therefore, with Oseen's viewpoint, we compute the fictitious *axial* current (Fig. 185), while the retarded electric scalar and magnetic vector potentials are on the cylindrical *surface* of the radiator. For the Hallén viewpoint, the current is taken on the radiator (cylindrical) *surface* and the potential difference along the *axis* of the cylindrical radiator. Therefore, in either model, *i.e.*, for either viewpoint, there is an inconsistency as regards the assumptions. We deal with V and I values, which are *not at* the *same point*, and V/I cannot be the impedance with respect to one and the same terminals. We have a condition as in filters, where we sometimes speak of a driving transfer impedance and mean that the voltage across the input terminals is divided by the current flowing into the load from the output terminals.

The *practical application* is then based on solving integral relations by the method of successive approximation, involving a quantity

$$\Omega = 4.606\log_{10}\left(\frac{l}{a}\right) \tag{856}$$

for which $1/\Omega$ is *small compared with unity*, l is the half length of a center-fed symmetrical linear radiator, and a is its radius. Such are also the assumptions for Abraham's radiator formula for the case of a *prolate spheroid*. Assuming that $1/\Omega = 0.1$ is small enough compared with

[1] Page 705.

unity, we have $\Omega = 10 = 4.604 \log_{10}(l/a)$, *i.e.*, $\log_{10}(l/a) = 2.17$. This gives a ratio $l/a \cong 148$, which result *corresponds to a customary linear radiator*. We have then the formula for a dipole radiator for the self-impedance

$$
\left.
\begin{aligned}
Z_{00}^{(\text{complex ohms})} &= -j60\,\Omega\,\frac{\cos\left(\frac{\beta l}{2}\right) + \frac{\alpha_1}{\Omega} + \frac{\alpha_2}{\Omega^2} + \cdots}{\sin\left(\frac{\beta l}{2}\right) + \frac{\beta_1}{\Omega} + \frac{\beta_2}{\Omega^2} + \cdots} \\
&= \underbrace{R_{00}}_{\substack{\text{self-radia-}\\\text{tion resist-}\\\text{ance}}} + \underbrace{jX_{00}}_{\substack{\text{associated}\\\text{reactance}}}; \qquad \Omega = 4.606 \log_{10}\left(\frac{l}{a}\right); \\
\alpha_1 &= \alpha_1^{\text{I}} + j\alpha_1^{\text{II}} + j\frac{l}{2} r_\alpha \frac{r_\iota + jx_\iota}{30} = A_1^{\text{I}} + jA_1^{\text{II}}; \\
\beta_1 &= \beta_1^{\text{I}} + j\beta_1^{\text{II}} + j\frac{l}{2} r_\beta \frac{r_\iota + jx_\iota}{30} = B_1^{\text{I}} + jB_1^{\text{II}} \\
\alpha_1^{\text{I}} &= \frac{1}{2}\left\{\cos\left(\frac{\beta l}{2}\right)[\overline{\text{Ci}}\,(2\beta l) - 2\,\overline{\text{Ci}}\,(\beta l)] - \sin\left(\frac{\beta l}{2}\right)\text{Si}\,(2\beta l)\right\} \\
\alpha_1^{\text{II}} &= \frac{1}{2}\left\{\cos\left(\frac{\beta l}{2}\right)[\text{Si}\,(2\beta l) - 2\,\text{Si}\,(\beta l)] + \sin\left(\frac{\beta l}{2}\right)\overline{\text{Ci}}\,(2\beta l)\right\} \\
\beta_1^{\text{I}} &= \frac{1}{2}\left\{\cos\left(\frac{\beta l}{2}\right)[4\,\text{Si}\,(\beta l) - \text{Si}\,(2\beta l)] + \sin\left(\frac{\beta l}{2}\right)[2\,\overline{\text{Ci}}\,(\beta l) - \overline{\text{Ci}}\,(2\beta l) + 4\log_e(2)]\right\} \\
\beta_1^{\text{II}} &= \frac{1}{2}\left\{\cos\left(\frac{\beta l}{2}\right)[\overline{\text{Ci}}\,(2\beta l) - 4\,\overline{\text{Ci}}\,(\beta l)] + \sin\left(\frac{\beta l}{2}\right)[2\,\text{Si}\,(\beta l) - \text{Si}\,(2\beta l)]\right\} \\
r_\alpha &= \frac{1}{2}\sin\left(\frac{\beta l}{2}\right) - \cos\left(\frac{\beta l}{2}\right)\frac{1 - \cos(\beta l/2)}{\beta l/2}; \\
r_\beta &= \frac{\sin(\beta l) - \sin(\beta l/2)}{\beta l} - \frac{1}{2}\cos\left(\frac{\beta l}{2}\right); \\
\text{Si}\,x &= \int_0^x \frac{\sin u}{u}\,du; \qquad \text{Ci}\,x = \int_0^x \frac{\cos u}{u}\,du; \\
\overline{\text{Ci}}\,(x) &= \int_0^x \frac{1 - \cos u}{u}\,du = 0.5772 + 2.303\log_{10}(x) - \text{Ci}\,(x)
\end{aligned}
\right\} \quad (856a)
$$

These are the expressions as given by R. King and F. G. Blake, Jr.,[1] including a good table and many useful curves. The coefficients α_1, α_2, β_1, and β_2 are the functions computed from Hallén's theory. The assumptions made in the classic theory for linear radiators are that a sinusoidal current distribution prevails. This is of use only if Ω is so large that all terms in α_1/Ω, α_2/Ω^2, β_1/Ω, β_2/Ω^2 in (856a) can be neglected.

The finite conductivity of 5.8×10^7 mhos across the square-meter faces of a meter cube separated by 1 m, in the case of copper, does not play an important part if compared with perfect conductivity. For a half-wave dipole, we have

$$\frac{\beta l}{2} = \frac{2\pi}{2\lambda} l = \frac{2\pi}{2\lambda}\frac{\lambda}{2} = 1.57; \qquad \alpha_1^{\text{I}} = -0.70908; \qquad \alpha_1^{\text{II}} = +1.21883$$

$$\beta_1^{\text{I}} = +1.81571; \qquad \beta_1^{\text{II}} = +1.14279$$

$$r_\alpha = +0.5; \qquad r_\beta = -0.31831$$

In (856a), the quantity $\overline{\text{Ci}}\,x$ denotes the *even* integral cosine function, which is easier to apply. The quantities r_i and x_i represent the *internal* ohmic resistance and reactance, respectively, of the cylindrical radiator per meter length. They are computed from well-known handbook formulas. If $1/\Omega$ is such a small quantity that

$$\left|\cos\left(\frac{\beta l}{2}\right) + \frac{\alpha_1}{\Omega}\right| \gg \left|\frac{\alpha_2}{\Omega^2}\right|; \qquad \left|\sin\left(\frac{\beta l}{2}\right) + \frac{\beta_1}{\Omega}\right| \gg \left|\frac{\beta_2}{\Omega^2}\right|$$

we have

$$Z_{00} = R_{00} + jX_{00} = -j60\Omega \frac{\cos\left(\frac{\beta l}{2}\right) + \frac{\alpha_1}{\Omega}}{\sin\left(\frac{\beta l}{2}\right) + \frac{\beta_1}{\Omega}}$$

$$= \underbrace{60\Omega \frac{A_1^{\text{II}}\left[\Omega\sin\left(\frac{\beta l}{2}\right) + B_1^{\text{I}}\right] - B_1^{\text{II}}\left[\Omega\cos\left(\frac{\beta l}{2}\right) + A_1^{\text{I}}\right]}{\left[\Omega\sin\left(\frac{\beta l}{2}\right) + B_1^{\text{I}}\right]^2 + (B_1^{\text{II}})^2}}_{= R_{00}\ \text{ohms}}$$

$$\underbrace{-\, j60\Omega \frac{\left[\Omega\cos\left(\frac{\beta l}{2}\right) + A_1^{\text{I}}\right]\left[\Omega\sin\left(\frac{\beta l}{2}\right) + B_1^{\text{I}}\right] + A_1^{\text{II}}B_1^{\text{II}}}{\left[\Omega\sin\left(\frac{\beta l}{2}\right) + B_1^{\text{I}}\right]^2 + (B_1^{\text{II}})^2}}_{-X_{00}\ \text{reactive ohms}} \qquad (856b)$$

If we restrict the general expression of Z_{00} of (856a) still more so that

$$\left|\Omega\sin\left(\frac{\beta l}{2}\right)\right| \gg \beta_1; \qquad \left|\Omega\cos\left(\frac{\beta l}{2}\right)\right| \gg \alpha_1$$

[1] King, R., and F. G. Blake, Jr., *Proc. IRE*, **30**, July, 1942.

we find

$$Z_{00} = R_{00} + jX_{00} = -j60\left[\Omega \cot\left(\frac{\beta l}{2}\right) + \frac{\alpha_1}{\sin(\beta l/2)} - \frac{\beta_1 \cos(\beta l/2)}{\sin^2(\beta l/2)}\right]$$

$$R_{00} = 60\,\frac{A_1^{\mathrm{II}} - B_1^{\mathrm{II}} \cot\left(\frac{\beta l}{2}\right)}{\sin\left(\frac{\beta l}{2}\right)} \quad \text{ohms;}$$

$$X_{00} = -60\left[\Omega \cot\left(\frac{\beta l}{2}\right) + \frac{A_1^{\mathrm{I}}}{\sin\left(\frac{\beta l}{2}\right)} - \frac{B_1^{\mathrm{I}} \cot\left(\frac{\beta l}{2}\right)}{\sin\left(\frac{\beta l}{2}\right)}\right] \quad \text{reactive ohms} \qquad (856c)$$

If R_s and X_s are the *external* resistive and reactive components of the total self-impedance Z_{00} and R_i and X_i the internal ohmic input resistance and input reactance, we find

$$\left.\begin{aligned}
R_{00} &= R_s + R_i, \qquad X_{00} = X_s + X_i\\
R_s &= 30\left\{\left[1 - \cot^2\left(\frac{\beta l}{2}\right)\right]\overline{\mathrm{Ci}}\,(2\beta l) + 4\cot^2\left(\frac{\beta l}{2}\right)\overline{\mathrm{Ci}}\,(\beta l)\right.\\
&\qquad\left. + 2\cot\left(\frac{\beta l}{2}\right)[\mathrm{Si}\,(2\beta l) - 2\,\mathrm{Si}\,(\beta l)]\right\}\\
X_s &= 30\left\{\left[1 - \cot^2\left(\frac{\beta l}{2}\right)\right]\mathrm{Si}\,(2\beta l) + 4\cot^2\left(\frac{\beta l}{2}\right)\mathrm{Si}\,(\beta l)\right.\\
&\qquad\left. + 2\cot\left(\frac{\beta l}{2}\right)\left[2\,\overline{\mathrm{Ci}}\,(\beta l) - \overline{\mathrm{Ci}}\,(2\beta l) - 2\log_e\left(\frac{l}{2a}\right)\right]\right\}\\
R_i &= \frac{r_i}{\beta}\underbrace{\left[\frac{\beta l}{2}\csc^2\left(\frac{\beta l}{2}\right) - \cot\left(\frac{\beta l}{2}\right)\right]}_{=M}; \qquad X_i = \frac{x_i}{\beta}M
\end{aligned}\right\} \qquad (856d)$$

The last restriction, leading to (856c) and (856d), is essentially *independent of the radius*. If we apply these results to a half-wave dipole, for which $l = \frac{1}{2}\lambda$ and $\frac{1}{2}\beta l = \pi/2$, so that $\cot(\frac{1}{2}\beta l) = \cot^2(\frac{1}{2}\beta l) = 0$, we find

$$R_s = 30\,\overline{\mathrm{Ci}}\,(2\beta l) = 30[0.5772 + 2.303\log_{10}(2\beta l) - \mathrm{Ci}\,(2\beta l)] = 73.4 \text{ resistive ohms}$$

$$X_s = 30\,\mathrm{Si}\,(2\beta l) = 42.9 \text{ reactive ohms}$$

This is exactly the result found for a thin linear radiator, since we arrive at the formulas given in Fig. 143*a*, where $R_{11} = R_s$ and $X_{11} = X_s$. The formulas of Fig. 143*a* are valid over the entire range, *i.e.*, for any value in terms of the wavelength, since a sinusoidal current distribution is assumed. This is not the case for (856*d*), since this is only an agreement

for $l = \lambda/2$ and would lead to favorable comparisons only near such a current distribution. The second restriction leading to (856*c*) and (856*d*) is not what is used in the curves by King and his coworkers. It is the formula (856*b*) that retains the Hallén functions α_1 and β_1, *i.e.*, the terms α_1/Ω and β_1/Ω, in addition to the cos $(\beta l/2)$ term in the numerator and

TABLE 32*

$\frac{\beta l}{2}$	$\frac{a}{\lambda} = 10^{-5}$		$\frac{a}{\lambda} = 3 \times 10^{-5}$		$\frac{a}{\lambda} = 10^{-4}$		$\frac{a}{\lambda} = 3 \times 10^{-4}$		$\frac{a}{\lambda} = 10^{-3}$	
	R_s, ohms	X_s, ohms	R_s, ohms	X_s, ohms	R_s, ohms	X_s, ohms	R_s, ohms	X_s, ohms	R_s, ohms	X_s, ohms
0.5	5.02	−1,800	5.00	−1,560	4.95	−1,310	4.9	−1,080	4.81	−825
1.0	22.2	−668	22.1	−585	21.9	495	21.7	−413	21.4	−324
1.5	60.7	−46.2	60.3	−37.5	59.8	−28.2	59.2	−20	58.2	−11.3
2.0	157	589	156	527	155	459	153	397	150	327
2.5	554	1,720	550	1,530	541	1,320	535	1,120	520	900
2.6	791	2,130	785	1,890	774	1,620	758	1,360	729	1,070
2.7	1,210	2,680	1,200	2,360	1,170	1,990	1,130	1,640	1,070	1,250
2.8	2,030	3,410	1,990	2,940	1,920	2,400	1,820	1,900	1,650	1,340
2.9	3,880	4,140	3,680	3,370	3,380	2,530	3,010	1,790	2,500	1,050
3.0	7,630	2,980	6,600	1,960	5,400	1,010	4,290	407	3,130	−48.6
3.1	8,250	−2,860	6,600	−2,500	5,020	−2,090	3,780	−1,710	2,640	−1,310
3.2	4,150	−4,770	3,510	−3,910	2,840	−3,060	2,270	−2,370	1,690	−1,710
3.3	1,970	−3,960	1,760	−3,340	1,510	−2,700	1,280	−2,150	1,020	−1,600
3.4	1,060	−3,100	974	−2,670	869	−2,200	765	−1,800	640	−1,370
3.5	637	−2,470	595	−2,140	543	−1,790	489	−1,190	422	−1,160
4.0	126	−943	122	−833	117	−713	112	−604	104	487
$\frac{\pi}{2} = 1.57$	69.2	35.9	68.8	35.2	68.2	34.3	67.4	33.3	66.4	31.8
$\pi = 3.14$	6,480	−4,360	5,260	−3,570	4,080	−2,800	3,130	−2,170	2,240	−1,580

* a is the radius and l the length of the center-driven linear radiator.

Remarks The self-resistance R_s and the self-reactance X_s due to the external radiation and reactive fields are in ohms. If we include also the internal resistance R_i and internal reactance X_i, due to the finite conductivity $\sigma = 5.8 \times 10^7$ mhos/m, in the case of a copper radiator, the effects for a half-wave dipole, $(\beta l/2) = 1.57$, are such that $R_{00} = 70.9$ ohms for $a = 10^{-5}\lambda$ and 66.4 ohms for $a = 10^{-3}\lambda$, while we have $X_{00} = 37.5$ and 31.9 reactive ohms for such radii. Therefore, $R_i = R_{00} - R_s = 70.9 - 69.2 = 1.7$ ohms *only* for $a = 10^{-5}\lambda$, and $X_i = X_{00} - X_s = 37.5 - 35.9 = 1.6$ reactive ohms *only*. The effect is, therefore, for a half-wave dipole smaller than could be detected by a measurement at VHF's and UHF's. Maximum radiation resistance is obtained for $l \cong \lambda$ or $\beta l/2 \cong \pi$, with a calculated value of 6,820 as against 4,840 ohms (experimental).

the sin $(\beta l/2)$ term in the denominator of the complex Z_{00} expression. This *assumes* that $1/\Omega$ must be so *small* compared with unity that terms of higher inverse power, such as $1/\Omega^2$, cannot play an important part in the final result. The papers by King, Blake, and Harrison[1] contain so many good curves and tables that reference is made to them. If we ignore the internal resistance R_i and the internal reactance X_i, we have, for instance, values for the self-resistance R_s and self-reactance X_s that are both external ohms for different values of the cylindric radiator

[1] *Proc. IRE*, **30**, 335, July, 1942.

radius a in terms of the wavelength λ as in Table 32. This table also shows, for instance, that for $a = 10^{-5}\lambda$, we have, according to (856), for a half-wave dipole,

$$\Omega = 4.606 \log_{10}\left(\frac{l}{a}\right) = 4.606 \log_{10}\left(\frac{\lambda/2}{10^{-5}\lambda}\right) = 4.606 \log_{10}(5 \times 10^4)$$
$$= 4.606[\log_{10}(5) + \log_{10}(10^4)] = 4.606(0.699 + 4) = 21.6$$

rendering $1/\Omega = 0.0463$, which satisfies the assumption for formula (856b) and the values of Table 32 that $1/\Omega$ is small compared with unity, being only 4.63 per cent of it. We deal, however, with a cylindrical half-wave radiator that is fed at the mid-gap, for which the total length is such that $a = 10^{-5}\lambda = 2 \times 10^{-5}l$. The diameter of the radiator is, therefore, only 4×10^{-5} the total dipole length, which is very small compared with the elongation. According to Table 32,

$$\left.\begin{array}{ll} Z_s = R_s + jX_s = 69.2 + j35.9 & \left\{\begin{array}{l}\text{based on the Hallén}\\ \text{distribution (Fig. 185)}\\ \text{and } a = 10^{-5}\lambda\end{array}\right. \\ Z_s = R_s + jX_s = 73.4 + j42.9 & \left\{\begin{array}{l}\text{for a sinusoidal current}\\ \text{distribution and a thin}\\ \frac{\lambda}{2}\text{ dipole}\end{array}\right. \\ Z_s = R_s + jX_s = 66.4 + j31.8 & \left\{\begin{array}{l}\text{based on the Hallén}\\ \text{distribution and } a =\\ 10^{-3}\lambda\end{array}\right. \end{array}\right\} \text{complex ohms}$$

The condition for the radius $a = 10^{-3}\lambda$, for $l = \frac{1}{2}\lambda$, yields

$$\Omega = 4.606 \log_{10}(5 \times 10^2) = 4.606[\log_{10}(5) + \log_{10}(10^2)]$$
$$= 4.606(0.699 + 2) = 12.4$$
$$\frac{1}{\Omega} = 0.0806; \qquad a = 10^{-3}\lambda = 2 \times 10^{-3}l$$

Therefore, the self-impedance of $73.4 + j42.9$ ohms is too high for the self- or radiation resistance R_s and the self-reactance X_s unless the radius a is negligible. The difference is, however, not so serious in the above illustrations, since, for instance, both values 69.2 and 66.4 ohms are close enough to 70 ohms as is also 73.4 ohms for a sinusoidal current distribution and a fine conductor for the half-wave dipole. This difference is not great as far as measurements are concerned. The reactances are not a serious matter, since they represent a wattless equivalent field and are neutralized by stubbing, or making the half-wave dipole somewhat shorter. Experimental results show that especially for frequencies near antiresonance, the above relations do not give satisfactory results.

Nevertheless, it must be admitted that we have made quite a step forward in treating radiators for which the diameter can be taken into account. The results for a half-wave dipole are close to experimental values.

As far as other details of (855*a*) are concerned, the functions $p_0(z) = \cos(\beta z)$ and $q_0(z) = \sin(\beta|z|)$ are used. If $I(z_0)$ represents a zero-order approximation for the expansion in powers for $1/\Omega$, we have for small values of $1/\Omega$ and $I(z_0)$ applied to the portion A of (855*a*),

$$I(z_0) = -j\frac{4\pi}{\Omega Z_0}\left[C_1p_0(z) + \frac{V_0}{2}q_0(z)\right];$$

$$I(z) = I(z_0) + \frac{I(z_1)}{\Omega}\cdots = -j\frac{4\pi}{\Omega Z_0}\left[C_1\left(p_0 + \frac{p_1}{\Omega}\cdots\right) + \frac{V_0}{2}\left(q_0 + \frac{q_1}{\Omega}\cdots\right)\right]$$

Using this in (855*a*), we find

$$p_1(z) = -p_0(z)\log_\epsilon\left(1 - \frac{z^2}{l^2/4}\right) - p_0(z)\Delta - \int_{-l/2}^{l/2}\frac{p_0(z_1)\epsilon^{-j\beta r_1} - p_0(z)}{r_1}\,dz_1$$

$$q_1(z) = -q_0(z)\log_\epsilon\left(1 - \frac{4z^2}{l^2}\right) - q_0(z)\Delta - \int_{-l/2}^{l/2}\frac{q_0(z_1)\epsilon^{-j\beta r_1} - q_0(z)}{r_1}\,dz_1$$

We find the constant C_1 from the condition that at $z = \frac{1}{2}l$ (the free end of the radiator), we have $I(z) = 0$. Hence,

$$C_1 = \tfrac{1}{2}V_0\frac{q_0(l/2) + (1/\Omega)q_1(l/2)}{p_0(l/2) + (1/\Omega)p_1(l/2)}; \qquad I(z) = j\frac{2\pi V_0}{\Omega Z_0}\frac{\sin[\beta(l/2 - |z|)] - (F/\Omega)}{\cos(\beta l/2) + G/\Omega}$$

$$F = F_1 + jF_2 = p_1(z)\sin\left(\frac{\beta l}{2}\right) - p_1\left(\frac{l}{2}\right)\sin(\beta|z|) + q_1\left(\frac{l}{2}\right)\cos(\beta z) - q_1(z)\cos\left(\frac{\beta l}{2}\right)$$

$$G = G_1 + jG_2 = p_1\left(\frac{l}{2}\right)$$

Then we have only to compute the values of $p_1(z)$ and $q_1(z)$ by means of the p_0 and q_0 functions. To do this, we neglect Δ and use, instead of $(1/r_1)\epsilon^{-j\beta r_1}$, the expression

$$\frac{\epsilon^{-j\beta|z-z_1|}}{|z - z_1|}, \qquad r_1 = \sqrt{(z - z_1)^2 + a^2}$$

This is the procedure of Hallén and King where, by means of the above-mentioned approximations, the integrals are expressed by well-known sine integrals and cosine integrals. For the sake of convenience, King and Blake use the *even* integral function $\overline{\text{Ci}}\,(x)$ in (856*a*). This function is then the Euler constant 0.5772 plus the natural logarithm of the argument x minus the Ci (x) value, as found in tables or on curves for the argument x.

Schelkunoff[1] has given a critical review of Hallén's integral equation for cylindrical radiators. This solution refers only to one integral and is given in (855). Proceeding the same way with the rigorous double-integral relation, in connection with Fig. 186, we have

[1] *Proc. IRE*, **33**, 872, December, 1945.

$$\int_{\varphi=0}^{\varphi=2\pi}\int_{z_1=-l/2}^{z_1=l/2}\frac{I(z_1)\epsilon^{-j\beta r_1}}{2\pi r_1}\,dz_1\,d\varphi = I(z)\int_0^{2\pi}\int_{-l/2}^{l/2}\frac{dz_1\,d\varphi}{2\pi r_1} + \int_0^{2\pi}\int_{-l/2}^{l/2}\frac{I(z_1)\epsilon^{-j\beta r_1}-I(z)}{2\pi r_1}\,dz_1\,d\varphi$$

where we have again such a resolution of the double integral that we can apply successive approximations. If the radius a is small and z_1 close to $z_1 = z$, we must also have r_1 small. Therefore, as a approaches zero, the first term on the right must

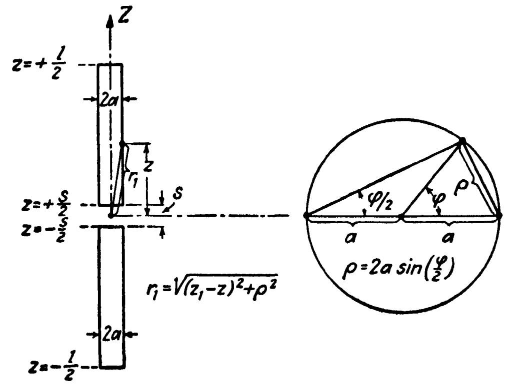


FIG. 186. Conditions for a cylindrical radiator that takes the cross dimension a into account.

become infinite. We can then neglect the second term, yielding the first approximation for $I(z)$. We obtain then successive corrections by replacing each preceding approximation to $I(z)$ in the second term of the above double-integral equation. The double integral

$$\Omega(z) = \int_0^{2\pi}\int_{-l/2}^{+l/2}\frac{dz_1\,d\varphi}{2\pi r_1}$$

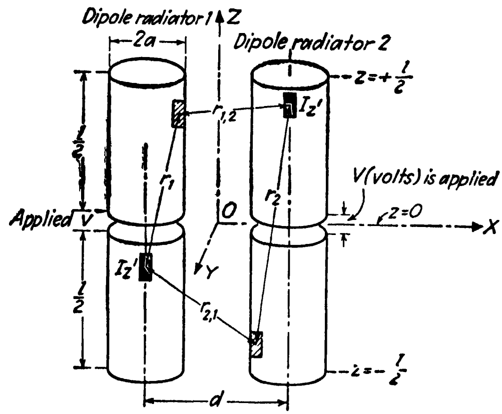
is an involved function of z. Using the simpler Hallén method, by writing

$$\Omega(z) = \Omega(0) + [\Omega(z) - \Omega(0)]$$

we find

$$I(z) = \underbrace{\frac{4\pi[A\cos(\beta z) + B\sin(\beta|z|)]}{\Omega_0}}_{=I_0(z)} + \left[1 - \frac{\Omega(z)}{\Omega(0)}\right]I(z) + \frac{1}{\Omega(0)}\int_0^{2\pi}\int_{-l/2}^{l/2}\frac{I(z_1)\epsilon^{-j\beta r_1}-I(z)}{2\pi r_1}\,dz_1\,d\varphi = I_0(z) + I_1(z) + I_2(z) + \cdots$$

where the series is an addition of descending powers of $\Omega(0) = 4.604\log_{10}(l/a)$ in the case of the Hallén single-integral derivation. The error is, however, only of the order of $(2a/l)^2$. Schelkunoff shows also that infinitesimal gaps must be avoided, since for an infinitesimal gap we have $I(0) = \infty$. It is shown by theoretical and experimental values for an $l \cong \lambda$ dipole that Hallén's method, if solved with enough care, can give results agreeing with procedures that are directly the outcome of Maxwell's equations.



$r_1 = \sqrt{(z - z')^2 + a^2} \cong |z - z'|$
$r_{12} = \sqrt{(z - z')^2 + d^2}$; $Z_0 = 376.7$ ohms
$\Omega = 4.604 \log_{10} \left(\frac{l}{a}\right)$
ψ_{12} is phase of current in dipole 2 with respect to phase of current in dipole 1
$\Delta = 2.302 \log_{10} (p)$
$p = 0.25 \left[\sqrt{1 + \left(\frac{a}{0.5l - z}\right)^2} + 1 \right] \left[\sqrt{1 + \left(\frac{a}{0.5l + z}\right)^2} + 1 \right]$

FIG. 187. Conditions for two cylindrical dipoles that are driven. O is radiation center, I_z is entrance current at dipole gap of radiator 2, I_1 is entrance current at the dipole gap of radiator 1; z' is along cylinder axis, shaded area refers to surface of cylinder and z values.

For many details regarding mutual effects based on the Hallén distribution, reference is made to the papers by King and Harrison.[1] For two parallel cylindrical radiators as in Fig. 187, we have then the current distribution I_z along the dipole 1, according to (855), as

$$I_z^{(\text{amp})} = \underbrace{-j \frac{4\pi}{\Omega Z_0} \left[C_1 \cos (\beta z) + \frac{V}{2} \sin (\beta|z|) \right]}_{= I(z)_0 \text{ is zeroth-order approximation}}$$

$$\frac{1}{\Omega} \left[I(z) \log_\epsilon \left(1 - \frac{4z^2}{l^2}\right) + I(z)\Delta + \int_{-l/2}^{l/2} \frac{I(z')\epsilon^{-j\beta r_1} - I_z}{r_1} dz' \right]$$

$$\underbrace{- \frac{1}{\Omega} \epsilon^{-j\psi_{12}} \int_{-l/2}^{l/2} I(z') \frac{\epsilon^{-j\beta r_{12}}}{r_{12}} dz'}_{\text{due to reaction of dipole 2 on dipole 1}} \quad (857)$$

A similar relation exists for $I(z')$, for the current distribution along dipole 2, and the constant C_1 is again found from the boundary condition at the free ends of each dipole, since for $I(z) = I(\pm l/2)$, we have $z = \pm \frac{1}{2} l$ and $I(z) = 0$.

[1] KING, R., and C. W. HARRISON, JR., *Jour. Appl. Phys.*, **15**, 481, June, 1944. HARRISON, C. W., *Proc. IRE*, **33**, 398, June, 1945; **33**, 892, December, 1945; **34**, 204P, April, 1946.

This is the outcome of the electric field relation at any space point, for which

$$\mathcal{E}_z = -j\frac{\omega}{\beta^2}\left(\frac{d^2A_z}{dz^2} + \beta^2 A_z\right) \qquad \text{volts/m}$$

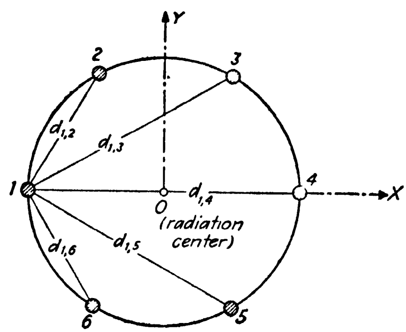
Due to a boundary condition between any two different electromagnetic mediums, the $\mathcal{E}_z$ field tangent to a boundary must be the *same* on each side of the boundary. For infinite conductivity of a dipole, there can be *no* $\mathcal{E}_z$ component along the conductor, which is also essentially the case for good conductors of finite conductivity (except the small tangential component which supplies the joulean heat loss). Therefore $\mathcal{E}_z \cong 0$ and

$$\frac{d^2A_z}{dz^2} + \beta^2 A_z = 0 \qquad \text{with a solution } A_z = K_1 \cos(\beta z) + K_2 \sin(\beta z)$$

where K_1 and K_2 are arbitrary constants and A_z is the only magnetic vector potential component in volt · seconds per meter. The vector potential at any point of either dipole surface is then given by

$$A_z = \frac{\mu}{4\pi}\int I(z')\frac{\epsilon^{-j\beta r}}{r}\,dz'; \qquad r = \sqrt{(z-z')^2 + a^2}$$

The quantity $I(z')$ is the *complex* current flowing in the dipole element dz'. Here r denotes the distance to the point θ, z, a from dz'. This point is the location for which A_z holds, and the integration is to be carried out over both dipoles.



Cylindrical dipole radiators parallel with the Z axis of the XYZ system. The circles marked 1, 2, 3, 4, 5, and 6 are the radiation centers at $z = 0$ of respective dipoles 1, 2, 3, 4, 5, and 6. General expression: $r_{1,p} = \sqrt{(z-z')^2 + d^2_{1,p}}$ [Eq. (857*a*)]. We have the equatorial spacing $d_{1,p} = d_{1,2}$ yielding

$$r_{1,p} = r_{1,2} = \sqrt{(z-z')^2 + d^2_{1,2}}$$

For the case of Fig. 187, we have $n = 2$ and $d_{1,2} = d$; $\psi_{1,p}$ is the phase of the current in dipole p referred to the current of dipole 1, and $p = 2, 3, 4, 5$, and 6 in the above illustration.

FIG. 188. Arrangement of six cylindrical radiators. For $n = 6$ identical dipoles.

If we have an array of n cylindrical dipoles of identical dimensions symmetrically disposed of as in Fig. 188 and all driven by V volts at the dipole mid-gap, we have

$$\left.\begin{aligned} I_m &= -\frac{1}{\Omega}\sum_{p=2}^{p=n}\epsilon^{-j\psi_{1p}}\int_{-l/2}^{l/2} I(z')\frac{\epsilon^{-j\beta r_{1p}}}{r_{1p}}\,dz' \\ &= \underbrace{-\frac{1}{\Omega}\left[\epsilon^{-j\psi_{12}}\int_{-l/2}^{l/2} I(z')\frac{\epsilon^{-j\beta r_{12}}}{r_{12}}\,dz' + \epsilon^{-j\psi_{13}}\int_{-l/2}^{l/2} I(z')\frac{\epsilon^{-j\beta r_{13}}}{r_{13}}\,dz'\right]}_{\text{if we had, for instance, three dipole radiators set on a circle in the equatorial } XY \text{ plane } (n = 3 \text{ in Fig. 188})} \\ r_{1p} &= \sqrt{(z-z')^2 + d^2_{1p}} \end{aligned}\right\} \quad (857a)$$

where d_{1p} is the perpendicular spacing with respect to the pth radiator

and radiator 1. The term I_m is then the mutual current effect on radiator 1. The zeroth-order approximation of (857) is used in the integrals for finding the first-order correction of the current distribution along dipole 1. The same procedure may be repeated over and over, since this is a procedure of successive approximations where each term depends upon the preceding one. Therefore, the *quality of the first approximation must affect all successive approximations.* It is, of course, taken for granted that the series converges. Hallén's procedure of successive approximations is usually expressed with only two terms in the expansion. The leading Hallén term assumes tacitly that the ratio $l/(2a)$ is small and Ω is large, where $\Omega = 4.604 \log_{10} (l/a)$, which is quite severe on following correction terms. The second approximation, used by M. C. Gray[1] and Schelkunoff, does not assume that $l/(2a)$ is small, which gives, therefore, a better second approximation and more reliable values for longer radiators.

The current I_1 amperes, flowing into the input terminals of dipole radiator 1 in Fig. 187, is found from (857) by putting $z = 0$, since this yields the position for the dipole gap. We have then the following details:

$C_1 = \cos(\beta z) + \tfrac{1}{2} V \sin(\beta|z|) = C_1, \qquad \Delta = 2 \log_\epsilon (p)$ according to Fig. 187

$$p = \frac{1}{4}\left[\sqrt{1 + \left(\frac{a}{0.5l - 0}\right)^2} + 1\right]\left[\sqrt{1 + \left(\frac{a}{0.5l + 0}\right)^2} + 1\right]$$

$$p = \frac{1}{4}\left(\sqrt{1 + 4\frac{a^2}{l^2}} + 1\right)^2 = \frac{1 + 4(a/l)^2 + 2\sqrt{1 + 4(a/l)^2} + 1}{4}$$

$$\cong \frac{1 + 0 + 2 \times 1 + 1}{4} = 1; \qquad \Delta \cong 2 \log_\epsilon (1) = 0$$

$$r_1 = \sqrt{(z - z')^2 + a^2} = \sqrt{(z')^2 + a^2} \cong |z'|;$$

$$r_{12} = \sqrt{(z - z')^2 + d^2} = \sqrt{(z')^2 + d^2}, \qquad \log_\epsilon\left(1 - \frac{4 \times 0}{l^2}\right) = 0$$

For *one boundary* condition, we have

$$I_1^{(\text{amp})} = -j\frac{4\pi}{\Omega Z_0} C_1 - \frac{1}{\Omega}\int_{z'=-l/2}^{z'=l/2} \frac{I(z')\epsilon^{j\beta|z'|} - I_1}{|z'|}\,dz' - \frac{1}{\Omega}\epsilon^{-j\psi_{12}}\int_{z'=-l/2}^{z'=l/2} I(z')\frac{\epsilon^{-j\beta\sqrt{(z')^2+d^2}}}{\sqrt{(z')^2 + d^2}}\,dz'$$

The *other boundary* condition is, at the free ends, $z = \pm \tfrac{1}{2} l$, for which $I(z) = 0$, and

$$I(z) = I\left(\pm\frac{1}{2}l\right) = 0 = -j\frac{4\pi}{\Omega Z_0}\left(C_1 \cos\frac{\beta l}{2} + \frac{1}{2} V \sin\frac{\beta l}{2}\right) - \frac{1}{\Omega}\int_{-l/2}^{l/2} I(z')\frac{\epsilon^{-j\beta r_1}}{r_1}\,dz' - \frac{1}{\Omega}\epsilon^{-j\psi_{12}}\int_{-l/2}^{l/2}\frac{I(z')\epsilon^{-j\beta r_{12}}}{r_{12}}\,dz'$$

for

$$r_1 = \sqrt{(0.5l - z')^2 + a^2} \cong |0.5l - z'|; \qquad r_{12} = \sqrt{(0.5l - z')^2 + d^2}$$

[1] GRAY, M. C., *Jour. Appl. Phys.*, **15**, 61, January, 1944.

Therefore, we must solve integrals with moduli signs as in $|(l/2) - z'|$, for instance. At the end of Sec. 129, example E gives the procedure. C. W. Harrison[1] has given solutions based on such a procedure and expressed them in Si and Ci functions as well as by the Euler constant and logarithmic terms. For the general case of n driven (symmetrically arranged as in Fig. 188), identical cylindrical dipoles with V volts impressed across each dipole gap, we have the following relations:

Abbreviations:

$$\mathrm{Ci}\,x = \int_\infty^x \frac{\cos(u)}{u}\,du; \qquad \overline{\mathrm{Ci}}\,x = \int_0^x \frac{1-\cos(u)}{u}\,du = \underbrace{0.5772}_{\text{Euler constant}} + 2.302\log_{10}(x) - \mathrm{Ci}(x)$$

$$\mathrm{Si}\,x = \int_0^x \frac{\sin(u)}{u}\,du; \qquad \text{Si } x \text{ and Ci } x \text{ found from tables or curves}$$

$$A = \epsilon^{-j\psi_{1p}}; \qquad B = \sqrt{\frac{l^2}{4} + d_{1p}^2}; \qquad C = \sqrt{l^2 + d_{1p}^2}$$

$A = 1$ for *inphase* excitations

$$b \cong -\int_{-l/2}^{l/2} \cos(\beta u)\,\frac{\epsilon^{-j\beta|u|} - 1}{|u|}\,du = \overline{\mathrm{Ci}}\,(\beta l) + j\,\mathrm{Si}\,(\beta l)$$

$$c \cong -\int_{-l/2}^{+l/2} \cos(\beta u)\,\frac{\epsilon^{-j\beta\left|\frac{l}{2}-u\right|}}{\left|\frac{l}{2}-u\right|}\,du = \frac{1}{2}\cos\left(\frac{\beta l}{2}\right)[\overline{\mathrm{Ci}}\,(2\beta l) + j\,\mathrm{Si}\,(2\beta l)] - \frac{1}{2}\sin\left(\frac{\beta l}{2}\right)[\mathrm{Si}\,(2\beta l) - j\,\overline{\mathrm{Ci}}\,(2\beta l)]$$

$$\delta \cong -\int_{l/2}^{+l/2} \sin(\beta|u|)\,\frac{\epsilon^{-j\beta|u|}}{|u|}\,du = -\mathrm{Si}\,(\beta l) + j\,\overline{\mathrm{Ci}}\,(\beta l)$$

$$e \cong -\int_{-l/2}^{+l/2} \sin(\beta|u|)\,\frac{\epsilon^{-j\beta\left|\frac{l}{2}-u\right|}}{|(l/2) - u|}\,du = \frac{1}{2}\cos\left(\frac{\beta l}{2}\right)[\mathrm{Si}\,(2\beta l) - 2\,\mathrm{Si}\,(\beta l) - j\,\overline{\mathrm{Ci}}\,(2\beta l) + j2\,\overline{\mathrm{Ci}}\,(\beta l)] - \frac{1}{2}\sin\left(\frac{\beta l}{2}\right)[\overline{\mathrm{Ci}}\,(2\beta l) + j\,\mathrm{Si}\,(2\beta l) - j2\,\mathrm{Si}\,(\beta l) - 2\,\overline{\mathrm{Ci}}\,(\beta l) - 9.208\log_{10}(2)]$$

$$g = -\sum_{p=2}^{p=n} A \int_{-l/2}^{+l/2} \cos(\beta u)\,\frac{\epsilon^{-j\beta\sqrt{u^2+d^2_{1p}}}}{\sqrt{u^2 + d_{1p}^2}}\,du$$

$$= \sum_{p=2}^{p=n} A\left\{-\mathrm{Ci}\left[\beta\left(B + \frac{l}{2}\right)\right] + \mathrm{Ci}\left[\beta\left(B - \frac{l}{2}\right)\right] - j\,\mathrm{Si}\left[\beta\left(B - \frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B + \frac{l}{2}\right)\right]\right\} \tag{857b}$$

[1] *Loc. cit.*

$$
\left.
\begin{aligned}
h &= -\sum_{p=2}^{p=n} A \int_{-l/2}^{+l/2} \cos(\beta u) \frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+d_{1p}^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+d_{1p}^2}}\,du \\
&= \sum_{p=2}^{p=n} A\Bigg(-\frac{1}{2}\cos\left(\frac{\beta l}{2}\right)\Big\{[\mathrm{Ci}\,\beta(C+l)] - \mathrm{Ci}\,[\beta(C-l)] - j\,\mathrm{Si}\,[\beta(C+l)] \\
&\quad + j\,\mathrm{Si}\,[\beta(C-l)]\Big\} + j\tfrac{1}{2}\sin\left(\frac{\beta l}{2}\right)\{2\,\mathrm{Ci}\,(\beta d_{1p}) - j2\,\mathrm{Si}\,(\beta d_{1p}) \\
&\quad - \mathrm{Ci}\,[\beta(C-l)] - \mathrm{Ci}\,[\beta(C+l)] + j\,\mathrm{Si}\,[\beta(C-l)] + j\,\mathrm{Si}\,[\beta(C+l)]\}\Bigg) \\
k &= -\sum_{p=2}^{p=n} A \int_{-l/2}^{+l/2} \sin(\beta|u|) \frac{\epsilon^{-j\beta\sqrt{u^2-d_{1p}^2}}}{\sqrt{u^2+d_{1p}^2}}\,du \\
&= \sum_{p=2}^{p=n} jA\left\{2\,\mathrm{Ci}\,(\beta d_{1p}) - j2\,\mathrm{Si}\,(\beta d_{1p}) - \mathrm{Ci}\left[\beta\left(B+\frac{l}{2}\right)\right]\right. \\
&\quad \left. - \mathrm{Ci}\left[\beta\left(B-\frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B+\frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B-\frac{l}{2}\right)\right]\right\} \\
m &= -\sum_{p=2}^{p=n} A \int_{-l/2}^{+l/2} \sin(\beta|u|) \frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+d_{1p}^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+d_{1p}^2}}\,du \\
&= \sum_{p=2}^{p=n} \frac{jA}{2}\Bigg(\epsilon^{-j(\beta l/2)}\left\{2\,\mathrm{Ci}\left[\beta\left(B-\frac{l}{2}\right)\right] - j2\,\mathrm{Si}\left[\beta\left(B-\frac{l}{2}\right)\right]\right. \\
&\quad \left. - \mathrm{Ci}\,[\beta(C-l)] + j\,\mathrm{Si}\,[\beta(C-l)] - \mathrm{Ci}\,(\beta d_{1p}) + j\,\mathrm{Si}\,(\beta d_{1p})\right\} \\
&\quad + \epsilon^{j(\beta l/2)}\left\{2\,\mathrm{Ci}\left[\beta\left(B+\frac{l}{2}\right)\right] - j2\,\mathrm{Si}\left[\beta\left(B+\frac{l}{2}\right)\right] - \mathrm{Ci}\,[\beta(C+l)]\right. \\
&\quad \left. + j\,\mathrm{Si}\,[\beta(C+l)] - \mathrm{Ci}\,(\beta d_{1p}) + j\,\mathrm{Si}\,(\beta d_{1p})\right\}\Bigg)
\end{aligned}
\right\} \quad (857b)
$$

The usefulness of the integral solutions of (857b) in connection with (857) for two parallel cylindrical dipoles (Fig. 187 and $n = 2$) and the general case of (857a), applying, for instance, to the circular cage array of Fig. 188 (where $n = 6$), will be shown now. Applying the functions of (857b) to the two boundary conditions $z = 0$ and $z = \pm l/2$, we find the entrance current I_1 at the dipole gap, since

$$I_1 = -j\frac{4\pi}{\Omega Z_0}\left[C_1\left(1+\frac{b+g}{\Omega}\right)+\frac{V(\delta+k)}{2\Omega}\right] \quad \text{amp;}$$

$$C_1 = -\frac{V}{2}\,\frac{\sin\left(\frac{\beta l}{2}\right)+\frac{e+m}{\Omega}}{\cos\left(\frac{\beta l}{2}\right)+\frac{c+h}{\Omega}} \quad \text{volts}$$

gives the factor C_1 for the dipole gap current I_1 of dipole 1. It is assumed that the quantity Ω is such that $1/\Omega$ is *small compared with unity*, so that terms in $1/\Omega^2$ may be ignored. The input impedance Z_1 is the ratio of the applied gap voltage V divided by the entrance current I_1 at the dipole gap of radiator 1 in the *presence* of all other n identical dipoles. Hence, we have the formula

$$\left.\begin{aligned} Z_1 &= -j60\,\frac{D+\Omega\cos(\beta l/2)}{(1+E)\sin(\beta l/2)-F\cos(\beta l/2)} = R_1 + jX_1 \quad \text{ohms;}\\ D &= c+h; \qquad E = \frac{b+g}{\Omega}; \qquad F = \frac{\delta+c+k+m}{\Omega}\\ \Omega &= 4.606\log_{10}\left(\frac{l}{a}\right) \end{aligned}\right\} \quad (857c)$$

where l is the total length of the dipole and a denotes its radius. The application of (857c) in connection with the integral functions b, c, δ, e, g, h, k, and m is not difficult, even though in some cases laborious. We have then, for two identical parallel half-wave dipoles that are spaced a half wavelength apart, the following details:

$n = 2$, since there are only two radiators; $l = \frac{1}{2}\lambda$ and $d_{1p} = d_{12}$, since there is only one separation distance $d_{12} = d$, as in Fig. 187 (simplification of Fig. 188 for radiator 1 and radiator 2 only present); $d = \frac{1}{2}\lambda$ if radius $a = 10^{-3}\lambda$ and $\psi_{1p} = \psi_{12} = 0$ deg (which is *inphase* driven dipoles), then

$$A = 1; \qquad \frac{l}{a} = \frac{\lambda/2}{10^{-3}\lambda} = 500$$

$$\begin{aligned}\Omega &= 4.606\log_{10}(500) = 4.606[\log_{10}(5)+\log_{10}(10^2)]\\ &= 4.606(0.69897+2) = 12.4\end{aligned}$$

satisfying $1/\Omega = 1/12.4 = 0.0806$, small compared to unity;

$$\frac{\beta l}{2} = \frac{2\pi}{\lambda}\frac{l}{2} = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

corresponds to 90 deg. The expressions $\cos(\beta l/2) = \cos 90\text{ deg} = 0$, $\sin(\beta l/2) = \sin 90\text{ deg} = 1$, and (857$c$) simplify to

$$Z_1 = -j60\,\frac{D+0}{(1+E)-0} = -j60\,\frac{D}{1+E} \quad \text{complex ohms} \qquad (857d)$$

requiring only computation of $D = c + h$ and $E = (b + g)/\Omega$, and

$$c = \underbrace{\frac{1}{2}\overbrace{\cos\left(\frac{\beta l}{2}\right)}^{=0}\left[\overline{\text{Ci}}\ (2\beta l) + j\ \text{Si}\ (2\beta l)\right]}_{\text{hence, }=0} - \underbrace{\frac{1}{2}\sin\left(\frac{\beta l}{2}\right)}_{=\frac{1}{2}}\left[\text{Si}\ (2\beta l) - j\ \overline{\text{Ci}}\ (2\beta l)\right];$$

$$\text{Si}\ (2\beta l) = \text{Si}\ (6.28) = 1.43$$

$$\text{Ci}\ (2\beta l) = \text{Ci}\ (6.28) = -0.023$$

$$\overline{\text{Ci}}\ (6.28) = \underbrace{0.5772}_{\substack{\text{Euler}\\ \text{constant}}} + 2.303\ \underbrace{\log_{10}(6.28)}_{=0.798} - \text{Ci}\ (6.28) = 0.5772 + 1.84 + 0.023 = 2.4402$$

$$c = -\tfrac{1}{2}[\text{Si}\ (2\beta l) - j\ \overline{\text{Ci}}\ (2\beta l)] = -\tfrac{1}{2}(1.43 - j2.4402) = -0.715 + j1.2201$$

$$h = \underbrace{-\underbrace{\frac{1}{2}\cos\left(\frac{\beta l}{2}\right)}_{=0}[\ ]}_{\text{hence, }=0} + \underbrace{j\frac{1}{2}\sin\left(\frac{\beta l}{2}\right)}_{=j0.5}\{\ \}$$

$$\{\ \} = 2\ \text{Ci}\ (\beta d) - j2\ \text{Si}\ (\beta d) - \text{Ci}\ [\beta(C - l)] - \text{Ci}\ [\beta(C + l)] + j\ \text{Si}\ [\beta(C - l)] + j\ \text{Si}\ [\beta(C + l)]$$

$$d = \frac{\lambda}{2}; \qquad \beta d = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi = 3.14$$

$$2\ \text{Ci}\ (\beta d) = 2\ \text{Ci}\ (3.14) = 2 \times 0.075 = 0.15; \qquad 2\ \text{Si}\ (3.14) = 2 \times 1.8515 = 3.703$$

$$C = \sqrt{l^2 + d^2} = \sqrt{\frac{\lambda^2}{4} + \frac{\lambda^2}{4}} = 0.7071\lambda; \qquad l = \frac{\lambda}{2}$$

$$C - l = (0.7071 - 0.5)\lambda = 0.2071\lambda; \qquad C + l = (0.7071 + 0.5)\lambda = 1.2071\lambda$$

$$\beta(C - l) = \frac{2\pi}{\lambda}0.2071\lambda = 6.28 \times 0.2071 = 1.3$$

$$\beta(C + l) = \frac{2\pi}{\lambda}1.2071\lambda = 6.28 \times 1.2071 = 7.58; \qquad \text{Si}\ [\beta(C - l)] = \text{Si}\ (1.3) = 1.184$$

$$\text{Ci}\ (1.3) = 0.4457; \qquad \text{Si}\ [\beta(C + l)] = \text{Si}\ (7.58) = 1.5264$$

$$\text{Ci}\ (7.58) = 0.10412; \qquad h = j0.5(0.15 - j3.703 - 0.4457 - 0.1042 + j1.184 + j1.5264) = j0.5(-0.39982 - j0.9926) = 0.4963 - j0.19991$$

$$D = c + h = -0.715 + j1.2201 + 0.4963 - j0.19991 = -0.2187 + j1.0202$$

$$E = \frac{b + g}{\Omega}; \qquad b = \overline{\text{Ci}}\ (\beta l) + j\ \text{Si}\ (\beta l); \qquad \beta l = \frac{2\pi}{\lambda}\frac{\lambda}{2} = 3.14$$

$$\text{Si}\ (3.14) = 1.8515; \qquad \text{Ci}\ (3.14) = 0.075$$

$$\overline{\text{Ci}}\ (3.14) = 0.5772 + 2.303\ \overbrace{\log_{10}(3.14)}^{0.49694} - \text{Ci}\ (3.14) = 0.5772 + 1.144 - 0.075 = 1.6462$$

$$b = 1.6462 + j1.8515; \qquad g = -\text{Ci}\left[\beta\left(B + \frac{l}{2}\right)\right] + \text{Ci}\left[\beta\left(B - \frac{l}{2}\right)\right] - j\ \text{Si}\left[\beta\left(B - \frac{l}{2}\right)\right] + j\ \text{Si}\left[\beta\left(B + \frac{l}{2}\right)\right]$$

$$B = \sqrt{\frac{l^2}{4} + d_{1p}^2} = \sqrt{\frac{l^2}{4} + d^2} = \sqrt{\frac{\lambda^2}{16} + \frac{\lambda^2}{4}} = 0.56\lambda$$

$$\frac{l}{2} = \frac{1}{4}\lambda; \qquad B + \frac{l}{2} = (0.56 + 0.25)\lambda = 0.81\lambda; \qquad B - \frac{l}{2} = (0.56 - 0.25)\lambda = 0.31\lambda$$

$$\beta\left(B + \frac{l}{2}\right) = \frac{2\pi}{\lambda}0.81\lambda = 6.28 \times 0.81 = 5.09; \qquad \beta\left(B - \frac{l}{2}\right) = 6.28 \times 0.31 = 1.95$$

$$\text{Ci}\left[\beta\left(B + \frac{l}{2}\right)\right] = \text{Ci}\,(5.09) = -0.18; \qquad \text{Ci}\left[\beta\left(B - \frac{l}{2}\right)\right] = \text{Ci}\,(1.95) = 0.43245$$

$$\text{Si}\left[\beta\left(B + \frac{l}{2}\right)\right] = \text{Si}\,(5.09) = 1.537; \qquad \text{Si}\left[\beta\left(B - \frac{l}{2}\right)\right] = \text{Si}\,(1.95) = 1.5816$$

$$g = -(-0.18) + 0.43245 - j1.5816 + j1.537 = 0.61245 - j0.0446$$

$$b + g = 1.6462 + j1.8515 + 0.61245 - j0.0446 = 2.25865 + j1.8069$$

$$E = \frac{b + g}{\Omega} = \frac{2.25865 + j1.8069}{12.4} = 0.182 + j0.146; \quad 1 + E = 1.182 + j0.146$$

We find then the input impedance of dipole 1 in presence of dipole 2 as

$$Z_1 = -j60\frac{D}{1 + E} = -j60\frac{-0.2187 + j1.0202}{1.182 + j0.146} \quad \text{ohms}$$

$$\frac{(-0.2187 + j1.0202)(1.182 - j0.146)}{1.182^2 + 0.146^2} = \frac{-0.259 + j0.032 + j1.205 + 0.149}{1.41844}$$

$$= \frac{-0.11 + j1.237}{1.41844} = -0.0766 + j0.875$$

$$Z_1 = -j60(-0.0776 + j0.875) = 52.5 + j4.656 \text{ ohms}$$

Inasmuch as in Fig. 187, we have two identical dipoles driven with the same inphase terminal voltage V at the respective mid-gaps and in our numerical example $l = \frac{1}{2}\lambda = d$, the input impedances at the gap of dipole 1 must be the same as at the gap of dipole 2 and equal $52.5 + j4.66$ ohms as just found. If we compare this with the self-impedance $Z_s = 66.4 + j31.8$, which exists if only one half-wave dipole is present (Table 32), we note that the mutual radiation resistance R_m from one radiator on the other and vice versa is $R_m = 52.5 - 66.4 = -13.9$ ohms and the associated mutual reactance is $X_m = 4.66 - 31.8 = -27.14$ reactive ohms. We have then the comparisons of Table 33.

For the radius a chosen, there is not so much real difference in the self-impedance Z_s value as well as the entrance impedance holding for each half-wave dipole if we think in terms of actual *experimental* checks. It is not exactly easy to carry on determinations of the active resistance and the reactive component of the entrance impedance without affecting the actual conditions by means of the measuring equipment. What is pleasing in this comparison is that the radiation resistance per dipole in presence of the other dipole is 52.5 ohms for the Hallén distribution and 60.92 ohms for a very fine half-wave dipole assuming sinusoidal current distribution. The reactive component is always stubbed out or neu-

tralized by shortening the dipole length somewhat, even though the X_m values of -27.14 and -30 ohms compare very well for a reactance that depends upon the frequency. The R_m values of -13.9 and -12.48 ohms are likewise rather close values. These results apply, of course, only to half-wave dipoles that are separated a half wavelength. But half-wave dipoles are what we actually use in short-wave work or $\frac{1}{4}\lambda$ poles or tips projecting out of a hole with a backed-up metal reference

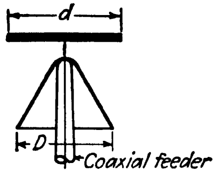
TABLE 33

For Hallén distribution, taking the *transverse* dimensions a of the dipole into account*	Classical sinusoidal current distribution (ignoring the effect of the radius a)
$Z_s = R_s + jX_s = 66.4 + j31.8$ $Z_1 = Z_2 = 52.5 + j4.66$ $Z_m = Z_1 - Z_s = R_m + jX_m = -13.9 - j27.14$ $R_m = -13.9$ ohms mutual radiation resistance $X_m = -27.14$ ohms mutual reactance	$Z_s = 73.4 + j42.9$ ohms $Z_1 = Z_2 = Z_s + Z_m = 60.92 + j12.9$ ohms $Z_m = -12.48 - j30$ ohms $R_m = -12.48$; $X_m = -30$ ohms
Z_s is the self-impedance, if only *one* half-wave dipole is present, for a radius $a = 10^{-3}\lambda$ meters, a total dipole length of $l = \lambda/2$ meters, $\Omega = 4.604 \log_{10}(l/a) = 12.4$, Z_1 and Z_2 are the entrance impedances at the gaps of dipole 1 and 2 in Fig. 187	Based on formulas by A. A. Pistolkors, R. Bechmann, P. S. Carter, G. H. Brown.† [Consult Figs. 143, 148, 149, 166, and formulas (851*b*), (851*c*), and (851*d*)]. For either distribution the two half-wave dipoles are parallel and separated by $d = \frac{1}{2}\lambda$ meters

* Based on formulas by R King F G Blake, C W Harrison.
† *Proc. IRE*, **25**, 78, January, 1937.

plane. The latter arrangement, as we shall see, is by the image concept a half-wave dipole in equivalence, except that radiation occurs only over a hemisphere, and the values computed for the full-dipole (half-wave radiator with a center gap) apply for the half-dipole ($\frac{1}{4}\lambda$ pole) if we divide the values by 2. Therefore, there are many short-wave cases where we can use the classic formulas in order to obtain an idea of what entrance impedance we can expect if dealing with half-wave dipoles. This is the reason why in this text the classic theory is presented as well as the more modern theories based on the Hallén distribution and on the biconical dipole, which is to follow. It must be admitted that we have at least now well-worked-out theories, which give a means of accounting in some way for the transverse dimension of a dipole radiator. This is a decided forward step in the antenna theory since the days of Abraham's announcements of natural oscillations on a perfectly conducting thin prolate spheroid, which was, in 1938, followed up by Page, Adams, and

Ryder[1] by forced oscillations on spheroids. It is of great benefit that the Harvard group under R. King and the ATT group under S. A. Schelkunoff have chosen separate roads to solve antenna problems of arbitrary shape, taking the transverse dimensions into account in addition to the longitudinal extension. These problems apply to the wide-band radiators, which are so much needed in television and other systems and which yield essentially the same input impedance over a wide carrier-frequency range (Fig. 189). The data of Fig. 189 are the ones given by A. G. Kandoian.[2]



Cutoff frequency, Mc/sec	d, in.	h, in.	D, in.
90	18	24	20
200	9	12 5	14

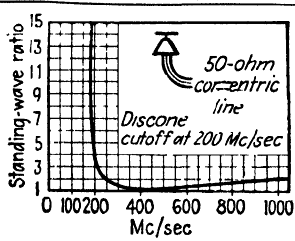


FIG. 189. Mismatch against frequency of a discone antenna for a cutoff frequency of 200 Mc/sec.

The method with Hallén distributions can, of course, also be applied to collinear dipoles and an alignment. Both cases for two collinear dipoles and three parallel dipoles have been treated in detail by C. W. Harrison.[3] The procedure is the same as described above, except that we have more integral functions than just evaluated in connection with Fig. 187. This is evident when examining a four-dipole alignment as in Fig. 190, where, for the sake of clearness, some of the coupling distances are not shown. If V_1 denotes the impressed voltage across the dipole gap of dipole 1, we have along this dipole a current distribution

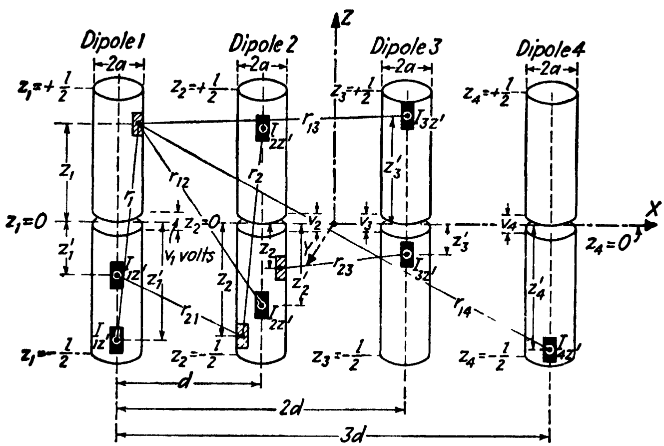
$$
\begin{aligned}
I_1(z) = {} & K\left[C_1 \cos(\beta z_1) + \frac{V_1}{2}\sin(\beta|z_1|)\right] \\
& - \frac{1}{\Omega}\left[I_1(z)\log_\epsilon\left(1 - \frac{4z_1^2}{l^2}\right) + I_1(z)\Delta_1 + \int_{-l/2}^{l/2} \frac{I_1(z')\epsilon^{-j\beta r_1} - I_1(z)}{r_1}\,dz_1'\right] \\
& - \underbrace{\frac{\int_{-l/2}^{l/2} I_2(z')\,\frac{\epsilon^{-j\beta r_{12}}}{r_{12}}\,dz_2'}{\Omega}}_{\text{due to reaction of dipole 2 on dipole 1}} - \underbrace{\frac{\int_{-l/2}^{l/2} I_3(z')\,\frac{\epsilon^{-j\beta r_{13}}}{r_{13}}\,dz_3'}{\Omega}}_{\text{reaction of dipole 3 on dipole 1}} - \underbrace{\frac{\int_{-l/2}^{l/2} I_4(z')\,\frac{\epsilon^{-j\beta r_{14}}}{r_{14}}\,dz_4'}{\Omega}}_{\text{reaction of dipole 4 on dipole 1}}
\end{aligned}
$$

[1] PAGE, LEIGH, and N. I. ADAMS, JR., *Phys. Rev.*, May 15, 1938, p. 819.

[2] *Elec. Commun.*, **23,** March, 1946.

[3] *Proc. IRE*, **34,** 204P, April, 1946.

This is evident, since we found with the classic sinusoidal distributions that for an alignment as in Fig. 189, with equal inphase driven dipoles, the impedance looking into the dipole gap of dipole 1 is $Z_s + Z_d + Z_{2d} + Z_{3d}$, where Z_s is the gap impedance with only one dipole present and



$r_1 = \sqrt{(z_1 - z_1')^2 + a^2} \cong |z_1 - z_1'|$, $r_2 = \sqrt{(z_2 - z_2')^2 + a^2} \cong |z_2 - z_2'|$
$r_3 \cong |z_3 - z_3'|$; $r_4 \cong |z_4 - z_4'|$, $r_{12} = \sqrt{(z_1 - z_2')^2 + d^2}$
$r_{21} = \sqrt{(z_2 - z_1')^2 + d^2}$, $r_{13} = \sqrt{(z_1 - z_3')^2 + 4d^2}$, $r_{14} = \sqrt{(z_1 - z_4')^2 + 9d^2}$
$r_{23} = \sqrt{(z_2 - z_3')^2 + d^2}$ etc

$\Omega = 4.604 \log_{10} \left(\frac{l}{a}\right)$, l is total length of dipole

$K = -j \frac{4\pi}{\Omega Z_0}$, $Z_0 = 376.7$ ohms

$$\Delta = \log_\epsilon \left\{0.25 \left[\sqrt{1 + \frac{a^2}{(0.5l - z)^2}} + 1\right]\left[\sqrt{1 + \frac{a^2}{(0.5l + z)^2}} + 1\right]\right.$$

Δ_1 for $z = z_1$, . . . ; $\rho_1, \rho_2, \rho_3, \rho_4, \rho_{12}, \rho_{21}, \rho_{23}$, . . . , are the corresponding r_1, r_2, . . . , values for the inner boundary $z_1 = z_2 = z_3 = z_4 = 0$, and ρ_1', ρ_2', . . . , are the corresponding r_1, r_2, . . . , values for the free-end boundary $z_1 = z_2 = z_3 = z_4 = \pm l/2$

FIG. 190. Conditions for an alignment of four cylindrical dipoles. I_1 is entrance current of dipole 1, z' refers to locations along respective cylinder axes, and z refers to shaded small area locations on the outside surface of respective dipole cylinders.

Z_d, Z_{2d}, and Z_{3d} are the mutual impedances due to dipoles 2, 3, and 4 on dipole 1. If, therefore, the same voltage is impressed across dipole gap 4 as is across all other dipoles, for symmetry reasons, the above impedance expression also denotes the entrance impedance of dipole 4. For symmetry reasons, we also have, for either dipole gap impedance of respective dipoles 2 and 3, the entrance value $Z_s + 2Z_d + Z_{2d}$ complex ohms, since the coupling is twice due to a distance d and once due to a distance $2d$.

For the distribution of current along the dipole 2, we have similarly

$$I_2(z) = K\left[C_2 \cos(\beta z_2) + \frac{V_2}{2}\sin(\beta|z_2|)\right]$$
$$\underbrace{-\frac{1}{\Omega}\left[I_2(z)\log_\epsilon\left(1-\frac{4z_2^2}{l^2}\right) + I_2(z)\Delta_2 + \int_{-l/2}^{l/2}\frac{I_2(z')\epsilon^{-j\beta r_2} - I_2(z)}{r^2}\,dz_2'\right]}_{\text{self-effect, due to dipole 2 only (other dipoles absent)}}$$
$$\underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_2(z')\epsilon^{-j\beta r_{21}}}{r_{21}}\,dz_1'\right]}_{\text{due to dipole 1 on dipole 2}} \underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_3(z')\epsilon^{-j\beta r_{23}}}{r_{23}}\,dz_3'\right]}_{\text{due to dipole 3 on dipole 2}} \underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_4(z')\epsilon^{-j\beta r_{24}}}{r_{24}}\,dz_4'\right]}_{\text{due to dipole 4 on dipole 2}}$$

For the current distribution along dipole 3, we find

$$I_3(z) = K\left[C_3 \cos(\beta z_3) + \frac{V_3}{2}\sin(\beta|z_3|)\right]$$
$$\underbrace{-\frac{1}{\Omega}\left[I_3(z)\log_\epsilon\left(1-\frac{4z_3^2}{l^2}\right) + I_3(z)\Delta_3 + \int_{-l/2}^{l/2}\frac{I_3(z')\epsilon^{-j\beta r_3} - I_3(z)}{r_3}\,dz_3'\right]}_{\text{self-effect, due to dipole 3 only (other dipoles absent)}}$$
$$\underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_1(z')\epsilon^{-j\beta r_{31}}}{r_{31}}\,dz_1'\right]}_{\text{due to dipole 1 on dipole 3}} \underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_2(z')\epsilon^{-j\beta r_{32}}}{r_{32}}\,dz_2'\right]}_{\text{due to dipole 2 on dipole 3}} \underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_4(z')\epsilon^{-j\beta r_{34}}}{r_{34}}\,dz_4'\right]}_{\text{due to dipole 4 on dipole 3}}$$

For the current distribution along dipole 4, we have

$$I_4(z) = K\left[C_4 \cos(\beta z_4) + \frac{V_4}{2}\sin(\beta|z_4|)\right]$$
$$\underbrace{-\frac{1}{\Omega}\left[I_4(z)\log_\epsilon\left(1-\frac{4z_4^2}{l^2}\right) + I_4(z)\Delta_4 + \int_{-l/2}^{l/2}\frac{I_4(z')\epsilon^{-j\beta r_4} - I_4(z)}{r_4}\,dz_4'\right]}_{\text{self-effect, due to dipole 4 only (other dipoles absent)}}$$
$$\underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_1(z')\epsilon^{-j\beta r_{41}}}{r_{41}}\,dz_1'\right]}_{\text{due to dipole 1 on dipole 4}} \underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_2(z')\epsilon^{-j\beta r_{42}}}{r_{42}}\,dz_2'\right]}_{\text{due to dipole 2 on dipole 4}} \underbrace{-\frac{1}{\Omega}\left[\int_{-l/2}^{l/2}\frac{I_3(z')\epsilon^{-j\beta r_{43}}}{r_{43}}\,dz_3'\right]}_{\text{due to dipole 3 on dipole 4}}$$

These are Hallén current distributions and by no means sinusoidal, even though the impressed gap voltages V_1, V_2, V_3, and V_4 are sinusoidal. These relations are all that is needed, since we know the dimensions (l and d in terms of λ), and can find the constants C_1, C_2, C_3, and C_4 from the boundary conditions $z_1 = \pm l/2 = z_2 = z_3 = z_4$. We know that the impedances at the gaps are different; and therefore for one and the same voltages across all gaps, the entrance currents I_1, I_2, I_3, and I_4 cannot be alike in magnitude and in phase with respect to the applied inphase voltages of same magnitude. As for the case of Fig. 187, we find the respective entrance currents for $z_1 = 0 = z_2 = z_3 = z_4$. Thus, we find, for dipole 1 and dipole 2, the respective entrance currents I_1 and I_2, as

$$I_1^{(\text{amp})} = \underbrace{KC_1 - \frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_1(z')\epsilon^{-j\beta\rho_1} - I_1}{\rho_1}\,dz_1'}_{\text{self-effect of dipole 1}} - \underbrace{\frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_2(z')\epsilon^{-j\beta\rho_{12}}}{\rho_{12}}\,dz_2'}_{\text{due to dipole 2}}$$
$$- \underbrace{\frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_3(z')\epsilon^{-j\beta\rho_{13}}}{\rho_{13}}\,dz_3'}_{\text{due to dipole 3}} - \underbrace{\frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_4(z')\epsilon^{-j\beta\rho_{14}}}{\rho_{14}}\,dz_4'}_{\text{due to dipole 4}}$$

$$I_2^{(\text{amp})} = \underbrace{KC_2 - \frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_2(z')\epsilon^{-j\beta\rho_2} - I_2}{\rho_2}\,dz_2'}_{\text{self-effect of dipole 2}} - \underbrace{\frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_1(z')\epsilon^{-j\beta\rho_{21}}}{\rho_{21}}\,dz_1'}_{\text{due to dipole 1}}$$
$$- \underbrace{\frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_3(z')\epsilon^{-j\beta\rho_{23}}}{\rho_{23}}\,dz_3'}_{\text{due to dipole 3}} - \underbrace{\frac{1}{\Omega}\int_{-l/2}^{l/2} \frac{I_4(z')\epsilon^{-j\beta\rho_{24}}}{\rho_{24}}\,dz_4'}_{\text{due to dipole 4}}$$

ρ for $(r)_{z=0}$; $\quad \rho_1 = (r)_{z_1=0} = |z_1'|$; $\quad \rho_{12} = \sqrt{(z_2')^2 + d^2}$, $\quad \rho_{13} = \sqrt{(z_3')^2 + 4d^2}$
$\rho_{14} = \sqrt{(z_4')^2 + 9d^2}$; $\quad \rho_2 = (r)_{z_2=0} = |z_2'|$, $\quad \rho_{21} = (r_{21})_{z_2=0} = \sqrt{(z_1')^2 + d^2}$
$\rho_{23} = \sqrt{(z_3')^2 + d^2}$; $\quad \rho_{24} = \sqrt{(z_4')^2 + 4d^2}$

Owing to the symmetry of the alignment, we have, for equal inphase impressed voltages, the constants $C_1 = C_4$ and the current distributions $I_1(z) = I_4(z)$ as well as the constants $C_2 = C_3$ and $I_2(z) = I_3(z)$, so that we need find only the relations for $I_1(z) = 0$ and $I_2(z) = 0$, which occur at the free ends of dipoles 1 and 2, *i.e.*, for $z_1 = \pm l/2 = z_2$. Therefore, the open-end boundary condition yields, for $V_1 = V_2 = V_3 = V_4 = V\epsilon^{j0}$ volts,

$$0 = I_1(z) = K\left[C_1 \underbrace{\cos\left(\frac{\beta l}{2}\right)}_{=A} + \frac{V}{2}\underbrace{\sin\left(\frac{\beta l}{2}\right)}_{=B}\right] - \frac{1}{\Omega}\left[\int_{-l/2}^{l/2} \frac{I_1(z')\epsilon^{-j\beta\rho_1'}}{\rho_1'}\,dz_1'\right.$$
$$\left. + \int_{-l/2}^{l/2} \frac{I(z')\epsilon^{-j\beta\rho_{12}'}}{\rho_{12}'}\,dz_2' + \int_{-l/2}^{l/2} \frac{I_3(z')\epsilon^{-j\beta\rho_{13}'}}{\rho_{13}'}\,dz_3' + \int_{-l/2}^{l/2} \frac{I_4(z')\epsilon^{-j\beta\rho_{14}'}}{\rho_{14}'}\,dz_4'\right]$$
$$0 = I_2(z) = K(AC_1 + B) - \frac{1}{\Omega}\left[\int_{-l/2}^{l/2} \frac{I_2(z')\epsilon^{-j\beta\rho_2'}}{\rho_2'}\,dz_2' + \int_{-l/2}^{l/2} \frac{I_1(z')\epsilon^{-j\beta\rho_{21}'}}{\rho_{21}'}\,dz_1'\right.$$
$$\left. + \int_{-l/2}^{l/2} \frac{I_3(z')\epsilon^{-j\beta\rho_{23}'}}{\rho_{23}'}\,dz_3' + \int_{-l/2}^{l/2} \frac{I_4(z')\epsilon^{-j\beta\rho_{24}'}}{\rho_{24}'}\,dz_4'\right]$$

$\rho' = (r)_{z=l/2}$; $\quad \rho_1' = (r_1)_{z_1=l/2} = \left|\frac{l}{2} - z_1'\right|$; $\quad \rho_{12}' = \sqrt{\left(\frac{l}{2} - z_2'\right)^2 + d^2}$

$\rho_{13}' = \sqrt{\left(\frac{l}{2} - z_3'\right)^2 + 4d^2}$; $\quad \rho_{14}' = \sqrt{\left(\frac{l}{2} - z_4'\right)^2 + 9d^2}$; $\quad \rho_2' = (r_2)_{z_2=l/2} = \left|\frac{l}{2} - z_2'\right|$

$\rho_{21}' = \sqrt{\left(\frac{l}{2} - z_1'\right)^2 + d^2}$; $\quad \rho_{23}' = \sqrt{\left(\frac{l}{2} - z_3'\right)^2 + d^2}$; $\quad \rho_{24}' = \sqrt{\left(\frac{l}{2} - z_4'\right)^2 + 4d^2}$

Inasmuch as we now have radiator couplings over distances d, $2d$, and $3d$, we have to

extend the table of integral functions in (857b). We have then in addition to b, c, δ, e, g, h, k, and m, where, for instance, k and m read for applications of Fig. 189,

$$k = -\int_{-l/2}^{l/2} \sin(\beta|u|)\frac{\epsilon^{-j\beta\sqrt{u^2+d^2}}}{\sqrt{u^2+d^2}}\,du = j\left\{2\,\mathrm{Ci}(\beta d) - j2\,\mathrm{Si}(\beta d) - \mathrm{Ci}\left[\beta\left(B+\frac{l}{2}\right)\right]\right.$$

$$\left. - \mathrm{Ci}\left[\beta\left(B-\frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B+\frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B-\frac{l}{2}\right)\right]\right\}; \qquad B = \sqrt{\frac{l^2}{4}+d^2}$$

$$m = -\int_{-l/2}^{l/2} \sin(\beta|u|)\frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+d^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+d^2}}\,du = \frac{j}{2}\left(\epsilon^{-j\frac{\beta l}{2}}\left\{2\,\mathrm{Ci}\left[\beta\left(B-\frac{l}{2}\right)\right]\right.\right.$$

$$- j2\,\mathrm{Si}\left[\beta\left(B-\frac{l}{2}\right)\right] - \mathrm{Ci}\,[\beta(C-l)] + j\,\mathrm{Si}\,[\beta(C-l)] - \mathrm{Ci}\,(\beta d) + j\,\mathrm{Si}\,(\beta d)\Big\}$$

$$+ \epsilon^{j\frac{\beta l}{2}}\left\{2\,\mathrm{Ci}\left[\beta\left(B+\frac{l}{2}\right)\right] - j2\,\mathrm{Si}\left[\beta\left(B+\frac{l}{2}\right)\right]\right.$$

$$\left.\left. - \mathrm{Ci}\,[\beta(C+l)] + j\,\mathrm{Si}\,[\beta(C+l)] - \mathrm{Ci}\,(\beta d) + j\,\mathrm{Si}\,(\beta d)\right\}\right)$$

$$C = \sqrt{l^2+d^2}; \qquad B = \sqrt{\frac{l^2}{4}+d^2}$$

We have also the functions

$$\left.\begin{aligned}
p &= -\int_{-l/2}^{l/2} \cos(\beta u)\frac{\epsilon^{-j\beta\sqrt{u^2+4d^2}}}{\sqrt{u^2+4d^2}}\,du; \\
q &= -\int_{-l/2}^{l/2} \cos(\beta u)\frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+4d^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+4d^2}}\,du \\
s &= -\int_{-l/2}^{l/2} \sin(\beta|u|)\frac{\epsilon^{-j\beta\sqrt{u^2+4d^2}}}{\sqrt{u^2+4d^2}}\,du; \\
t &= -\int_{l/2}^{l/2} \sin(\beta|u|)\frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+4d^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+4d^2}}\,du \\
p' &= -\int_{-l/2}^{l/2} \cos(\beta u)\frac{\epsilon^{-j\beta\sqrt{u^2+9d^2}}}{\sqrt{u^2+9d^2}}\,du; \\
q' &= -\int_{-l/2}^{l/2} \cos(\beta u)\frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+9d^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+9d^2}}\,du
\end{aligned}\right\} \quad (858)$$

$$\left.\begin{aligned} s' &= -\int_{-l/2}^{l/2} \sin(\beta|u|) \frac{\epsilon^{-j\beta\sqrt{u^2+9d^2}}}{\sqrt{u^2+9d^2}}\,du; \\ t' &= -\int_{-l/2}^{l/2} \sin(\beta|u|) \frac{\epsilon^{-j\beta\sqrt{\left(\frac{l}{2}-u\right)^2+9d^2}}}{\sqrt{\left(\frac{l}{2}-u\right)^2+9d^2}}\,du \end{aligned}\right\} \quad (858)$$

where, for the $4d^2$ cases, we have $B = \sqrt{0.25l^2 + 4d^2}$ and $C = \sqrt{l^2 + 4d^2}$ and, for the $9d^2$ cases, we have $B = \sqrt{0.25l^2 + 9d^2}$, $C = \sqrt{l^2 + 9d^2}$ and can then use g, h, k, and m functions with these B and C values. This means that the g, h, k, and m functions use everywhere either $2d$ or $3d$ in place of d. We note, therefore, that we can make the alignment of as many dipoles as we wish and just keep on adding $16d^2$, $25d^2$, $36d^2$, . . . in the g, h, k, and m functions. This is evident, since we have separations d, $2d$, $3d$, $4d$, $5d$, and $6d$ for six identical parallel dipoles, which are excited in phase and with the same voltage V. We have, therefore, the same method as for the thin sinusoidal linear radiators as far as the mutual effects from other radiators are concerned. This can be readily seen from the I_{1z}, I_{2z}, I_{3z}, I_{4z} current distributions and the corresponding entrance currents just given.

TABLE 34. FOR SEPARATION $d = \frac{1}{2}\lambda$

Integral	b	c	δ	e
$l = \lambda/2$	1.6462 + j1 8515	−0 71 + j1 22	−1 418 + j1 6484	1 8148 + j1 1429
$l = \lambda$	2.44 + j1 4175	−1 5569 − j0 746	−1 4185 + j2 44	−0 6721 + j0 88097

Integral	g	h	k	m
$l = \lambda/2$	0 61245 − j0 0446	0 4963 − j1 9991	0 59194 − j0 103	0 45568 − j0 2121
$l = \lambda$	0 3267 + j0 33605	−0 04905 − j0 396	0 99806 − j0 418	0 262 − j0 39615

Integral	p	q	s	t
$l = \lambda/2$	−0 31585 + j0 012	−0 29548 + j0 0671	−0 312 + j0 028	−0 28521 + j0 07747
$l = \lambda$	−0.0615 − j0 1308	−0 108 + j0 0881	−0 59102 + j0 134	−0 296 + 0 302

We then find the values for the various integral functions as in Table 34. The p', q', s', and t' functions are found in the same fashion as, for instance, for the s' integral for $l = \frac{1}{2}\lambda = d$. Since for s', we have $3d$ instead of d, we use the k solution with 2 Ci $(3\beta d)$, $j2$ Si $(3\beta d)$, and $B = \sqrt{0.25l^2 + 9d^2}$. Therefore,

$$s' = j\left\{2\,\mathrm{Ci}\,(3\beta d) - j2\,\mathrm{Si}\,(3\beta d) - \mathrm{Ci}\left[\beta\left(B + \frac{l}{2}\right)\right] - \mathrm{Ci}\left[\beta\left(B - \frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B + \frac{l}{2}\right)\right] + j\,\mathrm{Si}\left[\beta\left(B - \frac{l}{2}\right)\right]\right\}$$

$$3\beta d = 3\frac{2\pi}{\lambda}\frac{\lambda}{2} = 9.42; \qquad \text{Ci}(9.42) = +0.00495$$

$$\text{Si}(3\beta d) = \text{Si}(9.42) = +1.662; \qquad \frac{l^2}{4} = \frac{\lambda^2/4}{4} = \frac{\lambda^2}{16}; \qquad 9d^2 = 9\frac{\lambda^2}{4} = \frac{36\lambda^2}{16}$$

$$B = \sqrt{\frac{l^2}{4} + 9d^2} = \sqrt{\frac{37\lambda^2}{16}} = 1.52\lambda; \qquad B + \frac{l}{2} = 1.52\lambda + 0.25\lambda = 1.77\lambda$$

$$B - \frac{l}{2} = 1.52\lambda - 0.25\lambda = 1.27\lambda; \quad \beta\left(B + \frac{l}{2}\right) = \frac{2\pi}{\lambda}1.77\lambda = 11.1;$$

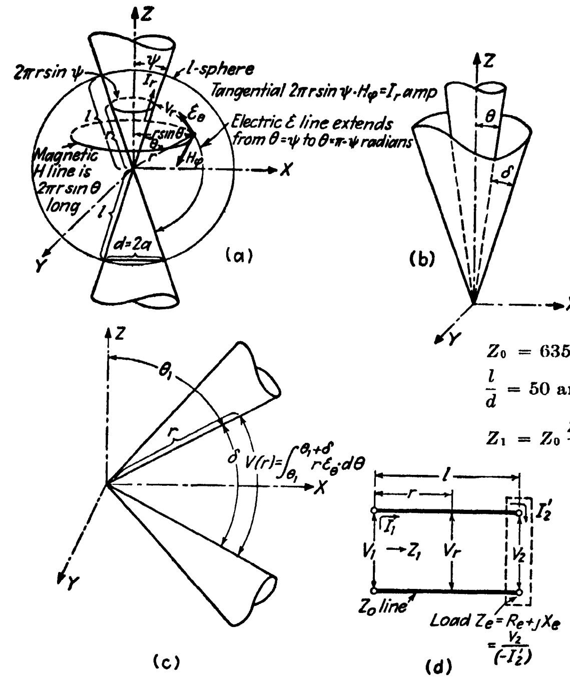
$$\text{Si}(11.1) = 1.57$$

Hence,

$$s' = j(2 \times 0.00495 - j2 \times 1.662 - 0.08558 - 0.1224 + j1.57 + j1.5742)$$
$$= j(-0.29808 - j0.1798) = 0.1798 - j0.29808$$

From the discussions given so far, we note that the Oseen-Hallén theory for the input impedance of cylindrical radiators is based on an approximate integral relation for the current distribution, which is no longer assumed sinusoidal. The Schelkunoff theory is based upon a *direct* approximation of Maxwell's field relations. It is, therefore, based on the electromagnetic field conditions[1] around a radiator and a subsequent solution of the current associated with this field. The biconical transmission line, which is the starting dipole configuration in the Schelkunoff antenna theory, consists of two conical open-ended conductors as in Fig. 191. First, the extension of these conductors is taken very long so as to meet the condition of an infinitely long transmission line with no electromagnetic wave energy being reflected back from the respective open ends. This means that we have only an outward-moving TEM wave. Figure 191*a* represents the conical dipole, Fig. 191*b* represents coaxial cones, and Fig. 191*c* shows diverging cones, since we deal with neither the coaxial case nor the antiaxial case. The conical double line is fed at the common apex. For an infinite length of each conical conductor, the electric $\mathcal{E}_\theta$ lines of force follow the meridians *of* spheres that are concentric with the common apex. This assumes perfect conductivity of the cones and a perfect dielectric in the medium in which the conical conductors are embedded. The H_φ lines of force are likewise circles. Since the H_φ circles keep in step with the circular $\mathcal{E}_\theta$ lines of force as we move outward along the conical conductors, we deal with a *uniform* transmission line with an inductance L_0 and capacitance C_0 per unit length as given in Fig. 191. For an infinite length, we have only an outgoing TEM wave with the incident values $\mathcal{E}_{\theta_i}$ and H_{φ_i}, since

[1] SCHELKUNOFF, S. A., *Proc. IRE*, **29**, 493, September, 1941; **34**, 23P, January, 1946; **34**, 265, May, 1946; "Electromagnetic Waves," Chap. XI, D. Van Nostrand Company, Inc., New York, 1943.



Z_0 = 635 ohms for

$\frac{l}{d}$ = 50 and Z_0 = 828 ohms for $\frac{l}{d}$ = 250

$$Z_1 = Z_0 \frac{Z_e \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + Z_e \sin(\beta l)}$$

Region I is enclosed by l sphere
Region II is outside the l sphere
d is diameter of cone at the end face of the cone of length l

Transverse voltage $V(r)$ is the line integral of the *meridian* $\mathcal{E}_\theta$ field, or

$$V(r) = -\int_{\theta=\psi}^{\theta=(\pi-\psi)} r\mathcal{E}_\theta d\theta \text{ volt}$$

Longitudinal current $I(r)$ is the line integral of the *tangential* H_φ field, or

$$I(r) = 2\pi r \sin\psi H_\varphi \text{ amp}$$

$$V(r) = V_i \epsilon^{j(\omega t - \overrightarrow{\beta r})} + V_r \epsilon^{j(\omega t + \overleftarrow{\beta r})}$$
$$I(r) = I_i \epsilon^{j(\omega t - \overrightarrow{\beta r})} + I_r \epsilon^{j(\omega t + \overleftarrow{\beta r})}$$
$$V_i = Z_0 I_i;\ V_r = -Z_0 I_r;\ \mathcal{E}_\varphi = 0 = H_\theta = \mathcal{E}_r = H_r$$
$$\mathcal{E}_{\theta i} = \frac{60 I_i \epsilon^{-j\beta r}}{r \sin\theta};\ H_{\varphi i} = \frac{I_i \epsilon^{-j\beta r}}{2\pi r \sin\theta}$$
$$\mathcal{E}_{\theta r} = -\frac{60 I_r \epsilon^{j\beta r}}{r \sin\theta};\ H_{\varphi r} = \frac{I_r \epsilon^{j\beta r}}{2\pi r \sin\theta}$$

} $\mathcal{E}$ and H fields and transverse voltages and longitudinal currents for the principal TEM wave

$$L_0 = \frac{\mu \log_\epsilon(q)}{\pi} \text{ henrys/m};\ C_0 = \frac{\pi\kappa}{\log_\epsilon(q)} \text{ farad/m};\ q = \cot\left(\frac{\psi}{2}\right) \cong \frac{2}{\pi} \text{ for small angles}$$

$$Z_0 = \frac{L_0}{C_0} = \frac{\sqrt{\mu/\kappa}}{\pi} \log_\epsilon(q) = 120 \log_\epsilon(q) = 276 \log_{10}\left[\cot\left(\frac{\psi}{2}\right)\right] \cong \underbrace{276 \log_{10}\left(\frac{4l}{d}\right)}_{\text{for } \psi \text{ small}} \text{ ohms}$$

$$\frac{dV}{dr} = j\omega L_0 I$$
$$\frac{dI}{dr} = -j\omega C_0 V$$

} holds for (a), (b), and (c), but C_0 and L_0 values are different for the three cases

Fig. 191. Relations for the biconical line. (a) Conical dipole along $+Z$ and $-Z$ axes (b) Coaxial bicones. (c) Diverging cones. (d) Equivalent conical dipole.

there can be no reflected fields $\mathcal{E}_{\theta_r}$ and H_{φ_r} coming from the open ends. As far as the classic line theory is concerned, we have, for a finite length l of the two conical conductors, transverse voltages $V(r) = V_r$ and longitudinal currents $I(r) = I_r$ across and along the cones, respectively, due to outgoing (incident) voltage V_i and current I_i wavelike disturbances and to end-reflected voltage V_r and current I_r disturbances. We have already found in this text that TEM waves belong to conductors, since we have never found anything like the principal waves in space propagation without the association of current-carrying conductors.[1] Inasmuch as in the well-known differential equations dV/dr and dI/dr for perfectly conducting cones in a perfect dielectric only the inductances and capacitances L_0 and C_0 assume different magnitudes, we have the feature of also including shapes that are degenerations of bicones. Thus, if we move the common apex of Fig. 191*b* to infinity, we arrive at the shape of two coaxial cylinders. Doing the same thing with the apex in the representation of Fig. 191*c*, we arrive at the shape of a Lecher wire system. Now we know that open-ended lines, as in Fig. 191, do radiate electromagnetic field energy. The classic transmission-line theory *cannot* account for it, nor, therefore, can the associated transverse electromagnetic fields of the principal TEM wave. Line equations are based only on voltage and current distributions existing on the biconical conductors. For this reason, Schelkunoff divides the space around a biconical conductor system of finite length l into a region I and a region II. Region I is a sphere about the common apex with a radius equal to the length l of each cone. Region II is the outside space of this sphere. We deal, therefore, with spherical coordinates. The sphere passing through the ends of the biconical radiators is a surface of discontinuity and the boundary between the space I of the antenna region and the outside space II. We have then an infinitesimal gap between the apexes across which the driving voltage is applied. The conical dipole may then be taken as equivalent network, as is shown in Fig. 191*d* with a finite impedance Z_e connected across the ends of the equivalent double line. As far as stationary principal waves (TEM waves) are concerned, we have

$$V_r = \{V_2 \cos[\beta(l-r)] + jZ_0 I_2 \sin[\beta(l-r)]\}\epsilon^{j\omega t} \quad \text{volts}$$

$$I_r = \left\{ I_2 \cos[\beta(l-r)] + j\frac{V_2}{Z_0}\sin[\beta(l-r)] \right\} \epsilon^{j\omega t} \quad \text{amp}$$

from Eqs. (607) and (608) for $\cosh(jx) = \cos x, \sinh(jx) = j \sin x$, V_1 and I_1 are the applied voltage and corresponding entrance current; and for the general condition, also a current I_2 at the far end is possible.

for the transverse voltage V_r and the longitudinal cone current I_r at any

[1] Except in the distant radiation region.

distance r from the common apex (infinitesimal gap between respective apexes across which V_1 is applied). According to the line theory (not accounting for radiation), the current I_2 *at* the *open ends* of cones l meters long *vanishes*, and $V_r = V_2 \cos [\beta(l - r)]$ and $I_r = j(V_2/Z_0) \sin [\beta(l - r)]$ if we omit the time function. If the l sphere were a perfect metallic sheet, the classic line theory would yield $V_r = jZ_0 I_2 \sin [\beta(l - r)]$ and $I_r = I_2 \cos [\beta(l - r)]$. Neither case prevails, but a condition for which we have an equivalent double line as in Fig. 191*d* with an impedance $Z_e = R_e + jX_e$ across the ends as far as the action of the TEM wave is concerned if we ignore the end capacitance effect.

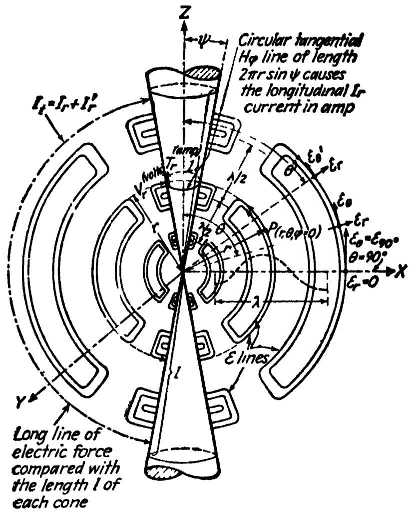


FIG. 192. The electric field configuration for the first order of TM wave.

The impedance Z_e *cannot* be V_2/I_2, since, for the principal wave with the far end open-ended, there can be no I_2 current. We can, however, assume a current flow that is $-I_2'$, which accounts for the discontinuity from region I into region II of the l sphere passing through the open ends. This is due to the fact that our conical conductors act like a radiator. We may then have to imagine that a wave guided by the conical surfaces moves along these surfaces as a TEM wave, experiences at the surface of the l sphere partial reflection back into region I and partial continuation into region II. Now region II cannot support a TEM wave, since there are no longer conductors. Quite a portion of the region II, *especially* in the neighborhood of the surface of the l sphere also has, in addition to the transverse $\mathcal{E}_\theta$ field, a pronounced radial $\mathcal{E}_r$ field. This means that at the surface of the l sphere, as far as the field energy penetration into region II is concerned, there is no TEM wave on the outer side of the l boundary to match the TEM wave coming from the inner side. We *must have* a condition of *continuity* at the surface of the l sphere. This condition is satisfied if we add suitable complementary fields in region I, which includes the required radial electric field. The added complementary waves then take care of the nonuniform reflection at the surface of the l sphere. If this sphere were a metal surface all over instead of only at the end surfaces of respective dipole radiators, uniform and complete wave reflection would occur at the spherical surface. Figure 192 gives an electric field configuration in the presence of a conical dipole, which is possible. It is the case of the first-order TM spherical wave. It is noted that in the equatorial XY plane, the radial $\mathcal{E}_r$ component vanishes. The tangential $\mathcal{E}_\theta$ component vanishes along the $\pm Z$ axis of the biconical radiator, while the radial component *remains* normal to the end surfaces. This is evident, since the conducting cone represents essentially the seat of the conduction current (displacement current in a conductor negligible in comparison with the conduction current,) while the electric lines

of force normal to the end surfaces constitute part of the displacement current of the pure dielectric *near region* about the cones.

Inasmuch as the *total* longitudinal current of I_t amperes flowing over the conical surface at a distance of r meters from the common apex is $2\pi r H_\varphi \sin\psi$ amperes and the circular H_φ field about the cones comes from the electromagnetic field due to the principal TEM wave and the additional electromagnetic field of the complementary waves, we have to add the effect of the respective magnetic fields H_φ and H'_φ and find

$$I_t = \underbrace{2\pi r H_\varphi \sin\psi}_{\text{due to TEM wave}} + \underbrace{\Sigma 2\pi r H'_\varphi \sin\psi}_{\text{due to complementary waves}} = I_r + \Sigma I'_r \qquad \text{amp} \qquad (859)$$

The current contribution $\Sigma I'_r$ is then caused by the *higher order* of complementary *waves*. Since the voltage V_r between two conical radiators, as in Fig. 192, is the integral of the tangential meridian component over the surface of r = constant, we find

$$V_r = \underbrace{-r\int_{\theta=\psi}^{\theta=\pi-\psi} \mathcal{E}_\theta \, d\theta}_{\text{due to TEM wave}} - \underbrace{\Sigma\left(r\int_{\psi}^{\pi-\psi} \mathcal{E}'_\theta \, d\theta\right)}_{\text{due to complementary waves (gives zero value)}} \qquad \text{volts} \qquad (859a)$$

We note, therefore, that for the conical dipole, we no longer have a *single* mode condition, as in commercial feeder lines, where the distance across respective conductors is small compared with the length dimension and also small compared with the operating wavelength. For the conical dipole of Fig. 192, the circular meridian path of radius $r = l$ from one conductor surface to the other conductor surface is much longer than the l dimension of each cone. As a matter of fact, most paths accounting for the different V_r values of the transverse voltage are appreciable in terms of the wavelength. Therefore, we no longer have, as for Lecher wires and concentric lines, short and straight electric lines of force from one conductor to the other conductor, except for the curved and small electric field lines at the free ends of such lines. Therefore, for most of the conductor length of the conical dipole, we have a configuration of lines of force that is expressible *not* by a single transmission mode of the classical TEM wave and well-known propagation constant $\Gamma = \alpha + j\beta \cong j\beta = j\omega\sqrt{C_0L'_0} = j\omega/c'$, where c' is the phase velocity, *but expressible by a multiple mode*. We have then possibilities well known from the theory of the electromagnetic horn (Barrow). This means that we can have waves with several characteristic impedances and several propagation constants, due to transverse dimensions that are no longer negligible in terms of the operating wavelength.

From this discussion, it is clear that for very small r values along the respective conical conductors, we have essentially the effect of the TEM wave, since the transverse dimension can be only small and negligible with respect to the wavelength. This means then that at the input terminals ($r = 0$), the entrance current is $I_t = I_1 + I'_1 = I_1 + 0 = I_1$ amperes, since the current contribution I'_1 must vanish. We found that

the transverse voltage V_r is the *same all along the conical dipoles* [Eq. (859*a*)], as for the classical TEM waves, but $\Sigma I'_r$ must be added and becomes all the more significant in comparison with the I_r value (due to the principal wave) the larger the transverse path between corresponding conductor elements. The relative magnitude reaches its maximum effect at the very open ends, since then $I_t = I_2 + I'_2$. For a small cross section at the open ends, we have $I_t = 0$, since both currents must be coming from the source, and we have $I_2 = -I'_2$. For a large cross section at the very ends, some current will creep over the circular edge into the end face, and we have, in effect, two admittances in parallel, since

$$\frac{1}{Z_t} = \frac{1}{Z_2} + \frac{1}{Z'_2} = \frac{1}{Z_2} + \frac{1}{Z_e} = \frac{I_2}{V_2} + \frac{I'_2}{V_2} \qquad \text{mhos}$$

This expresses that $Z_2 = V_2/I_2$ represents the loading in complex ohms due to large end faces and $Z_e = -V_2/I'_2$ represents the loading due to the electromagnetic field escaping from the conical conductors into region II.

For perfectly conducting dipole cones, we have $\Sigma I'_r = I'_{r_1} + I'_{r_3} + I'_{r_5} + I'_{r_7} + \cdots$, since only *odd* complementary waves play a part. Due to the imperfect reflection, the voltage maximum will not occur at the very end, but without in any way affecting the attenuation constant α of the principal wave. In region II, it is assumed that only TM waves with radial $\mathcal{E}_r$, tangential $\mathcal{E}_\theta$, and circular H_φ components will be produced, where the dipole axis along the $\pm Z$ direction refers to $\theta = 0$ and $\theta = \pi$. Since region II extends all the way out to infinity, the second Hankel function can be used, and we have

$$\mathcal{E}_\theta^{(\text{volts/m})} = \frac{K_n P'_n(\cos\theta)}{j\omega\kappa r^{1.5}} [nH^{(2)}_{n+\frac{1}{2}}(\beta r) - \beta r H^{(2)}_{n-\frac{1}{2}}(\beta r)];$$

$$H_\varphi^{(\text{amp/m})} = \frac{K_n}{r^{0.5}} P'_n(\cos\theta) H^{(2)}_{n+\frac{1}{2}}(\beta r)$$

$$\mathcal{E}_r^{(\text{volts/m})} = -\frac{K_n n H_{n+\frac{1}{2}}(\beta r)}{j\omega\kappa r^{1.5}\sin\theta}[\cos\theta\, P'_n(\cos\theta) - P'_{n+1}(\cos\theta)]$$

Therefore, *only* $P'_n(\cos\theta)$ functions are needed for distances $r > l$ from the radiation center. The quantity n is an integer, and we have an *infinite number of possible TM waves.* We need only such a number n, *i.e.*, number of TM waves, as will satisfy the boundary conditions. Schelkunoff has shown that if the characteristic impedance Z_0 is large (fine conical antenna wire) with zero total current I_t at the ends of the cones, we have

$$Z_e = R_e + jX_e = \frac{Z_0^2}{G(\beta l) + jF(\beta l)} \qquad \text{ohms;} \qquad L = \beta l = \frac{2\pi}{\lambda} l$$

$$\left.\begin{aligned} G(L) &= 60[0.5772 + 2.303 \log_{10}(2L) - \text{Ci}(2L)] \\ &\quad + 30[0.5772 + 2.303\log_{10}(L) - 2\,\text{Ci}(2L) + \text{Ci}(4L)]\cos(2L) \\ &\quad + 30[\text{Si}(4L) - 2\,\text{Si}(2L)]\sin(2L) \\ F(L) &= 60\,\text{Si}(2L) + 30[\text{Ci}(4L) - 2.303\log_{10}(L) - 0.5772]\sin(2L) \\ &\quad - 30\,\text{Si}(4L)\cos(2L) \end{aligned}\right\} \qquad (860)$$

Inasmuch as we have for the conical dipole of length l of each cone, the equivalent network of Fig. 191d, we find, for a fine conical dipole for $l = \lambda/4$, that the equivalent $\frac{1}{2}\lambda$ wavelength section acts as an impedance transformer for which $Z_0 = \sqrt{Z_1 Z_e}$ ohms and the input impedance Z_1 yields

$$Z_1 = \frac{Z_0^2}{Z_e} = G(L) + jF(L) = R_1 + jX_1 \quad \text{ohms} \tag{861}$$

We find then

$$l = \frac{\lambda}{4};\ L = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2} = 1.57;\ 2L = 3.14;\ 4L = 6.28;\ \sin(2L) = \sin(\pi) = 0$$

$\cos(2L) = \cos(\pi) = -1$

and

$$\begin{aligned} R_1 &= 60[0.5772 + 2.303 \log_{10}(3.14) - \mathrm{Ci}(3.14)] \\ &\quad - 30[0.5772 + 2.303 \log_{10}(1.57) - 2\,\mathrm{Ci}(3.14) + \mathrm{Ci}(6.28)] \quad \text{resistive ohms} \\ X_1 &= 60\,\mathrm{Si}(3.14) + 30\,\mathrm{Si}(6.28) \quad \text{reactive ohms}; \quad \log_{10}(3.14) = 0.49693 \\ \log_{10}(1.57) &= 0.1959; \quad \mathrm{Ci}(3.14) = 0.075 \\ \mathrm{Ci}(6.28) &= -0.023; \quad \mathrm{Si}(3.14) = 1.8515; \quad \mathrm{Si}(6.28) = 1.43 \\ X_1 &= 60 \times 1.8515 + 30 \times 1.43 = 153.99 \text{ ohms} \\ R_1 &= 60(0.5772 + 2.303 \times 0.49693 - 0.075) \\ &\quad - 30(0.5772 + 2.303 \times 0.1959 - 2 \times 0.075 - 0.023) \\ &= 60 \times 1.6472 - 30 \times 0.8552 = 73.176 \text{ ohms} \end{aligned}$$

From this numerical result, we note that the radiation resistance of a fine half-wave conical dipole is again about the value found for the thin cylindrical half-wave radiator but the associated reactance is much larger than 42.9 ohms (Table 33), since it is 153.99 ohms. The formulas for the functions $G(L)$ and $F(L)$, given in (860), are also given by the series

$$\left.\begin{aligned} G(L) &= \sum_{m=0}^{m=\infty} k_m J_{2m+\frac{3}{2}}(L); \qquad F(L) = -\sum_{m=0}^{m=\infty} k_m J_{2m+\frac{3}{2}}(L) N_{2m+\frac{3}{2}}(L); \\ k_m &= \frac{30\pi L(4m+3)}{(m+1)(2m+1)} \end{aligned}\right\} \tag{861a}$$

and the curves in Fig. 193. It is the half-wave dipole that is of greatest interest in short-wave work. For other lengths, the input impedance Z_1 is found from the equivalent dipole network of Fig. 191d and Eq. (609) as

$$Z_1 = Z_0 \frac{Z_e \cosh(\alpha + j\beta l) + Z_0 \sinh(\alpha + j\beta l)}{Z_0 \cosh(\alpha + j\beta l) + Z_e \sinh(\alpha + j\beta l)} \cong Z_0 \frac{Z_e \cosh(j\beta l) + Z_0 \sinh(j\beta l)}{Z_0 \cosh(j\beta l) + Z_e \sinh(j\beta l)}$$
$$= Z_0 \frac{\cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_e \sin(\beta l)};$$

$$Z_1 = Z_0 \frac{\cos(L) + jZ_0 \sin(L)}{Z_0 \cos(L) + jZ_e \sin(L)} \text{ complex ohms}; \quad Z_e = \frac{Z_0^2}{G(L) + jF(L)} \text{ complex ohms}$$

Substituting the value for Z_e in the expression for Z_1 and rearranging give

$$Z_1^{(\text{complex ohms})} = R_1 + jX_1$$

$$= \frac{\overbrace{G(L) - j\{0.5Z_0 \sin(2L) + F(L)\cos(2L) - [F^2(L) + G^2(L)][\sin(2L)]/(2Z_0)\}}^{A}}{\sin^2(L) + \dfrac{F(L)}{Z_0}\sin(2L) + [F^2(L) + G^2(L)][\cos^2(L)]/Z_0^2} \tag{861b}$$

The fine biconical dipole is at resonance when the reactance vanishes, *i.e.*, the factor $A = 0$, yielding

$$\tan(2L) = -\frac{2Z_0F(L)}{Z_0^2 - G^2(L) - F^2(L)} = -\frac{2F(L)}{Z_0 - [G^2(L) + F^2(L)]/Z_0} \cong -\frac{2F(L)}{Z_0}$$

with solution

$$2L = k\pi - \frac{2F(k\pi/2)}{Z_0}; \qquad k = 1, 2, 3, \cdots$$

showing that the input reactance of the conical dipole vanishes whenever

$$\frac{4l}{k\lambda} = 1 - \frac{2F(k\pi/2)}{kZ_0} = 1 - \frac{120\,\text{Si}(k\pi) + 60(-1)^{k+1}\,\text{Si}(2k\pi)}{\pi k Z_0}$$

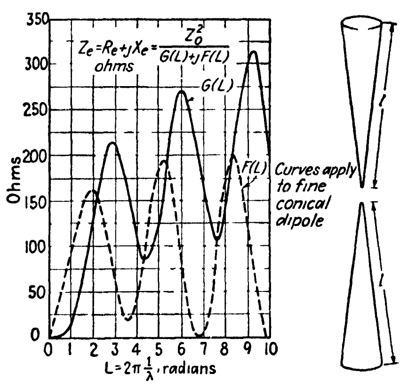


FIG. 193. Curves for the radiation impedance of a conical dipole.

Hence, the resonance length is $2l = k\lambda/2$. Owing to the end effect, the dipole must be somewhat shorter to resonate. For instance, in the case of two Lecher wires with an interaxial spacing s and a radius a, we have $Z_0 = 276 \log_{10}(s/a)$ ohms, and

$C = \dfrac{120(s - a)}{cZ_0^2}$ farads capacitance effect across the open end (due to warped electric field)

$G = \dfrac{120\pi^2 s^2}{Z_0^2\lambda^2}$ mhos conductance in parallel with C, due to *radiated power* $\qquad c = 3 \times 10^8$ m/sec

$$\lambda = 4l + 4cCZ_0 = 4l + 4 \times 3 \times 10^8 \frac{120(s-a)}{3 \times 10^8 Z_0} = 4l + \frac{480(s-a)}{Z_0} \quad \text{m}$$

The quantity $480(s - a)/Z_0$ denotes then the *shortening effect.*

Figure 194 gives the input resistance R_1 and the input reactance X_1 of the fine conical dipole, where each small segment of the radiator is

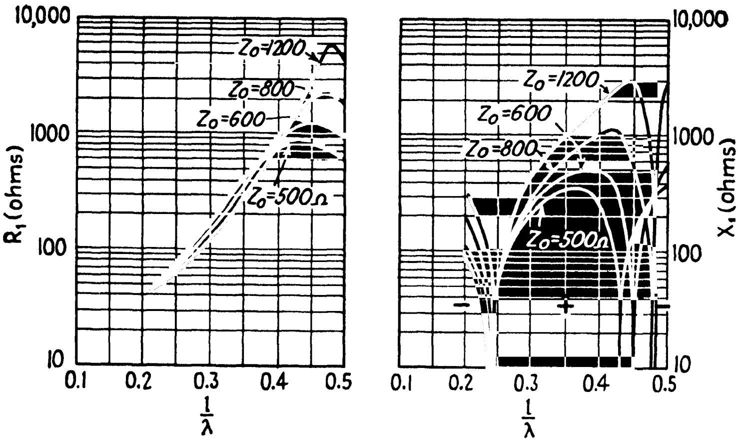


FIG. 194. Input resistance and reactance for a hollow cylindrical dipole in unobstructed radiation space. For a vertical dipole over a perfect reflector divide Z_0 by 2.

taken as a section of a cone for the capacitance and inductance per unit length as

$$L_0 = \frac{\mu \log_\epsilon (p)}{\pi} \text{ henrys/m}; \qquad C_0 = \frac{\pi\kappa}{\log_\epsilon (p)} \text{ farads/m}; \qquad p = \frac{4r}{d}$$

$$\mu \cong 1.257 \times 10^{-6} \text{ henry/m}; \qquad \kappa = 8.854 \times 10^{-12} \text{ farad/m}$$

if the conical conductors are embedded in air. According to Fig. 191, the conical dipole of Fig. 191*a* can be changed to coaxial cones as in Fig. 191*b*, with the coaxial line as an asymptotic approximation, or can be changed to Fig. 191*c*, with the parallel line as an approximation. The well-known relations for the principal or TEM wave are

$$\frac{dV}{dr} = -j\omega L_0 I; \qquad \frac{dI}{dr} = -j\omega C_0 V; \qquad \frac{d\left(\frac{1}{C_0}\frac{dI}{dr}\right)}{dr} = -\omega^2 LI;$$

$$V(r) = \sqrt{Z_0}(K_1\epsilon^{-j\beta r} + K_2\epsilon^{j\beta r})$$

$$I(r) = \frac{K_1\epsilon^{-j\beta r} - K_2\epsilon^{j\beta r}}{\sqrt{Z_0}}; \qquad Z_0 = 276 \log_{10} (p)$$

where d denotes the diameter of the cross section of the conical conductor at the distance r from the apex. The differential relations for dV/dr

and dI/dr hold for all configurations, such as, for instance, Fig. 191*a*, *b*, and *c*, if the proper C_0 and L_0 values are used. The relations for $V(r)$ and $I(r)$ are symmetrical and predict sinusoidal V and I distributions in contradiction to the Hallén and King solutions, for which only V is sinusoidal. The formulas for the entrance impedance Z_1 at the infinitesimal gap of both cones are based on approximations that must be taken as they stand. Nevertheless, the results of the curves given for R_1 and

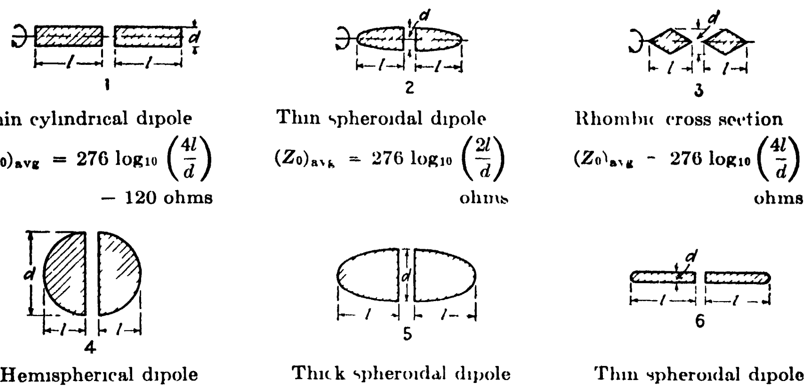


FIG. 195 Different forms of dipole radiators Cross sections of surfaces of revolution about the dipole axis

X_2 are in fair agreement with experiment, even though applying only to fine conical radiators. The reactance curves show that for lower characteristic impedance values Z_0, the resonance is less sharp and somewhat off the condition that would be expected for integer multiples of half waves. This does not matter, since even in the fine-wire theory, we can "ideally" meet the conditions of only $l = \lambda/2$, etc., distributions and the dipole length is always a few per cent shorter for actual resonance. The condition for *less sharp* resonance indicates, however, that for *thicker* dipoles, we have a wide-band termination feature, even though the formulas given cannot claim great accuracy for thick radiators. Schelkunoff uses the curves for the fine biconical radiator as a first approximation. The characteristic impedance is then taken as an average value, defined by

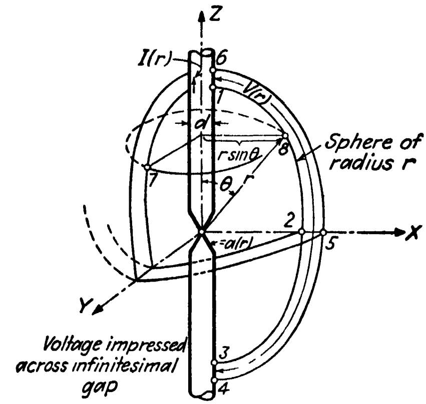
$$(Z_0)_{avg} = \frac{1}{l}\int_0^l Z_0\,dr = \frac{1}{l}\int_0^l Z_0(r, d)dr \qquad \text{ohms}$$

and given in Fig. 195.

The case of the spheroidal dipole (prolate spherical radiators) has been treated by Stratton and Chu,[1] where it is also assumed that the

[1] CHU, L. J., and J. A. STRATTON, *Jour. Appl. Phys*, March, 1941, p. 241.

dipole voltage is applied across a very small gap. It is shown that the hemispheres yield very flat reactance maxima (wide-band feature). For small values of d/l, the resonance for possible oscillation modes becomes rather sharp (narrow-band input impedance). For very small values of d/l, we have essentially the results as found for thin cylindrical dipoles



$$\frac{dV}{dr} = -j\omega L_0 I \text{ volt/m}; \quad \frac{dI}{dr} = -j\omega C_0 V \text{ amp/m}$$

$$L_0 = \frac{\mu \log_\epsilon [\cot(\psi/2)]}{\pi} \text{ henrys/m}; \quad C_0 = \frac{\pi\kappa}{\log_\epsilon [\cot(\psi/2)]} \text{ farads/m}$$

if electric $\mathcal{E}$ lines are (as in this illustration) nearly along meridians and $\sin\psi = a/r$ (cone angle ψ with the Z axis, varies continuously with r).

For the curvilinear rectangle 1-8-2-3-4-5-6-1, we have

$$dV = \underbrace{\left(-j\omega\mu \int_{\theta=\psi}^{\theta=\pi-\psi} rH_\varphi d\theta\right) dr \text{ volts}}_{\text{Faraday's induction law } (\text{curl } \mathcal{E} = -j\omega\mu H)}$$

For the magnetic 7-8 line, taken for the entire circulation, we have

$$\underbrace{\underbrace{2\pi r \sin\theta}_{\substack{\text{four times the} \\ \text{length 7 to 8 in meters}}} H_\varphi = I(r) + jI_d \text{ amp}}_{\text{Ampère-Maxwell law } [\text{curl } H = I(r) + jI_d]}$$

$I(r)$ is the conduction current and

$$jI_d = j2\pi\omega\kappa r^2 \int_{\theta=\psi}^{\theta=\theta} \mathcal{E}_r \sin\theta \, d\theta \text{ is the radial displacement current.}$$

FIG. 196. Cylindrical linear radiator (dipole) of $d = 2a$ diameter and tapered input terminals (conical). $d = 2a$ for cylinder, since $a(r) = a$.

and the classic theory applies again, since practically sinusoidal current distribution prevails. For larger values of d/l, a condition may be reached for which the dipole length $2l$ has to be somewhat longer than $\lambda/2$ in the case of a $\lambda/2$ radiator, in order to neutralize the reactance of the input impedance of the dipole gap. For the upper limit, of spheroidal dipoles, *i.e.*, the case of a two-hemisphere dipole, we have then no condi-

tion for which the capacitive reactance of the input impedance can be canceled, without departing from the shape of the limiting case, or using stubbing.

S. A. Schelkunoff[1] has also recently applied his theory based on complementary and principal waves to an analysis of cylindrical and other nonconical radiators. This theory also includes the case for which the diameter of the cylindrical dipole radiator is not small compared with the operating wavelength. Two cylindrical conductors with tapered input terminals (Fig. 196) can then be used for causing electric $\mathcal{E}_\theta$ lines along meridians as required by the TEM waves. Applying Maxwell's curl equations, we obtain the complete solution and find the gradient

$$\frac{dV}{dr} = -j\omega\mu \int_{\theta=\psi}^{\theta=\pi-\psi} rH_\varphi \, d\theta \qquad \text{volts/m}$$

from Faraday's induction law. But the amperian conduction current $I(r)$ and the Maxwellian displacement current I_d, as found from the curl H relation of Eq. (1), give the total possible current contribution. For the principal, or TEM, wave, which holds for the classic transmission-line relations, the radial displacement current is not accounted for, since we have only transverse field components with respect to the direction of propagation. Introducing the complete value for H_φ, given by the *Ampère-Maxwell law* in Fig. 196, we find the *complete* expression

$$\frac{dV}{dr} = -j\omega\mu \int_{\psi}^{\pi-\psi} r\left[\frac{I(r)}{2\pi r \sin\theta} + \frac{j2\pi\omega\kappa r^2}{2\pi r \sin\theta}\int_{\psi}^{\theta} \mathcal{E}_r \sin\theta \, d\theta\right] d\theta \qquad \text{volts/m}$$

and the comparisons

$$\left.\begin{aligned} \frac{dV}{dr} &= -j\omega LI + \underbrace{\mu\kappa\omega^2 r^2 \int_{\theta=\psi}^{\theta=\pi-\psi} \frac{d\theta}{\sin\theta} \int_{\theta=\psi}^{\theta=\theta} \mathcal{E}_r \sin\theta \, d\theta}_{\text{effect of radial intensity } \mathcal{E}_r} \qquad \textit{complete solution} \\ \frac{dV}{dr} &= -j\omega LI \qquad \text{solution based on the } \textit{classic transmission-line} \text{ theory (TEM waves)} \end{aligned}\right\} \text{volts/m}$$

The complete expression for the dI/dr relation is found by using from the curl H relation the components in spherical coordinates, as given in Eq. (139). The only portion that remains for the case of a perfect dielectric medium is then

$$\frac{\partial(rH_\varphi)}{\partial r} = -j\omega\kappa r\mathcal{E}_\theta$$

Introducing the expression for H_φ of the Ampère-Maxwell law, as given in Fig. 196, we find

$$-j\omega\kappa r\mathcal{E}_\theta = \frac{1}{2\pi\sin\theta}\frac{dI}{dr} + \frac{\partial\left(\dfrac{j\omega\kappa r^2}{\sin\theta}\displaystyle\int_{\theta=\psi}^{\theta=\theta} \mathcal{E}_r \sin\theta \, d\theta\right)}{\partial r}$$

[1] *Proc. IRE*, **34**, 23P, January, 1946.

But

$$V(r) = \int_{\theta=\psi}^{\theta=\pi-\psi} r\mathcal{E}_\theta \, d\theta \qquad \text{volts}$$

Therefore, integrating the $-j\omega\kappa r\mathcal{E}_\theta$ equation from $\theta = \psi$ to $\theta = \pi - \psi$ yields

$$-j\omega\kappa V = -\frac{\log_\epsilon\left[\cot\left(\frac{\psi}{2}\right)\right]}{\pi}\frac{dI}{dr} + \frac{d\left(\int_{\theta=\psi}^{\theta=\pi-\psi} \frac{j\omega\kappa r^2}{\sin\theta}\,d\theta \int_{\theta=\psi}^{\theta=\theta} \mathcal{E}_r \sin\theta\,d\theta\right)}{dr}$$

and we obtain the comparisons

$$\left.\begin{aligned} \frac{dI}{dr} &= -j\omega CV - j\omega C\,\frac{d\left(\int_{\theta=\psi}^{\theta=\pi-\psi} \frac{r^2\,d\theta}{\sin\theta} \int_{\theta=\psi}^{\theta=\theta} \mathcal{E}_r \sin\theta\,d\theta\right)}{dr} \quad &&\textit{complete solution} \\ \frac{dI}{dr} &= -j\omega CV \quad &&\text{solution based on the \textit{classic transmission-line} theory (TEM waves)} \end{aligned}\right\} \text{amp/m}$$

The values of the inductance L_0 and capacitance C_0 per unit length for TEM waves on a cylinder of any radius, except for the correction for the small radial electric field intensity in the direction of wave propagation, is found from

$$\frac{dV}{dr} = -j\omega\mu \int_0^\pi rH_\varphi \, d\theta \qquad \text{and}$$

$$\frac{dV}{dr} = -j\omega I \underbrace{\frac{\mu}{2\pi} \int_{\theta=0}^{\theta=\pi} \frac{r\,d\theta}{a + r\sin\theta}}_{=L_0} = -j\omega I L_0 \qquad \text{volts/m}$$

and C_0 can be found from $L_0C_0 = \mu\kappa = 1/v^2$, where v is the velocity of propagation in meters per second. Upon integration, Schelkunoff finds that in free space ($\mu = \mu_0$ and $\kappa = \kappa_0$), for the radius a of the cylindrical dipole where it is intercepted by the sphere of radius r

$$\left.\begin{aligned} Z_0 &= \sqrt{\frac{L_0}{C_0}} = \frac{240(r/a)}{\sqrt{1-(r/a)^2}} \tan^{-1}\left(\sqrt{\frac{a-r}{a+r}}\right) \text{ for } r < a; \\ Z_0 &= 120 \text{ ohms} \qquad \text{for } r = a \\ Z_0 &= \frac{240(r/a)}{\sqrt{(r/a)^2 - 1}} \tanh^{-1}\left(\sqrt{\frac{r-a}{r+a}}\right) \qquad \text{for } r > a \\ \int_r^a Z_0\,dr &= 240a\left[\sqrt{1-\left(\frac{r}{a}\right)^2} \tan^{-1}\left(\sqrt{\frac{a-r}{a+r}} - \frac{a-r}{2a}\right)\right] \\ &\qquad \text{m}\cdot\text{ohms for } r < a \\ \int^r Z_0\,dr &= 240a\left[\sqrt{\left(\frac{r}{a}\right)^2 - 1}\, \tanh^{-1}\left(\sqrt{\frac{r-a}{r+a}} - \frac{r-a}{2a}\right)\right] \\ &\qquad \text{m}\cdot\text{ohms for } r > a \end{aligned}\right\} \quad (861c)$$

The integral expressions are of great importance in the evaluation of the input impedance of dipole radiators, since we no longer have a fixed Z_0 value. The curves of Fig. 197 show the variations of the nominal characteristic impedance Z_0 with the radius r of the sphere for a fixed radius a of the cylindrical dipole. From it, the average value can be found.

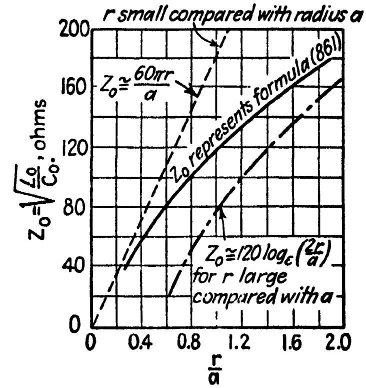


FIG. 197. Nominal impedance $\sqrt{L_0/C_0}$ against the ratio of the radius a of the cylinder to the radius r of the sphere. a is radius of cylindrical dipole, and r is radius of sphere (Fig. 196).

141. Radiation from Circular Antennas. Radiation of circular-shaped aerials with appreciable wave distribution, such that for the radius R of the loop radiator, we have $2\pi R = \lambda/2$, λ, 2λ, 3λ, have been discussed before in current literature,[1] with solutions for the electromagnetic field in the plane of the circular antenna and normal to it. For a maximum value of I_0 amperes of the circular radiator of Fig. 198, we find for a line element dl, expressed in meters, at a distant space point P located at r, φ, θ

$$d\mathcal{E}^{(\text{volts/m})} = \frac{188.35 I_0}{r\lambda} (\sin \psi)[\cos (\beta R\varphi)]\epsilon^{j\beta(r_0-r)}\, dl \tag{862}$$

if we use the electric current element expression of Eq. (161). The quantity $r_0 - r$ denotes the path difference between P to the origin and P to the particular current element dl of length $R\, d\varphi$ meters. The quantity δ is the angle of the projection into the XY plane. For the horizontal XY plane, we have $\theta = 90$ deg and $\sin \psi = \sqrt{1 - \sin^2 \delta} = \cos \delta$. Putting $k = 188.35 I_0 R/(r\lambda)$ and expressing the elementary field contribution in terms of angles φ, θ, and δ, we find

$$\begin{aligned} d\mathcal{E}^{(\text{volts/m})} &= k[\cos (\beta R\varphi)] \sqrt{1 - \sin^2 \theta \sin^2 \delta}\, \epsilon^{j\beta R \sin \theta \cos \delta}\, d\varphi \\ &= k \sqrt{1 - \sin^2 \theta \sin^2 \delta}\, \epsilon^{j\rho \sin \theta \cos \delta} \cos (\rho\varphi)\, d\varphi \end{aligned} \tag{862a}$$

The total field $\mathcal{E}$ for points P in the horizontal XY plane (plane of the radiating circular conductor) is found by putting in (862a) the angle θ equal to 90 deg, yielding $\sin \theta = 1 = \sin^2 \theta$, $\sqrt{1 - \sin^2 \theta \sin^2 \delta} = \sqrt{1 - \sin^2 \delta} = \cos \delta$, and integrating from 0 to 2π, since we have to cover the entire circumference. Therefore, in the plane of the radiator,

$$\mathcal{E}^{(\text{volts/m})} = k \int_0^{2\pi} \epsilon^{j\rho \cos \delta} \cos \delta \cos (\rho\varphi)\, d\varphi \tag{862b}$$

[1] SHERMAN, J. B., *Proc. IRE*, **32**, 534, September, 1944.

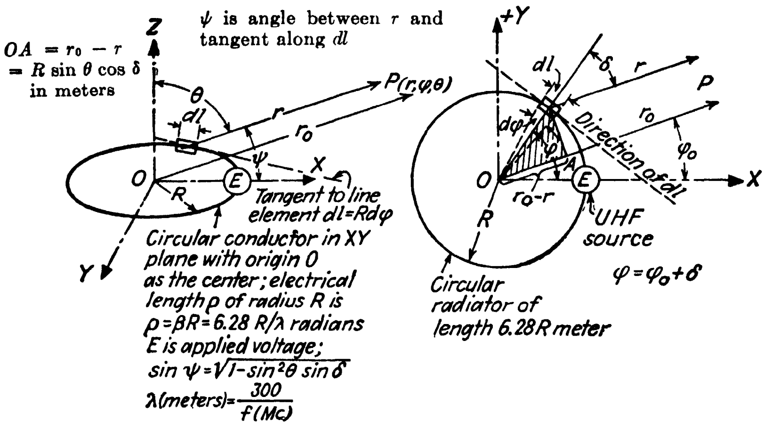
If φ_0 denotes the angle from the location of the HF source, where the HF voltage E volts is applied toward the direction r_0, which is fixed (independent of φ of any radiator element dl), we have $\varphi = \varphi_0 + \delta$; and for the value q under the integral in $\mathcal{E} = k\int_0^{2\pi} q$ in (862b), we find

$$q = \epsilon^{j\rho\cos\delta}\cos\delta\cos[\rho(\varphi_0+\delta)]\,d(\varphi_0+\delta)$$

where, because φ_0 is constant, we deal only with the differential operator $d\delta$.

$$k = \frac{94.75 I_0 R}{r\lambda}$$

λ, R, r in meters
I_0 in amp



Circular conductor in XY plane with origin O as the center;
Electrical length ρ of radius R is $\rho = \beta R = 6.28(R/\lambda)$ radians
E is applied voltage; $\sin\psi = \sqrt{1-\sin^2\theta\sin\delta}$

$$\lambda^{(m)} = \frac{300}{f^{(Mc)}}$$

FIG. 198. Conditions for a circular radiator.

Since

$$\cos[\rho(\varphi_0+\delta)] = \cos(\rho\varphi_0)\cos(\rho\delta) - \sin(\rho\varphi_0)\sin(\rho\delta)$$

we find from (862b)

$$\mathcal{E}^{(\text{volts/m})} = k\int_0^{2\pi} q$$
$$= k\int_0^{2\pi}\epsilon^{j\rho\cos\delta}\cos(\delta)[\cos(\rho\varphi_0)\cos(\rho\delta) - \sin(\rho\varphi_0)\sin(\rho\delta)]\,d\delta \quad (862c)$$

where the value of the integral is *not* affected by the change of limits required, by changing from the variable $d\varphi$ to the variable $d(\varphi_0+\delta)$, since φ_0 is fixed. Since

$$\cos(\rho\varphi_0)\cos(\rho\delta) = \tfrac{1}{2}[\cos(\rho\varphi_0+\rho\delta) + \cos(\rho\varphi_0-\rho\delta)]$$
$$\sin(\rho\varphi_0)\sin(\rho\delta) = \tfrac{1}{2}[-\cos(\rho\varphi_0+\rho\delta) + \cos(\rho\varphi_0-\rho\delta)]$$

we find

$$\cos(\rho\varphi_0)\cos(\rho\delta) - \sin(\rho\varphi_0)\sin(\rho\delta) = \cos(\underbrace{\rho\varphi_0 + \rho\delta}_{=m})$$

$$\cos\delta[\cos(\rho\varphi_0 + \rho\delta)] = \tfrac{1}{2}[\cos(\underbrace{\delta + m}_{=p}) + \cos(\delta - m)]$$

$$= \tfrac{1}{2}\{\cos[\rho\varphi_0 + \delta(\rho+1)] + \underbrace{\cos[-\rho\varphi_0 - \delta(\rho-1)]}_{\cos[\rho\varphi_0 + \underbrace{\delta(\rho-1)}_{=q}]}\}$$

and therefore

$$\cos\delta[\cos(\rho\varphi_0 + \rho\delta)]$$
$$= \tfrac{1}{2}[\cos(\rho\varphi_0)\cos p - \sin(\rho\varphi_0)\sin p + \cos(\rho\varphi_0)\cos q - \sin(\rho\varphi_0)\sin q]$$
$$= \tfrac{1}{2}(\cos(\rho\varphi_0)\{\cos[\delta(\rho+1)] + \cos[\delta(\rho-1)]\}$$
$$- \sin(\rho\varphi_0)\{\sin[\delta(\rho+1)] + \sin[\delta(\rho-1)]\})$$

where $\cos(\rho\varphi_0)$ and $\sin(\rho\varphi_0)$ are fixed and can be taken out of the integral of Eq. (862c). For $K = 0.5k$ and this expansion in (862c), we find

$$\mathcal{E}^{(\text{volts/m})} = K\left(\cos(\rho\varphi_0)\left\{\int_0^{2\pi}\epsilon^{j\rho\cos\delta}\cos[\delta(\rho+1)]\,d\delta + \int_0^{2\pi}\epsilon^{j\rho\cos\delta}\cos[\delta(\rho-1)]\,d\delta\right\}\right.$$
$$\left. - \sin(\rho\varphi_0)\left\{\int_0^{2\pi}\epsilon^{j\rho\cos\delta}\sin[\delta(\rho+1)]\,d\delta + \int_0^{2\pi}\epsilon^{j\rho\cos\delta}\sin[\delta(\rho-1)]\,d\delta\right\}\right) \quad (862d)$$

For the exponential factor $\epsilon^{j\rho\cos\delta}$, we have the well-known relation

$$\epsilon^{j\rho\cos\delta} = J_0(\rho) + 2\sum_{n=1}^{n=\infty} j^n J_n(\rho)\cos(n\delta) \quad (862e)$$

This means that we arrive at Bessel functions $J_0(\rho)$ and $J_n(\rho)$ of the first kind. Making use of this relation, multiplying each term by $\cos(n\delta)$, and integrating render the circular functions equal to zero, with the result

$$\int_0^{2\pi}\epsilon^{j\rho\cos\delta}\cos(n\delta)\,d\delta = 2\pi j^n J_n(\rho); \qquad \int_0^{2\pi}\epsilon^{j\rho\cos\delta}\sin(n\delta)\,d\delta = 0 \quad (862f)$$

Relations (862d) and (862f) yield the field intensity formula

$$\mathcal{E} = j^{(\rho+1)}\,591\,\frac{I_0R}{r_0\lambda}[J_{\rho+1}(\rho) - J_{\rho-1}(\rho)]\cos(\rho\varphi_0) \qquad \text{volts/m};$$
$$\rho = 6.28\frac{R}{\lambda} \quad (862g)$$

where r_0 is the distance to the center of the circular radiator in meters. The horizontal pattern is, therefore, given by a cosine function, according to

$$D_\varphi = \cos(\rho\varphi_0) \quad (862h)$$

It is the circumference $2\pi R$ and the corresponding electrical length $6.28R\beta$

that are of concern. For a complete wavelength distribution along a thin circular radiator, we have $6.28R = \lambda$ and $\beta R = \rho = 2\pi R/\lambda = \lambda/\lambda = 1$. Equation (862$g$) for such a single-wavelength distribution reads

$$\mathcal{E} = -591 \frac{I_0 R}{r_0 \lambda} [J_2(1) - J_0(1)] \cos \varphi_0 \qquad \text{volts/m} \qquad (862i)$$

But $J_2(1) = 0.1149$, $J_0(1) = 0.7651$, $R = \lambda/6.28$, and we find, for $591/6.28 = 94.25$ and $J_2(1) - J_0(1) = -0.6502$, the field intensity at P located at $\theta = 90$ deg, φ_0, r in the XY plane of Fig. 198, as

$$\mathcal{E} = +61.2 \frac{I_0}{r_0} \cos \varphi_0 \qquad \text{volts/m} \qquad (862j)$$

with a maximum value of $61.2 I_0/r_0$ along the X axis, a distance r_0 meters away from the center of the circular radiator in the XY plane. For a 2-wavelength distribution along the circular radiator, we have $6.28R = 2\lambda$ and $R = \lambda/3.14$ and $\rho = 6.28R/\lambda = 2$, and (862g) reads

$$\begin{aligned} \mathcal{E} &= j^3 591 \frac{I_0 R}{r_0 \lambda} [J_3(2) - J_1(2)] \cos (2\varphi_0) \\ &= -j591 \frac{I_0(\lambda/3.14)}{r_0 \lambda} [0.12894 - 0.57672] \cos (2\varphi_0) \\ &= +j84.5 \frac{I_0}{r_0} \cos (2\varphi_0) \qquad \text{volts/m} \qquad (862k) \end{aligned}$$

For a 3-wavelength radiator, we find in the same way

$$\mathcal{E}^{(\text{volts/m})} = -100 \frac{I_0}{r_0} \cos (3\varphi_0) \qquad (862l)$$

We note, therefore, that we have horizontal directivities (in the XY plane or plane of the circular radiator) $\cos \varphi_0$, $\cos (2\varphi_0)$, $\cos (3\varphi_0)$, $\cos (4\varphi_0)$, $\cdots$, for circular loop 1, 2, 3, 4, $\cdots$ wavelengths long.

TABLE 35. RADIATION PATTERN IN PLANE OF CIRCULAR RADIATOR

φ_0^0	0	10	20	30	40	50	60	70	80	90
$\cos (\varphi_0)$	1.000	0.985	0.939	0.866	0.766	0.643	0.500	0.342	0.1719	0.0000
$\cos (2\varphi_0)$	1.000	0.939	0.766	0.5	0.1719	0.1719	0.5	0.766	0.939	1.0000

Remarks: $\cos (\varphi_0)$ is the radiation pattern in the plane of the circular radiator for a full-wavelength distribution and $\cos (2\varphi_0)$ for $6.28R = 2\lambda$.

Since symmetry exists, we only have to look up values for $\varphi_0 = 0$ to 90 deg and find the results of Table 35 for the λ and the 2λ current distribution. Figures 199 and 200 give the radiation patterns in the plane of the circular radiator. If we therefore arrange two 3λ circular

radiators, which are parallel with their planes, so that the six maximum positions of one loop occur at angles where we have zero field positions for the other circular radiator, we should expect a resultant radiation pattern as is shown in Fig. 201. In the pattern for the plane of the radiator, the angle φ_0 denotes the angular departure from the location of the applied HF voltage E and cosinoidal current distribution is assumed for thin radiators. The resultant pattern of Fig. 201 can also be obtained by the arrangement of Fig. 202, where a "backfolded" line of electrical length βl causes a phase lag of $\rho\varphi_0$ electrical degrees without appreciable radiation. This phase shifter is arranged normal to the plane of the circular radiator. It is to be borne in mind, however, that there are also mutual effects from one radiator on the other, as was explained in connection with linear radiators. Fortunately, the mutual effects in the case of two equal radiators cause the same effective input impedance if the applied voltages are of the same amplitude. With respect to assumed sinusoidal current distributions, reference is made to Fig. 203. It is noted that for an *open* gap diametrically opposite the feeder terminals a and b, the current distribution must be such that a current node exists at all places where Fig. 203 shows a current maximum. Therefore, the directivity patterns in the plane of the circular radiator, as shown in Figs. 199 to 201, are then angularly shifted halfway between the maxima for the case of open-ended loops. It can also be seen from Fig. 203 that a current flow symmetry exists with respect to the radiation along the Z axis of the circular radiator. It can also be seen that half-wavelength sections 1 and 2 for the $6.28R = \lambda$ distribution are an *additive* multiple with respect to the field vector in the direction of the axis, yielding $F_1 + F_2$. For the $6.28R = 2\lambda$ distribution, the $F_1 - F_3$ and $F_2 - F_4$ effects are each in antiparallel and cancel. For the $6.28R = 3\lambda$ distribution, we have three field vectors $F_3 + F_6$, $F_2 + F_5$, and $F_1 + F_4$ on the axis under 120 deg. For the $6.28R = 4\lambda$ distribution, we have again, canceling, antiparallel vectors $F_1 - F_5$, $F_2 - F_6$, $F_3 - F_7$, and

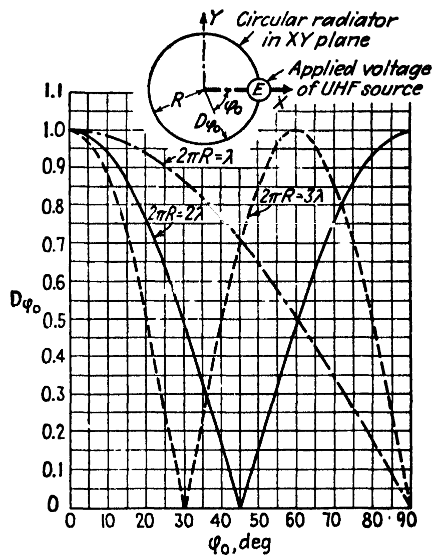


FIG. 199. Directivity patterns in the plane of a circular radiator.

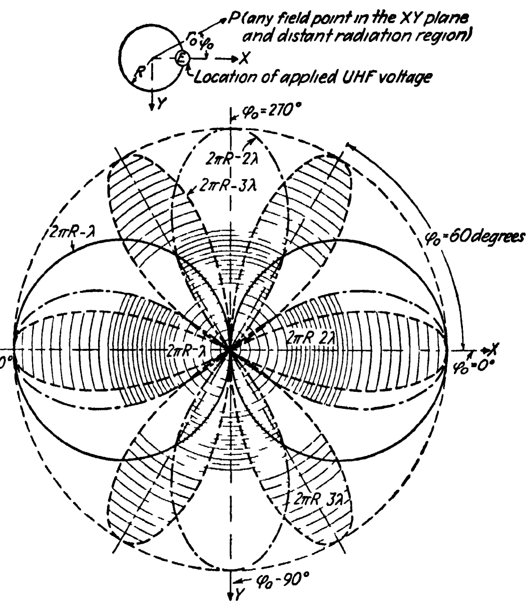


FIG. 200. Polar directivity in the plane of a circular radiator.

$F_4 - F_8$. Moreover, since along the $\theta = 0$ axis we have the factor $\sqrt{1 - \sin^2\theta\cos^2\delta} = 1$ and $\epsilon^{i\rho\sin\theta\cos\delta} = 1$, the total field along the Z axis becomes

$$\left.\begin{aligned} \mathcal{E}_{\theta=0^\circ} &= 2k\int_{\varphi=-90^\circ}^{\varphi=+90^\circ}\cos(\rho\varphi)\,d\varphi = \frac{2k}{\rho}\Big|\sin(\rho\varphi)\Big|_{\varphi=-90^\circ}^{\varphi=+90^\circ} \\ &= \frac{4k\sin(\rho 90^\circ)}{\rho} \qquad \text{volts/m} \\ k &= \frac{188\ 35 I_0 R}{r_0\lambda} \end{aligned}\right\} \qquad (862m)$$

For the distant radiation region ($r_0 \cong r$) and for $6.28R = \lambda$, *i.e.*, $\rho = 1$, we find

$$k = \frac{188.35 I_0 R}{r_0\lambda} = \frac{188.35 I_0 R}{2\pi R r_0} = 30\frac{I_0}{r_0}$$

$$\mathcal{E}_{\theta=0} = \frac{4k\sin(90\rho)^\circ}{\rho} = 120\frac{I_0}{r}\sin 90^\circ$$

$$\mathcal{E}_{\theta=0} = 120\frac{I_0}{r_0} \qquad \text{volts/m} \qquad (862n)$$

For a 2-wavelength circular radiator, we have

$$6.28 = 2\lambda; \qquad \rho = \frac{2\pi R}{\pi R} = 2; \qquad k = \frac{188.35 I_0 R}{\pi R r_0} = 60\,\frac{I_0}{r_0}$$

$$\mathcal{E}_{\theta=0} = \frac{4k \sin (180^\circ)}{\rho} = 0 \qquad \text{volts/m} \qquad (862o)$$

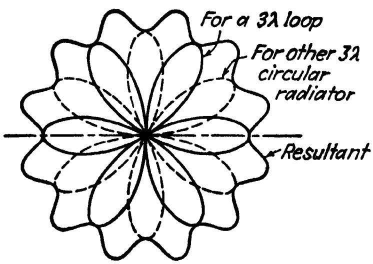


FIG. 201. Resultant effect of two circular radiators excited in the three-wavelength distribution with the fields interlaced.

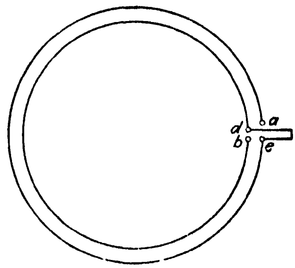


FIG. 202. Two circular radiators driven by a voltage across terminals a and b; the length d to e acts as a phase delay section. Length d to e is equal to l. Voltage E is applied across a and b, and length l is so chosen that βl corresponds to 90° phase lag for the 3λ distribution.

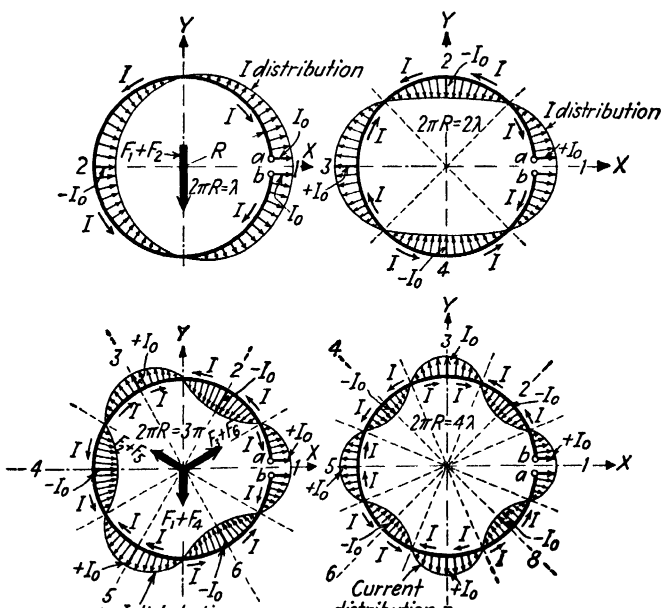


FIG. 203. Current distributions along a circular radiator in the XY plane (Z is axis of radiator). UHF voltage E is applied across terminals a and b. F are field vectors.

Therefore, no radiation exists along the axis of the circular radiator. This can also be understood from Fig. 203, since antiparallel current elements cancel their radiation effects along the Z axis. For a 3λ circular radiator, we have

$$2\pi R = 3\lambda; \qquad \rho = \frac{2\pi R}{2\pi R/3} = 3; \qquad k = \frac{188.35 I_0 R}{2\pi R r_0/3} = 90\,\frac{I_0}{r_0}$$

$$\mathcal{E}_{\theta=0°} = \frac{4k \sin 270°}{3} = -\frac{4 \times 90}{3}\frac{I_0}{r_0} = -120\,\frac{I_0}{r_0} \qquad \text{volts/m} \qquad (862p)$$

For $2\pi R = 4\lambda$ or any other *even* number of wavelengths, we have $\sin (90\rho) = 0$ and, therefore, *no axial radiation* effect; while for $2\pi R = 5\lambda$ or any

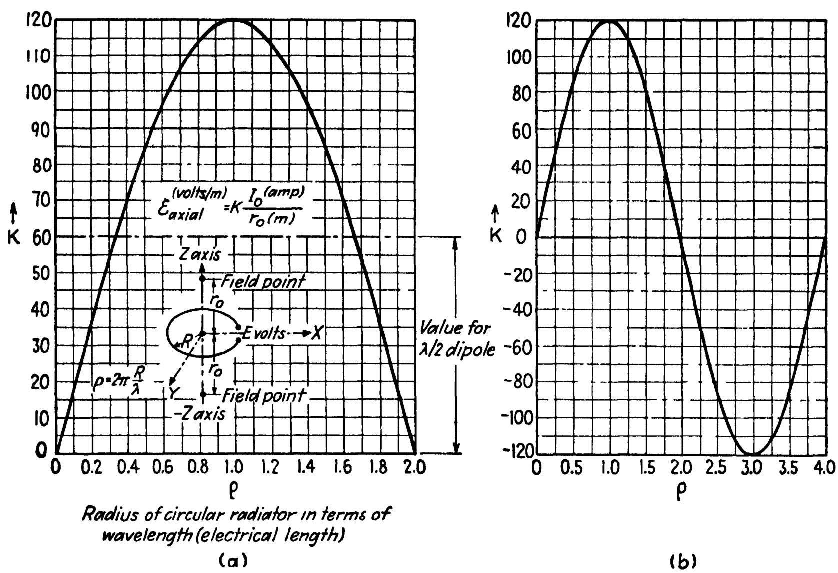


FIG. 204. The value of the constant K for circular radiators corresponding to the value of 60 in the case of a linear half-wave radiator.

other odd wavelength distribution, we always have an axial electric field intensity of

$$\mathcal{E}_{axial} = 120 \sin (p 90°)\,\frac{I_0}{r_0} \qquad \text{volts/m} \qquad (862q)$$

for I_0 amperes, r_0 meters and where p stands for odd integers such as 1, 3, 5, 7, . . . in $2\pi R = p\lambda$. Comparing these results with the maximum field effects in the plane of the circular radiator, we find that for a full-wavelength radiator, the axial effect is $120/61.2 = 1.96$ times larger, while it is $^{120}\!/\!_{100}$, or only 1.2 times larger for the 3-wavelength radiator. Inasmuch as there can be no axial field for *even* wavelength distribution,

there must be a transition between even and odd distributions. Table 36 gives the values and the computations. Figure 204 gives the numerical constant K of (862*d*), which corresponds to the factor 60 in the case of a half-wave dipole radiator. In customary field intensity measurements,

TABLE 36

Length of circular radiator in terms of λ $\rho = \frac{2\pi R}{\lambda}$	$\rho 90°$, deg	$\sin(\rho 90°)$	$\frac{4k}{\rho}$	$\mathcal{E}_{\text{axial}} = \frac{4k}{\rho}\sin(\rho 90°)$	$\mathcal{E}_{\text{axial}}^{(\text{volts/m})} \times \frac{I_0}{r_0}$
0.000	0.000	0.000	0.000*k*	0.000	0.000
0.01	0.9	0.0157	400*k*	6.29*k*	1.88
0.05	4.5	0.0785	80*k*	6.3*k*	9.45
0.1	9	0.1564	40*k*	6.3*k*	18.9
0.2	18	0.309	20*k*	6.18*k*	37.08
0.3	27	0.454	13.3*k*	6.03*k*	54.27
0.4	36	0 588	10*k*	5.88*k*	70.5
0.5	45	0.707	8*k*	5.65*k*	84.5
0.6	54	0.809	6.67*k*	5.39*k*	97
0.7	63	0.891	5.71*k*	5.1*k*	107
0.8	72	0.951	5*k*	4.75*k*	114
0.9	81	0.988	4.44*k*	4.36*k*	118
1.0	90	1.000	4*k*	4*k*	120
1.2	108	0.951	3.325*k*	3.17*k*	114
1.4	126	0.809	2.86*k*	2.31*k*	97
1.6	144	0.588	2.5*k*	1.47*k*	70.5
1.8	162	0.309	2 22*k*	0.685*k*	37.08
2.0	180	0.000	2.00*k*	0.000	0.000

Remarks: $k = 188.35 I_0 R/r_0\lambda$, $\rho = (2\pi/\lambda)R$. Length of circular radiator is $2\pi R = \rho\lambda$; $\lambda = 2\pi R/\rho$; $k = 188.35 I_0 R\rho/(2\pi R r_0) = 30\rho(I_0^{(\text{amp})}/r_0^{(\text{m})})$.

Note: 60 would be the factor for a $\lambda/2$ dipole; $\mathcal{E}_{\text{axial}}$ is Eq. (862*m*); Fig. 204 gives curves.

we use circular loops of about 14 in. diameter. Therefore, $R = 7$ in., and the circular radiator length is $6.28 \times 2.54 \times 10^{-2} = 1.12$ m. Suppose we work with a current distribution defined by $\rho = 0.1$; then $\rho = 6.28R/\lambda = 0.1$ and $\lambda = 11.2$ m, corresponding to a frequency of 300/11.2 = 26.75 Mc/sec. Therefore, when a single-turn circular loop is used for direction finding, we have, in addition to the customary figure-eight pattern holding for $\rho = 0.1$ with maxima along the plane of the circular radiator, also according to Table 36, an axial field of $18.9 I_0/r_0$ in volts per meter for I_0 in amperes and r_0 in meters. The direction-finder loop will then not have zero effect normal to its plane. According to (862*g*), we have for $\rho = 0.1$ only one-tenth of a wavelength distribution along the circular radiator, and the largest current value must exist where the HF

voltage E in Fig. 203 is impressed. Hence, $\cos(\rho\varphi_0) = 1$ is the condition for obtaining the maximum field value in the plane of the circular radiator. We have then $6.28R = \rho\lambda$ or $\lambda = 6.28R/0.1 = 62.8R$ with the following details:

According to (862*g*), we find the maximum radiation effect

$$\mathcal{E}^{(\text{volts/m})} = j^{1.1}591 \frac{I_0 R}{\lambda r_0} [J_{1.1}(0.1) - J_{-0.9}(0.1)] = j^{1.1}591 \frac{I_0 R}{20\pi R r_0} [J_{1.1}(0.1) - J_{-0.9}(0.1)]$$

$$= j^{1.1}9.45 \frac{I_0}{r} [J_{1.1}(0.1) - J_{-0.9}(0.1)] \quad (862r)$$

Now curves for such order as 1.1 and -0.9 are available in the 1943 edition of Jahnke and Emde "Tables of Functions," page 152 (Fig. 84). We find from such curves $J_{1.1}(0.1) \cong 0.05$, but for $J_{-0.9}(0.1) \cong 1.5$, which could also be $\cong 2.0$ or $\cong 2.3$, since respective fixed $J_n(\rho)$-value contours run all together. Assuming that 1.9 is the value, we have the difference $J_{1.1}(0.1) - J_{-0.9}(0.1)$ essentially equal to -1.9, since we cannot read off anywhere with nearly such accuracy as 0.05. With such a value, we find then

$$\mathcal{E} \cong -j^{1.1}9.45 \times 1.9 \frac{I_0}{r_0} = -j18 \frac{I_0}{r_0} \quad \text{volts/m} \quad (862s)$$

which is somewhat smaller than the axial value with the constant $K = 18.9$ as computed in Table 36. The formula for the Bessel function is

$$J_n(\rho) = \frac{(0.5\rho)^n}{0!n!} - \frac{(0.5\rho)^{n+2}}{1!(n+1)!} + \frac{(0.5\rho)^{n+4}}{2!(n+2)!} - \frac{(0.5\rho)^{n+6}}{3!(n+3)!} + - \cdots$$

if n is *not an integer* for the Bessel function $J_n(\rho)$ of the first kind. If n is a positive integer, we have the expansion

$$J_n(\rho) = \frac{\rho^n}{2^n n!}\left[1 - \frac{\rho^2}{2(2n+2)} + \frac{\rho^4}{2 \times 4(2n+2)(2n+4)} - \cdots\right]$$

which, for $J_1(0.1)$, yields

$$J_1(0.1) = \frac{0.1}{2}\left(1 - \frac{0.01}{8} + \frac{0.0001}{192} - \cdots\right) = 0.049937 \cong \frac{0.1}{2}$$

The approximation $J_n(\rho) \cong 0.5\rho$ for $\rho \leqq 0.1$ holds all the more the smaller ρ is in comparison with the argument $\rho = 0.1$, corresponding to a tenth of a wavelength distribution along the circular radiator. Hence, if we deal in (862*g*) with small values of ρ, *i.e.*, $\rho = 6.28R/\lambda \to 0$, we can assume

$$J_{p+1}(\rho) - J_{p-1}(\rho) \cong J_1(\rho) - J_1(\rho)$$

Utilizing the relation $J_{-1}(\rho) = -J_1(\rho)$, we find

$$J_1(\rho) - J_{-1}(\rho) = 2J_1(\rho)$$

which yields, according to $J_n(\rho) \cong 0.5\rho$ for $\rho \leqq 0.1$,

$$J_1(\rho) - J_{-1}(\rho) = \rho \qquad \text{for } \rho \leqq 0.1$$

The Bessel function of the first kind and nth order is for n and ρ *without restrictions, i.e.*, the order n as well as the argument ρ, real or complex, integer or noninteger, given by the most important expression

$$J_n(\rho) = \sum_{m=0}^{m=\infty} (-1)^m \frac{(0.5\rho)^{n+2m}}{m!\Gamma(n+m+1)} \tag{863}$$

Nothing is changed when n is replaced by $-n$ Therefore, we have an expression by means of the *Gamma function* $\Gamma(n + m + 1)$, which applies to the solution of expressions as obtained in (862r), where n is equal to 1.1 and equal to -0.9, respectively. We have then *no integer for n*. Hence, in formula (862g), the Bessel functions $J_n(\rho)$ and $J_{-n}(\rho)$ are generally two linearly independent solutions; and for $n = 0$, the solutions become identical. But when n is an integer, $J_n(\rho)$ *becomes* a *single-valued* function and the function $\Gamma(n + m + 1)$ in (863) is replaced by the factorial $(n + m)!$, yielding

$$J_n(\rho) = \sum_{m=0}^{m=\infty} \frac{(-1)^m(0.5\rho)^{n+2m}}{m!(n+m)!}; \qquad n = 0, 1, 2, 3, 4, 5, \cdots;$$

$$J_{-n}(\rho) = (-1)^n J_n(\rho) \tag{863a}$$

Now let us *apply the general unrestricted formula* (863) to (862r). We find then for $J_{1.1}(0.1) = J_n(\rho)$, the operating values $n = 1.1$ and $\rho = 0.1$, and

$$\left.\begin{aligned} J_{1.1}(0.1) &= \sum_{m=0}^{m=\infty} (-1)^m \frac{(0.5 \times 0.1)^{1.1+2m}}{m!\Gamma(1.1+m+1)} = 1 \times \frac{0.05^{1.1+0}}{1 \times \Gamma(1.1+0+1)} \\ &- 1 \times \frac{(0.05)^{1.1+2}}{1 \times \Gamma(1.1+1+1)} + 1 \times \frac{(0.05)^{1.1+4}}{1 \times 2\Gamma(1.1+2+1)} - + \cdots \\ &= A - B + C - D + \cdots; \quad A = \frac{0.05^{1.1}}{\Gamma(2.1)}; \quad B = \frac{0.05^{3.1}}{\Gamma(3.1)}; \\ & \qquad C = \frac{0.05^{5.1}}{2\Gamma(4.1)} \end{aligned}\right\} \tag{863b}$$

where the following terms D, E, F, etc., decrease *very rapidly*, since the C term is already controlled by a factor $0.05^{5.1}$, which is very small in comparison with the factor $0.05^{1.1}$ of the A term. We find then $0.05^{1.1} = 0.0466$, $0.05^{3.1} = 0.000116$, $0.05^{5.1} = 2.96 \times 10^{-13}$. Now the Γ functions[1] of x are usually given in tables as factorial functions, where $\Gamma(x + 1)$ is given for the different values of x and $x! = \prod(x) = \Gamma(x + 1)$ is basic, and $\prod$ refers to the Gauss function. To find $\Gamma(2.1)$, we have to look then under $x = 2.1 - 1 = 1.1$ and find $\Gamma(2.1) = 1.0465$; for $x = 3.1 - 1 = 2.1$, we find $\Gamma(3.1) = 2.198$; for $x = 4.1 - 1 = 3.1$, we find $\Gamma(4.1) = 6.813$ and Eq. (863b) yields

$$J_{1.1}(0.1) = \frac{0.0466}{1.0465} - \frac{0.000116}{2.198} + \underbrace{\frac{2.96 \times 10^{-13}}{13.626}}_{\text{negligible}} - + \cdots$$

$$\cong 0.045 - 0.000053 = 0.044947 \tag{863c}$$

[1] In Figs. 389 and 390 Gamma functions are plotted. (Consult also Tables 91 and 92, pp. 1257 and 1261.)

Moreover, we have the solution

$$\left.\begin{aligned} J_{-0.9}(0.1) &= \sum_{m=0}^{m=\infty} (-1)^m \frac{(0.5 \times 0.1)^{0.9+2m}}{m!\Gamma(-0.9+m+1)} = \frac{0.05^{-0.9}}{1 \times \Gamma(-0.9+0+1)} \\ &- \frac{0.05^{-0.9+2}}{1 \times \Gamma(-0.9+1+1)} + \underbrace{\frac{0.05^{-0.9+4}}{1 \times 2\Gamma(-0.9+2+1)}}_{\text{negligible}} - \cdots = A - B \\ A &= \frac{0.05^{-0.9}}{\Gamma(0.1)}; \quad B = \frac{0.05^{1.1}}{\Gamma(1.1)}; \quad 0.05^{-0.9} = \frac{1}{0.05^{0.9}} = \frac{1}{0.0546}; \\ & \qquad 0.05^{1.1} = 0.0466 \end{aligned}\right\} \quad (863d)$$

In the $\Gamma(x + 1)$ table, we have to look up $\Gamma(0.1)$ under $x = 1.1 - 1 = -0.9$, which negative argument is *not* in the table. We then use the formula

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \quad (863e)$$

and find accordingly

$$\Gamma(0.1) = \frac{\Gamma(0.1+1)}{0.1} = 10\Gamma(1.1)$$

Looking under $x = 1.1 - 1 = 0.1$, the Γ function table yields $\Gamma(1.1) = 0.9514$ and $\Gamma(0.1) = 10 \times 0.9514 = 9.514$. Formula (863*d*) then yields

$$J_{-0.9}(0.1) = A - B = \underbrace{\frac{1}{0.0546 \times 9.514}}_{0.52} - \frac{0.0466}{0.9514} = 1.92 - 0.049 = 1.871$$

We have then

$$J_{1.1}(0.1) - J_{-0.9}(0.1) = 0.044947 - 1.871 = -1.82605$$

which accurate result checks the approximate value of -1.9. Hence, the correct result for the approximation of (862*s*) is

$$\mathcal{E}^{(\text{volts/m})} = -j9.45 \times 1.826 \frac{I_0}{r_0} = 17.3 \frac{I_0^{(\text{amp})}}{r_0^{(\text{m})}}$$

for the maximum effect in the plane of the circular radiator. If we are interested in an expression for very small ρ values, *i.e.*, $6.28R/\lambda$ or R/λ very small, as is customary in *DF* loops in the 2- to 20-Mc range, for instance, then we can simplify (862*g*) greatly. One simplification is that the factor $\cos(\rho\varphi_0)$ is essentially unity even for operating angles as large as $\rho\varphi_0 = 4$ deg, since then $\cos(\rho\varphi_0) = 0.9976$. From our computations of patterns in the plane of the circular radiator, we found that for integer λ distributions, we have symmetrical lobe conditions around the Z axis normal to the XY plane and through the center of the radiator. Therefore, only angles of φ_0 between 0 and 90 deg are of concern. For small values of ρ, we do not deal with such large λ distributions, and we have to compute the radiation effect for the entire angular range. But with the source E as in Fig. 198 on the $+X$ axis side of the radiator, the two semicircles must have the same current distributions and the physical range of $\varphi_0 = 0$ to 180 deg gives the status of affairs. The severest limit of $\rho\varphi_0 = \rho 180° = 4$ deg gives a means of finding what value of ρ renders $\cos(\rho\varphi_0)$

essentially equal to unity. The limiting value then gives $\rho = 4/180 = 0.0222$; and for values of $\rho \leqq 0.0222$, we have $\cos(\rho\varphi_0) \cong 1$. For such values of ρ, we can also use the approximation $J_1(\rho) - J_{-1}(\rho) = 2J_1(\rho)$, leading to approximation $J_1(\rho) - J_{-1}(\rho) = \rho$, since $\rho \leqq 0.0222$ is well below the requirement utilized in $J_n(\rho) \cong 0.5\rho$ for $\rho \leqq 0.1$. We find then from (862g), by using $J_1(\rho) - J_{-1}(\rho) = 2J_1(\rho)$

$$\mathcal{E} = j1{,}182\,\frac{I_0R}{r_0\lambda}\,J_1(\rho) = j188\rho\,\frac{I_0}{r}\,J_1(\rho) \qquad \text{volts/m}; \qquad \rho \leqq 0.0222 \qquad (863f)$$

This expression can be used for computing the field intensity, which is the same in all radial directions of the radiator for a radius $R = \rho\lambda/6.28$ meters. For the upper limit of $\rho = 0.0222$, we have $J_1(\rho) = 0.0099995$, which is essentially 0.01, and $\mathcal{E}^{(\text{volts/m})} = 0.0417I_0/r_0$. When we also use the approximation $J_n(\rho) \cong 0.5\rho$, leading to $J_1(\rho) - J_{-1}(\rho) = \rho$, Eq. (863f) simplifies to

$$\mathcal{E}^{(\text{volts/m})} = j94\rho^2\,\frac{I^{(\text{amp})}}{r^{(\text{m})}}$$

which relation can be used for somewhat smaller values as given by the upper limit of (863f). The latter expression applies to the case where $\rho \to 0$, and the circular radiator is driven by currents in the lower HF range.

142. The Theory of the Disk Radiator. Such a radiator is a circular disk antenna (Fig. 205) of copper or some other highly conducting material, which is fed at the center. The current flows from the center in the same way in all directions toward the open-ended circular edge of the disk and decreases from the center of the disk to zero value at the edge. The outer circular edge is a metal contour of high potential referred to the central feed point of the disk. The current from the edge along a radial sector element has then roughly a rectangular distribution, since a steep increase occurs next to the outer edge. The elementary ring of inner and outer radii ρ and $\rho + d\rho$ of total current I_r is then composed of elementary current contributions $I_e = (I_r\,d\rho\,d\varphi)/6.28$ with components $I_e \cos\varphi$ and $I_e \sin\varphi$. On account of the symmetry, with respect to the vertical XZ plane we have four such current elements in the disk in the horizontal XY plane. This also holds, of course, for any other vertical reference plane, causing, for instance, the ab intersection with the XY plane. Along the X direction, we have $\cos\varphi$ functions,

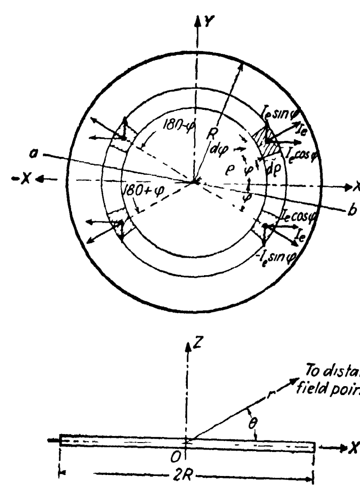


FIG. 205. Four symmetrical radiation elements of a disk radiator. $d\varphi$ is in radians.

which are additive vectorially; and along the Y direction, we have sin φ functions, which cancel. With respect to the field point P of the distant radiation field, we have, for an elevation θ, the elementary electric field contribution

$$d\mathcal{E}_\theta^{(\text{volts/m})} = -\frac{2\omega}{\pi r} I_r \, d\rho \, d\varphi \cos\varphi \sin\theta \sin(\beta\rho\cos\varphi\cos\theta)$$

where $\beta = \omega/c = 2\pi/\lambda$. Putting $(2\omega/\pi r)I_r \sin\theta = A \sin\theta$, we have, for the total field at P,

$$\mathcal{E}_\theta = -A\sin\theta \int_{\varphi=0°}^{\varphi=90°} \int_{\rho=0}^{\rho=R} \cos\varphi \sin(\beta\rho\cos\varphi\cos\theta)\, d\varphi\, d\rho \qquad \text{volts/m}$$

since, for symmetry reasons, the integration of φ is to be extended only over one quadrant.

Integrating with respect to ρ, we find

$$\mathcal{E}_\theta = +A\sin\theta \int_{\varphi=0°}^{\varphi=90°} \cos\varphi \left|\frac{\cos(\beta\rho\cos\varphi\cos\theta)}{\beta\cos\varphi\cos\theta}\right|_{\rho=0}^{\rho=R} d\varphi$$

$$\left|\frac{\cos(\beta\rho\cos\varphi\cos\theta)}{\beta\cos\varphi\cos\theta}\right|_{\rho=0}^{\rho=R} = \frac{\cos(\beta R\cos\varphi\cos\theta) - 1}{\beta\cos\varphi\cos\theta}$$

Hence,

$$\mathcal{E}_\theta = \frac{A\tan\theta}{\beta} \int_{\varphi=0°}^{\varphi=90°} [\cos(\beta R\cos\varphi\cos\theta) - 1]\, d\varphi \qquad \text{volts/m} \qquad (864)$$

Putting $A/\beta = K$, for λ and r in meters, f in megacycles per second, and I_r in amperes, because of $\omega/r = 188.35/(r\lambda)$, we find

$$A = 0.637 \times 188.35 I_r = \frac{120 I_r}{r\lambda} = \frac{0.4 f I_r}{r} \qquad (864a)$$

rendering the factor K

$$K = \frac{A}{\beta} = \frac{A\lambda}{2\pi r} = \frac{120}{6.28}\frac{I_r}{r} = 19.2\frac{I_r}{r} \qquad (864b)$$

for $k = \beta R\cos\theta$, we find from (864)

$$\mathcal{E}_\theta = \frac{19.2 I_r \tan\theta}{r}\left[\int_{\varphi=0}^{\varphi=\pi/2} \cos(k\cos\varphi)\, d\varphi - \int_{\varphi=0}^{\varphi=\pi/2} d\varphi\right]$$

$$= 19.2 I_r \tan\theta \cdot \tan(p - q) \qquad \text{volts/m} \qquad (864c)$$

where the p term is a well-known Bessel function of zeroth order and the first kind, since

$$p = \frac{\pi}{2} J_0(k); \qquad q = \frac{\pi}{2}$$

$$\mathcal{E}_\theta = 1.57 \times 19.2 \frac{I_r}{r}[J_0(k) - 1]\tan\theta = 30\frac{[J_0(k) - 1]I_r}{r}\tan\theta$$

$$\mathcal{E}_\theta = 30\frac{[J_0(\beta R\cos\theta) - 1]I_r}{r}\tan\theta \qquad \text{volts/m} \qquad (864d)$$

The quantity R is the radius of the disk in meters, λ and the distance r from field point P to the center of the disk are also in meters, I_r is in amperes, and $\beta = 6.28/\lambda$. The directivity D_θ, in any vertical plane, is then

$$D_\theta = [J_0(\beta R \cos \theta) - 1] \tan \theta \tag{864e}$$

We find for any elevation θ above the plane of the disk radiator, the values of Table 37. After finding the value k denoting the argument of the Bessel function $J_0(k)$, we note when looking up the Bessel values (Fig. 182)

TABLE 37. FOR RADIUS OF DISK $R = 2\lambda$

Elevation θ, deg	From trig. table $\cos \theta$	Computed $k = \beta R \cos \theta = 12.56 \cos \theta$	Computed $J_0(k) - 1$	From trig. table $\tan \theta$	Computed $D_\theta = [J_0(k) - 1] \tan \theta$	From Bessel table $J_0(k)$
0	1.0000	12.56	−0.8435	0.0000	0.00000	0.1565
5	0.9962	12.4	−0.8704	0.0875	−0.076	0.1296
10	0.9848	12.35	−0.8797	0.1763	−0.153	0.1203
15	0.9659	12.1	−0.9303	0.2679	−0.25	0.0697
20	0.9387	11.8	−0.998	0.364	−0.364	0.002
25	0.9063	11.4	−1.0902	0.4663	−0.51	−0.0902
30	0.866	10.85	−1.1959	0.5774	−0.688	−0.1959
35	0.8192	10.25	−1.249	0.7002	−0.88	−0.249
40	0.766	9.6	−1.209	0.8391	−1.02	−0.209
45	0.7071	8.85	−1.0523	1.0000	−1.0523	−0.0523
50	0.6428	8.1	−0.8525	1.1918	−1.015	0.1475
55	0.5736	7.2	−0.7049	1.4281	−1.005	0.2951
60	0.5	6.28	−0.7804	1.7321	−1.36	0.2196
65	0.4226	5.3	−1.0758	2.1445	−2.3	−0.0758
70	0.342	4.3	−1.361	2.7475	−3.74	−0.361
75	0.2588	3.25	−1.3328	3.7321	−4.96	−0.3328
80	0.1736	2.18	−0.8785	5.6713	−4.97	0.1215
85	0.0872	1.09	−0.002	11.43	−0.02286	0.998
90	0.0000	0.00	0.000	∞	0.0000	1.0000

Remarks: $\beta R = 4\pi = 12.56$ for $R/\lambda = 2$. D_θ is directivity in normal planes through center of the disk radiator of radius R meters.

$J_0(k)$, that we cover a range of some 2 complete to-and-fro swings of the Bessel curve. It is, therefore, necessary to locate on the Bessel curve or in the corresponding Bessel table the arguments k for which the extreme values (maxima and zero values) of $J_0(k)$ occur. This is done in Table 38. The value $J_0(k)$ vanishes in the 0 to 12.56 range for $k = 2.4048$, 5.5201, 8.6537, and 11.7915, while a positive maximum occurs for $k = 7.15$ with a value of $J_0(7.15) = 0.3001$. We find, according to Bessel curves or tables, the two negative extremes of $J_0(3.83) = -0.4028$ and $J_0(10.175) = -0.2497$. Figure 206 gives the radiation pattern. As we shall learn from the method of perfect images for obstructed radiation of Chap.

TABLE 38. FOR $R = 2\lambda$

$J_0(k)$ from Bessel curve or table	k from curve or table (Fig. 182)	Computed $\cos\theta = \frac{k}{\beta R} = \frac{k}{12.56}$	From trig. table θ, deg	From trig. table $\tan\theta$	Computed $J_0(k) - 1$	Computed $D_\theta = [J_0(k) - 1] \times \tan\theta$
0.000	2.4048	0.192	78.9	5.097	−1.0000	−5.097
−0.4028	3.83	0.305	72.2	3.1146	−1.4028	−4 38
0.000	5.5201	0.441	63.8	2.0323	−1.0000	−2.0323
+0.3001	7.15	0.57	55.2	1.4388	−0.6999	−1.005
0.000	8.6537	0.69	46.4	1.0501	−1.0000	−1.0501
−0.2497	10.175	0.81	35 9	0 7239	−1.2497	−0.91
0.0000	11 7915	0.94	19 9	0 362	−1.0000	−0.362

Remarks: Directivity for the extreme amplitude swings and $J_0(k) = 0$.

VII, a disk a distance of h meters over a metal plane with sufficient extension gives a directivity such that we have only to multiply (864*c*) and (864*d*) by 2 sin (βh sin θ) and we have

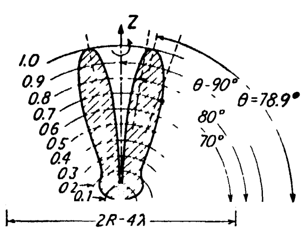
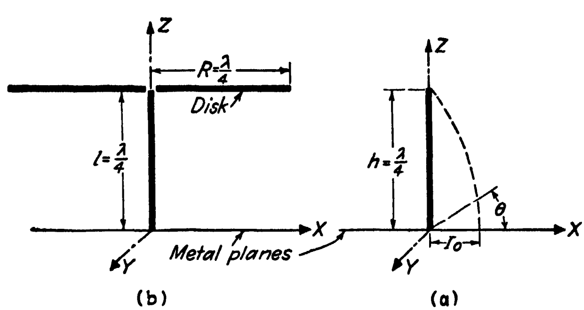


FIG. 206. Radiation pattern of a disk radiator.

$$D'_\theta = 2\, D_\theta \sin(\beta h \sin\theta) \qquad (864f)$$

where D_θ is the directivity given in (864*d*). The value $H = \beta h \sin\theta$ is then the electrical separation of the disk from the reference metal sheet obstruction. If we compare (864*c*) with the expression

$$\mathcal{E}^{(\text{volts/m})} = 60\,\frac{I_0}{r}\,\frac{\cos(90^\circ \sin\theta)}{\cos\theta} = 30\,\frac{I_0}{r}\,D''_\theta \qquad (865)$$



$$\beta R = \frac{2\pi}{\lambda}\,\frac{\lambda}{4} = 1.57 \text{ radians}$$

Feeder h and horizontal disk radiating

FIG. 207. Arrangement for a circular disk and vertical linear feeder (both radiating).

of a vertical $\frac{1}{4}\lambda$ linear radiator (Fig. 207*a*), we note that the directivity D''_θ of such a radiator has a maximum effect D_{max} for zero elevation and

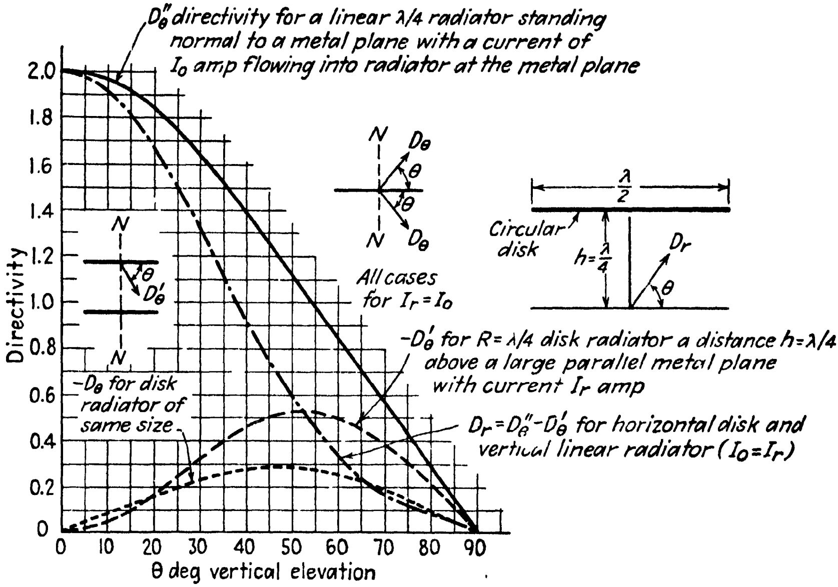


FIG. 208. Directivities for disk radiator and disk radiator together with vertical linear radiator.

zero effect along the vertical radiator direction. Therefore, by combining a disk with $R = \frac{1}{4}\lambda$ meters and a linear $\frac{1}{4}\lambda$ radiator ($h = \frac{1}{4}\lambda$ meters) as in Fig. 207*b*, we must add up the radiation effects, according to the field expressions as given under "Remarks" in Table 39. The curves in Fig. 208 show the resultant directivity D_r where the current I_r is taken equal to I_0. Inasmuch as the component pattern D'_θ holding for the disk by itself and the pattern D''_{θ_1} holding for the linear radiator by itself are such that D''_θ is negative, the resultant D_r pattern is found by the difference of both effects and by the proper choice of the ratio I_0/I_r. We can cause cancellation as is done in Fig. 209 at an elevation of about $\theta = 52$ deg. Figure 210 gives the cor-

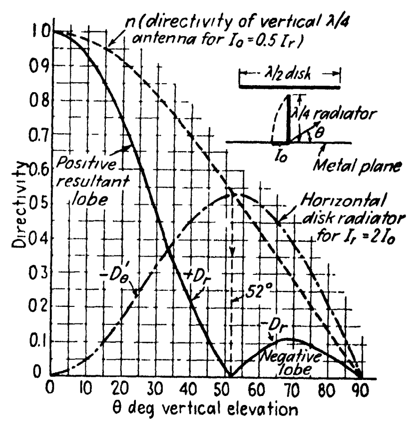


FIG. 209. Resultant D_r pattern for a disk radiator and a normal linear radiator.

TABLE 39. HORIZONTAL DISK RADIATOR $R = \frac{1}{4}\lambda$ PARALLEL WITH A HORIZONTAL REFERENCE METAL PLANE OF SUFFICIENT EXTENSION WITH A SPACING $h = \frac{1}{4}\lambda$ AND A VERTICAL $\frac{1}{4}\lambda$ LINEAR RADIATOR (FIG. 208)

θ, deg	For *horizontal* disk $R = h = \lambda/4$; $\beta R = \frac{2\pi R}{\lambda} = \frac{\pi}{2}$; $\beta h = 90°$; $H = \beta h \sin\theta = 90° \sin\theta$										For *vertical* linear radiator $h = \frac{\lambda}{4}$ (Fig. 208)				
	$\cos\theta$	$k = \beta R \cos\theta = 1.57 \cos\theta$	$J_0(k)$	$J_0(k) - 1 = m$	$\tan\theta$	From (864d) $D_\theta = m \tan\theta$	$\beta h \sin\theta = 90° \sin\theta$, deg	$\sin(\beta h \sin\theta) = \sin H$	$2 \sin H$	From (864f) $D_H' = 2D_\theta \sin H$	$\cos(90° \times \sin\theta)$	$n = \frac{\cos(90° \sin\theta)}{\cos\theta}$	$D_\theta'' = 2n$	$I_r = I_0$ Resultant directivity $D_r = D_\theta' + D_\theta''$	D_r' for $I_0 = 0.5I_r$
0	1.0000	1.57	0.4725	−0.5275	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	2.0000	2.0000	1.0000
10	0.9848	1.54	0.4894	−0.5106	0.1763	−0.0881	15.6	0.2689	0.5378	−0.0475	0.9632	0.98	1.96	1.9125	0.9325
20	0.9387	1.47	0.5285	−0.4715	0.364	−0.171	30.7	0.5105	1.0210	−0.1745	0.8599	0.915	1.83	1.6555	0.7405
30	0.866	1.36	0.5884	−0.4116	0.5774	−0.236	45	0.7071	1.4142	−0.334	0.7071	0.815	1.63	1.296	0.481
40	0.766	1.2	0.6711	−0.3289	0.8391	−0.275	58	0.848	1.696	−0.465	0.5299	0.69	1.38	0.915	0.225
50	0.6428	1.01	0.7608	−0.2392	1.1918	−0.285	69	0.9336	1.8672	−0.531	0.3584	0.555	1.11	0.579	0.024
55	0.5736	0.9	0.8075	−0.1925	1.4281	−0.275	73.5	0.9588	1.9176	−0.53	0.284	0.492	0.984	0.454	−0.038
60	0.5	0.785	0.8518	−0.1482	1.7321	−0.256	78	0.9781	1.9562	−0.501	0.2079	0.417	0.834	0.333	−0.084
70	0.342	0.538	0.929	−0.071	2.7475	−0.195	84.6	0.9956	1.9912	−0.389	0.0941	0.275	0.55	0.161	−0.114
80	0.1736	0.274	0.9812	−0.0188	5.6713	−0.107	88.5	0.9997	1.9994	−0.214	0.0262	0.151	0.302	0.088	−0.063
90	0.0000	0.000	1.0000	0.0000	∞	0.000	90	1.0000	2.0000	0.000	0.0000	0.000	0.000	0.000	0.000

Remarks: D_θ holds for the directivity of the disk in unobstructed radiation space and is due to Eq. (864c), where $\mathcal{E}_\theta = 30\,(I_r/r)\,m \tan\theta = \frac{30I_r}{r}\underbrace{m \tan\theta}$ volts per meter for I_r in amperes and r in meters. D_θ' is the directivity for the same disk with a large parallel metal plate at a distance $h = \lambda/4$ from the disk; D_θ'' is the directivity for a $\lambda/4$ linear radiator normal to the extended metal plate (Fig. 208). Formula (865) is basic since $\mathcal{E}_\theta = (60I_0/r)[\cos(90° \sin\theta)/\cos\theta] = (30I_0/r)D_\theta''$ volts per meter holds.

responding polar patterns in any vertical plane about the vertical quarter-wave linear radiator pointing toward the center of the radiating circular disk.

143. Radiation Effects from Cylindrical Surfaces. Similarly to the procedure employed in connection with disk radiators, we can select in Fig. 211 four current elements I_c, which are symmetrically disposed of with respect to radiation effects in the XZ plane. The sine terms cancel

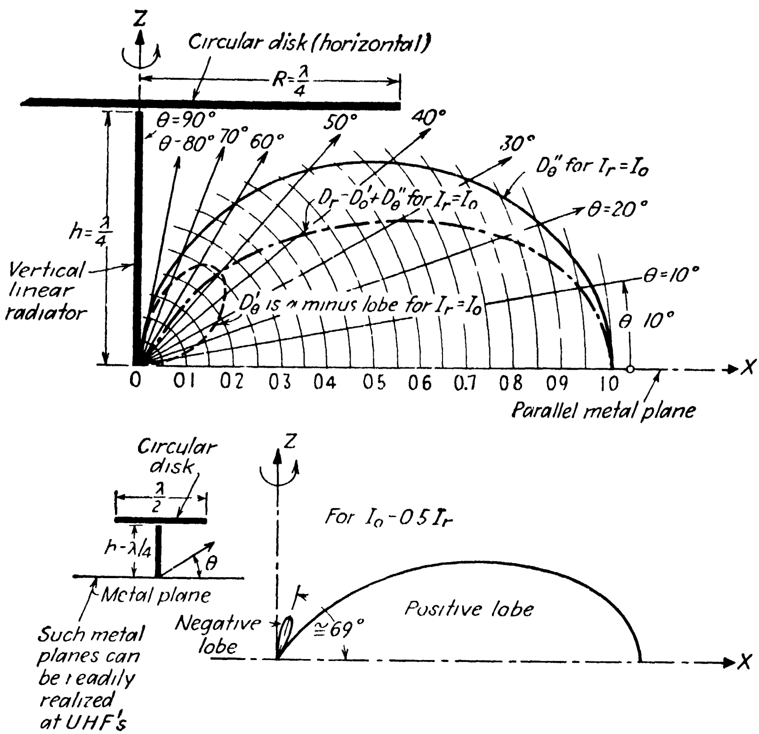


FIG 210 Amplitude directivity for a disk and perpendicular linear radiator with equal and unequal antinode currents

with respect to such an XZ plane. This also holds, of course, for any other plane through the cylinder axis Z with such symmetrical current elements. Inasmuch as the current I_z along any infinitesimal dz line on the cylindrical surface is constant, such a linear element dz acts with an electric current moment of $I_z\,dz$ meter · amperes. For an elementary surface ring of elementary height dz expressed in meter units and width $R\,d\varphi$ also expressed in meters, we have the surface moment $I_z\,dz\,2\pi R$ square meter · amperes. For an elementary surface, we have

then a current moment

$$I_e = \frac{R\,d\varphi}{2\pi R} I\,dz = \frac{I\,dz}{2\pi}\,d\varphi$$

For fine wires, the cylindrical surface of the wire coincides practically with the Z axis (cylinder axis). The distances r_z and r' of Fig. 211 are

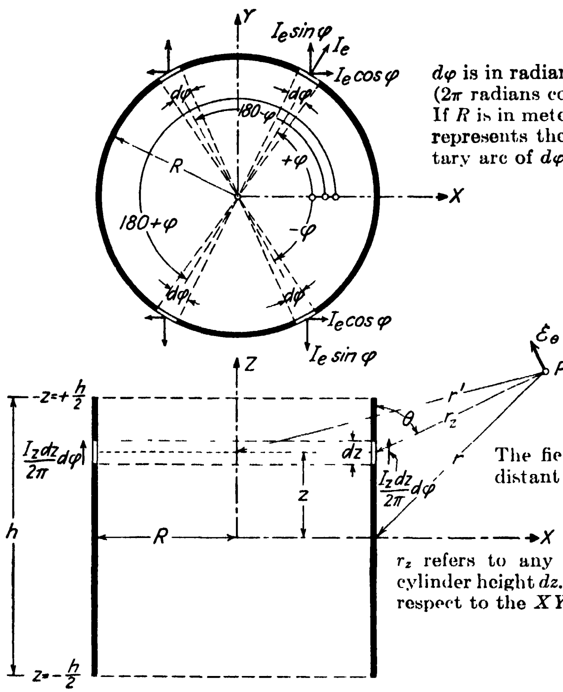


FIG. 211. Radiation surface elements on the outside cylinder surface for a cylindrical radiator (hollow cylinder).

then identical. In our case, r_z is the distance from any surface current element I_e to a space point $P(r_z, \varphi = 0, \theta)$, as far as the point P in the figure is concerned. For a very fine copper tube, the field contribution at distant point P is

$$d\mathcal{E}_\theta = \frac{188.35 I\,dz}{r_z \lambda} \sin\theta \qquad \text{volts/m}$$

$$I = I_z \sin(\omega t - \beta r_z) = I_z\left(\omega t - \frac{\omega r_z}{c}\right) \qquad \text{volts/m}$$

The cylindrical surface element, for R/λ very small, becomes identical with a current element $I_z\,dz$ along the Z axis. We have then the case shown in Fig. 212, which applies to a linear radiator. Suppose we assume a half-wave sinusoidal current

distribution (since a filament radiator). As far as the distant field point P is concerned, we have two symmetrical current elements $\pm I_z\, dz$. This case has already been treated, and we found the solution

$$\mathcal{E}_\theta = \frac{188.35 I_0 \sin\theta}{r\lambda} \int_{z=0}^{z=\lambda/4} \left[\sin\omega\left(t - \frac{r - z\cos\theta}{c}\right) + \sin\omega\left(t - \frac{r + z\cos\theta}{c}\right)\right] \cos(\beta z)\, dz = \frac{60 I_0}{r} D_0 \sin\left[\omega\left(t - \frac{r}{c}\right)\right] \quad \text{volts/m}$$

$$D_0 = \frac{\cos(90^\circ \cos\theta)}{\sin\theta}; \qquad c = 3 \times 10^8 \text{ m/sec}$$

The directivity D_0 must then also play a part when we deal with the general case of

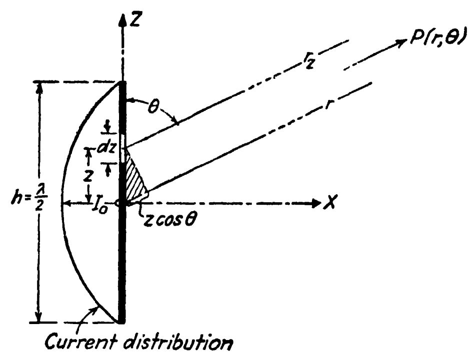


FIG. 212. Conditions for a thin linear filament half-wave radiator. Diameter of linear radiator assumed very small (filament radiator). $I_z = I_0 \cos(\beta z)$ for the $\lambda/2$ distribution.

Fig. 211, for which R/λ is no longer a negligible quantity. We find the field contribution at P, due to the four elementary I_z radiators, as

$$d\mathcal{E}_\theta = \frac{4 \times 188.35 I\, dz \sin\theta}{r_z\lambda} \cos(\beta R \cos\varphi \sin\theta)\, d\varphi \quad \text{volts/m}$$

$$I = I_z \sin(\omega t - \beta r_z) \quad \text{amp}$$

for any I_z distribution along the cylindrical surface. For an infinitely short cylinder of height dz, as well as a cylinder at least so short that $h = \Delta z$ has essentially a fixed I_z current distribution along Δz, we find

$$\mathcal{E}_\theta = \frac{754 I_z\, dz \sin\theta}{r\lambda} \int_{\varphi=0^\circ}^{\varphi=90^\circ} \cos(\beta R \cos\varphi \sin\theta)\, d\varphi$$

$$= \frac{754 I_z\, dz \sin\theta}{r\lambda} \int_0^{\pi/2} \cos(k \cos\varphi)\, d\varphi \qquad \text{volts/m}$$

$$k = \beta R \sin\theta$$

But

$$\int_0^{\pi/2} \cos(k\cos\varphi)\, d\varphi = \frac{\pi}{2} J_0(k) \tag{866}$$

and

$$\mathcal{E}_\theta = \frac{1.57 \times 754 I_z\, dz \sin\theta}{r\lambda} J_0(k) = \frac{1185 I_z \sin\theta}{r\lambda} J_0(k) \quad \text{volts/m}; \qquad k = \beta R \sin\theta \tag{866a}$$

In this expression, I_z denotes the current value in amperes along dz and $J_0(k) \sin\theta$ represents the directivity for a cylinder of infinitesimally short height dz or small height Δz, for which the I_z distribution remains practically constant. Investigating a practical application of a half-wavelength cylindrical radiator, for which $z = \frac{1}{4}\lambda$, we have $I_z = I_0 \cos(\beta z)$. Hence, at the distant field point P of Fig. 211,

$$\mathcal{E}_\theta = \frac{754 I_0 \sin\theta}{r\lambda} \int_{\varphi=0}^{\varphi=\pi/2} \int_{z=0}^{z=\lambda/4} \cos(\beta R \cos\varphi \sin\theta) q\, d\varphi \qquad \text{volts/m}$$

$$q = \left[\sin\omega\left(t - \frac{r_z - z\cos\theta}{c}\right) + \sin\omega\left(t - \frac{r_z + z\cos\theta}{c}\right)\right] \cos(\beta z)\, dz$$

According to (866), the integration with respect to φ results in a function proportional to $J_0(k)$, for $k = \beta R \sin\theta$, and

$$\mathcal{E}_\theta = \frac{754 \frac{\pi}{2} I_0 \sin\theta}{r\lambda} J_0(k) \int_{z=0}^{z=\lambda/4} \left\{ \sin\left[\omega\left(t - \frac{r - z\cos\theta}{c}\right)\right] + \sin\left[\omega\left(t - \frac{r + z\cos\theta}{c}\right)\right]\right\} \cos(\beta z)\, dz$$

$$= \frac{754\pi I_0 J_0(k) \sin\theta}{r\lambda} \sin\left[\omega\left(t - \frac{r}{c}\right)\right] \int_{z=0}^{z=\lambda/4} \underbrace{\cos(\beta z)}_{=a} \cos\underbrace{(\beta z\cos\theta)}_{=b}\, dz \qquad \text{volts/m} \tag{866b}$$

But

$$\cos a \cos b = \tfrac{1}{2}[\cos(a+b) + \cos(a-b)] = \tfrac{1}{2}\{\cos[\beta(1+\cos\theta)z] + \cos[\beta(1-\cos\theta)z]\}$$

$$\beta z = \frac{2\pi z}{\lambda}; \qquad \beta\frac{\lambda}{4} = \frac{\pi}{2} \text{ or } 90°; \qquad \int \cos(mz)\, dz = \frac{\sin(mz)}{m}$$

$$\int_{z=0}^{z=\lambda/4} \cos a \cos b\, dz = \tfrac{1}{2}\left\{\int_0^{\lambda/4} \cos[\beta(1+\cos\theta)z]\, dz + \int_0^{\lambda/4} \cos[\beta(1-\cos\theta)z]\, dz\right\}$$

$$= \frac{1}{2}\left|\frac{\sin[\beta(1+\cos\theta)z]}{\beta(1+\cos\theta)} + \frac{\sin[\beta(1-\cos\theta)z]}{\beta(1-\cos\theta)}\right|_{z=0}^{z=\lambda/4}$$

$$= \frac{1}{2}\left\{\frac{\sin\overbrace{[90°(1-\cos\theta)]}^{=d}}{\beta(1+\cos\theta)} + \frac{\sin\overbrace{[90°(1-\cos\theta)]}^{=e}}{\beta(1-\cos\theta)}\right\}$$

$$= \frac{1}{2}\left\{\frac{(1-\cos\theta)\sin d + (1+\cos\theta)\sin e}{\beta(1-\cos^2\theta)}\right\}$$

$$1 - \cos^2\theta = \sin^2\theta$$

For $m = 90$ deg $\cos\theta$, we find

$$\sin d = \sin (90° + m) = \sin 90° \cos m + \cos 90° \sin m$$
$$\sin e = \sin (90° - m) = \sin 90° \cos m - \cos 90° \sin m$$
$$(1 - \cos\theta)\sin d + (1 + \cos\theta)\sin e = \sin d - \cos\theta \sin d + \sin e + \cos\theta \sin e$$
$$= \sin d + \sin e - \cos\theta(\sin d - \sin e) = 2(\sin 90° \cos m - \overbrace{\cos\theta \cos 90° \sin m}^{=0})$$

$$\int_{z=0}^{z=\lambda/4} \cos a \cos b \, dz = 0.5\,\frac{2 \sin 90° \cos m}{\beta \sin^2\theta} = \frac{\cos (90° \cos\theta)}{\beta \sin^2\theta} = \frac{\cos (90° \cos\theta)}{2\pi \sin^2\theta}\,\lambda$$

which, inserted in (866b), yields

$$\mathcal{E}_0 = \frac{376.7 I_0 \sin\left[\omega\left(t - \frac{r}{c}\right)\right]}{r} D_0 J_0(k) \quad \text{volts/m}; \qquad D_0 = \frac{\cos (90° \cos\theta)}{\sin\theta};$$
$$k = \beta R \sin\theta \qquad (866c)$$

where D_0 is the vertical directivity when the cylindrical radiator has a radius R so small that the cylinder and its axis coincide. The directivity D_θ of the cylindrical radiator is, therefore,

$$D_\theta = D_0 J_0(k) = \frac{\cos (90° \cos\theta)}{\sin\theta} J_0(\beta R \sin\theta) \qquad (866d)$$

From (866d), we note that for $R = 0$, the Bessel function $J_0(k) = J_0(0) = 1$, and the cylindrical ½λ radiator represents a linear fine conductor with the customary D_0 directivity in any meridian plane, in the case of unobstructed radiation space. For the thin linear half-wave radiator, we have always maximum field effect along the equatorial XY plane of the radiator. This is *generally not* true for a hollow cylindrical half-wave radiator with R/λ a noticeable ratio. As a matter of fact, $J_0(k)$ becomes zero (Fig. 182) for

$$k = \beta R \sin\theta = 2.4048,\ 5.5201,\ 8.6537,\ 11.7915,\ 14.9309,\ 18.0711,\ \cdots$$

since the Bessel curve cuts through the k reference axis for such values. The equatorial plane is characterized by the value $\theta = 90$ deg or $k = \beta R$, and we have then no electromagnetic field along all directions in the equatorial XY plane for the above-mentioned k values. Since $\beta = 6.28/\lambda$ and $k = 6.28R/\lambda$, zero equatorial field occurs, for instance, if $R/\lambda = 2.404/6.28 = 0.382$. Table 40 gives the computation for $R/\lambda = 0.25$, 0.382, and 0.5. The D_0 values are taken from the D_θ column of Table 25. Figure 213a and b gives the respective relative directivities of $\lambda/2$ cylindrical heights for different radii. For $R = 0.5\lambda$, we obtain two space patterns of which one lobe is of opposite polarity with respect to the other lobe. We also note that for $R = 0.382\lambda$, we have no radiation effects along the equatorial plane ($\theta = 90$ deg). In order to see from which radius R on, in terms of the wavelength λ, the magnitude of

TABLE 40

θ, deg	D_0 numeric	$\sin\theta$ numeric	For $\frac{R}{\lambda} = 0.25$; $\beta R = 6.28 \times 0.25 = 1.57$; $k = \beta R \sin\theta = 1.57 \sin\theta$			For $\frac{R}{\lambda} = 0.382$; $\beta R = 6.28 \times 0.382 = 2.4$; $k = 2.4 \sin\theta$			For $\frac{R}{\lambda} = 0.5$; $\beta R = 6.28 \times 0.5 = 3.14$; $k = 3.14 \sin\theta$		
			$k = 1.57 \sin\theta$	$J_0(k)$	$D_\theta = D_0 J_0(k)$	$k = 2.4 \sin\theta$	$J_0(k)$	$D_\theta = D_0 J_0(k)$	$k = 3.14 \sin\theta$	$J_0(k)$	$D_\theta = D_0 J_0(k)$
00	0.000	0.0000	0 000	1.000	0.000	0.000	1.0000	0.0000	0.0000	1.0000	0.000
10	0.142	0.1736	0.274	0.985	0.149	0.417	0.9575	0.144	0 546	0.925	0.14
20	0.28	0.342	0.539	0.928	0.255	0.821	0.839	0.23	1.07	0.734	0.202
30	0.417	0.5	0.785	0.8517	0 356	1.2	0.671	0.28	1.57	0.472	0.197
40	0.556	0.6428	1.01	0.761	0.424	1.54	0.489	0.271	2.02	0.2124	0.118
50	0.693	0.766	1.2	0.671	0.465	1 84	0.3167	0.22	2.4	0.0000	0.000
60	0.816	0.866	1 36	0.588	0.478	2.075	0.18	0.1465	2.71	−0.147	−0.12
70	0.912	0.9397	1.475	0 528	0.484	2 25	0.0827	0.0755	2.94	−0.239	−0.219
80	0.98	0.9848	1.545	0.489	0 48	2.36	0.0235	0.023	3.09	−0.289	−0.284
90	1.000	1.0000	1.57	0 472	0 472	2.4	0.0000	0.0000	3.04	−0.314	−0.314

Remarks: **R is the radius of the hollow cylinder of height $h = \frac{1}{2}\lambda$ and the Bessel function $J_0(k)$ is found from tables or curves (page 704).**

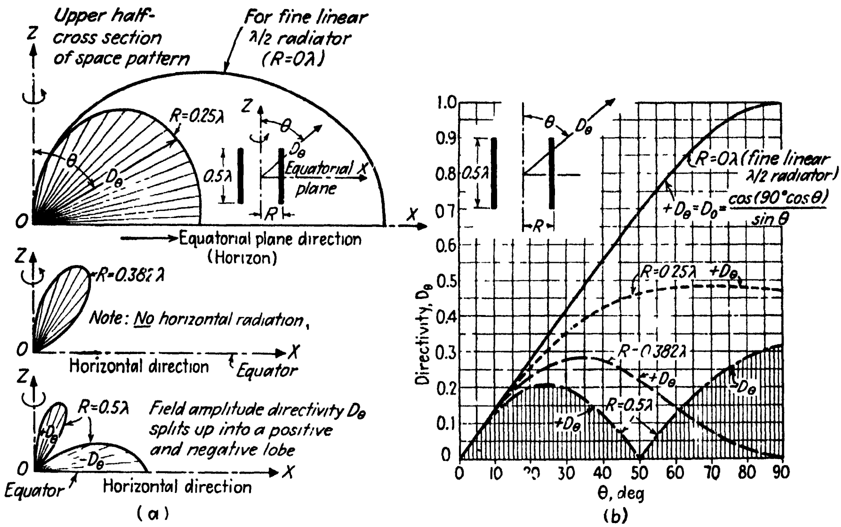


FIG. 213. Field amplitude directivity (relative) of hollow cylindrical radiator compared with a thin linear radiator. (*a*) Polar plot. (*b*) Rectangular plot.

R affects the radiation pattern, if compared with a thin linear radiator, the computations of Table 41 are added. Figure 214 shows the results for different cylinder radii. It is seen that for angles up to $\theta = 30$ deg,

TABLE 41

θ, deg	D_0	sin θ	$R = \frac{\lambda}{8}$; $\beta R = 0.785$			$R = 0.1\lambda$; $\beta R = 0.625$		
			$k =$ 0.785 sin θ	$J_0(k)$	$D_\theta = D_0 J_0(k)$	$k =$ 0.625 sin θ	$J_0(k)$	$D_\theta = D_0 J_0(k)$
00	0 000	0.000	0 0000	1 000	0 0000	0.000	1.000	0 0000
10	0.142	0.174	0 136	0 995	0 15	0.108	0.997	1.505
20	0 28	0.342	0 269	0 982	0 27	0.214	0.988	0 271
30	0.417	0 5	0 3925	0 962	0 401	0 3125	0.976	0.406
40	0 556	0.643	0.501	0 938	0.52	0.402	0.96	0.532
50	0.693	0.766	0.6	0 912	0.63	0.479	0.943	0.65
60	0.816	0.866	0 68	0 888	0.72	0.54	0.928	0.758
70	0.912	0 9397	0 74	0 868	0.79	0.588	0.915	0.835
80	0.98	0.985	0.77	0 857	0.836	0 615	0 907	0.885
90	1.000	1 000	0 785	0 85	0 785	0 625	0.904	0.904

Remarks: Some of the columns are taken from Table 40 such as the D_0 and sin θ values.

i.e., an elevation of 60 deg and more, the radius of the hollow cylinder does not affect much the directivity for the magnitudes shown in this

figure. Now for a 100-Mc excitation, we have a wavelength of $300/100 =$ 3 m in free space and one-tenth of it is 30 cm, or $30/2.54 = 11.8$ in. The diameter of the hollow cylinder is then about 2 ft for the $R = 0.1\lambda$ condition. Therefore, in cases where we use quarter-wave sleeves as partial radiators, it can be said that at frequencies below 100 Mc/sec, the customary diameters of sleeves should not materially affect the field pattern.

144. Notes on Concentric Cage Aerials. This has reference to Fig. 215, where we have a linear radiator along the axis and many such radiators symmetrically surrounding it, so that radiation interference is

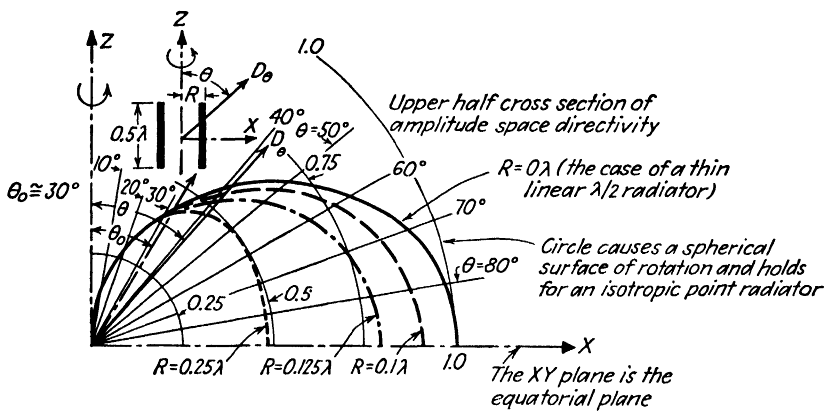


FIG. 214. Relative amplitude directivity D_θ for half-wave hollow cylindrical radiators in comparison with a thin linear half-wave radiator.

possible into the outside space not only through respective ends but also all around the array. If the outside cage were a hollow cylinder, we should have the case of a concentric line radiator. Suppose the cage radiators are on a circle of radius R and h meters in length. It is assumed that we have indicated current distributions and $I_1 = mI_0$ amperes. When both currents are in phase, then m is real and positive; while for antiphase excitations, m is real and negative. Thus for $m = 1$ and antiphase condition, $I_1 = -I_0$. The radiation directivity for the axial linear radiator is, for indicated $\lambda/2$ length,

$$D_0 = \frac{\cos (90° \cos \theta)}{\sin \theta}$$

For the outside array approximating the condition for a solid hollow cylinder, the directivity becomes

$$D_1 = mD_0J_0(k) = mD_0J_0(\beta R \sin \theta)$$

yielding a resultant directivity

$$D_\theta = D_0 + D_1 = D_0\left[1 + mJ_0\left(6.28\frac{R}{\lambda}\sin\theta\right)\right]$$

where $J_0(k)$ stands again for the Bessel function of the first kind and zeroth order. For the general case, the respective axial and the circular array are not driven in phase and we have $I_1 = (m + jn)I_0$, where m

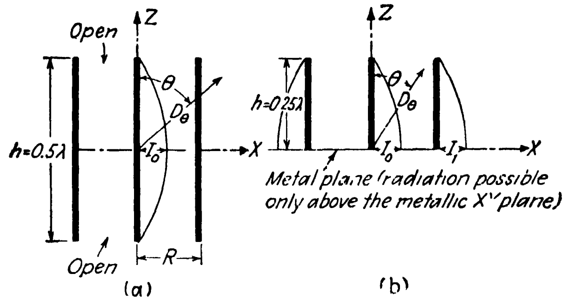


FIG. 215. Conditions for coaxial radiators.

and n are real numbers and mI_0 denotes the inphase and jnI_0 the quadrature component with respect to the phase of mI_0. Therefore, $\tan^{-1}(n/m)$ is the phase angle between inside and outside excitations. The magnitude of I_1 is then $I_0\sqrt{m^2 + n^2}$, with a resultant directivity

$$D_\theta = D_0\sqrt{[1 + mJ_0(k)]^2 + [nJ_0(k)]^2}; \qquad k = 6.28\frac{R}{\lambda}\sin\theta;$$
$$D_0 = \frac{\cos(90^\circ\cos\theta)}{\sin\theta}.$$

since

$$D_\theta = D_0 + D_1 = D_0 + D_0[(m + jn)J_0(k)] = D_0[\underbrace{1 + mJ_0(k)}_{\text{real}} + jnJ_0(k)]$$

An application is as follows:

For $R = 0.25\lambda$, we have $k = (6.28R/\lambda)\sin\theta = 1.57\sin\theta$. Suppose the total current I_1 flowing through the XY plane is twice the value of the axial value I_0 and 30 deg out of phase. Then $m = 2\cos 30^\circ = 2 \times 0.866 = 1.732$ and $n = 2\sin 30^\circ = 2 \times 0.5 = 1$. Therefore,

$$D_\theta = \frac{\cos(90^\circ\cos\theta)}{\sin\theta}\sqrt{[1 + 1.732J_0(1.57\sin\theta)]^2 + J_0^2(1.57\sin\theta)}$$

The directivity in the equatorial XY plane is found from $\theta = 90$ deg, as

$$D_{\theta-90^\circ} = \sqrt{[1 + 1.732J_0(1.57)]^2 + J_0^2(1.57)} = \sqrt{(1 + 1.732 \times 0.472)^2 + 0.472^2}$$
$$= \sqrt{3.3015 + 0.22784} = 1.88$$

since $\cos(90^\circ\cos 90^\circ) = 1$, $\sin 90^\circ = 1$, $D_0 = 1$, instead of unity value as in the case of a linear radiator. For $\theta = 30$ deg, we have $\sin\theta = 0.5$, $\cos\theta = 0.866$,

$\cos (90° \cos \theta) = \cos (90° \times 0.866) = \cos 78° = 0.2079,$

$$D_0 = \frac{0.2079}{0.5} = 0.4158; \qquad k = 1.57 \sin 30° = 1.57 \times 0.5 = 0.785$$

$$J_0(0.785) = 0.8518; \qquad J_0^2(0.785) = 0.7259$$

$$D_{30°} = 0.4158 \sqrt{(1 + 1.732 \times 0.8518)^2 + 0.7259} = 0.4158 \times 2.62 = 1.09$$

Table 42 and Fig. 216 give the results for all other angles.

TABLE 42. FOR $R = \lambda/4$, $h = \lambda/2$

$I_1 = 2I_0/\underline{30°}$; $m = 2 \cos 30° = 1.732$; $n = 2 \sin 30° = 1$, $D_\theta = D_0 \sqrt{[1 + 1.732 J_0(k)]^2 + J_0^2(k)}$

θ, deg	D_0	$k = 1.57 \times \sin \theta$	$J_0(k)$	$J_0^2(k) = p$	$1.732 \times J_0(k)$	$1 + 1.732 \times J_0(k)$	$q = [1 + 1.732 \times J_0(k)]^2$	$p + q$	$s = \sqrt{p + q}$	$D_\theta = sD_0$
00	0.000	0.000	1.000	1.000	1.732	2.732	7.453	8.7453	2.95	0.000
10	0.142	0.279	0.985	0.97	1.7	2.7	7.29	8.26	2.88	0.435
20	0.28	0.539	0.928	0.861	1.6	2.6	6.76	7.62	2.75	0.755
30	0.417	0.785	0.8518	0.7259	1.475	2.475	6.126	6.85	2.62	1.09
40	0.556	1.01	0.761	0.579	1.32	2.32	5.38	5.96	2.44	1.35
50	0.693	1.2	0.671	0.45	1.16	2.16	4.66	5.11	2.25	1.56
60	0.816	1.36	0.588	0.346	1.02	2.02	4.08	4.426	2.1	1.71
70	0.912	1.475	0.528	0.279	0.91	1.91	3.65	3.93	1.98	1.82
80	0.98	1.545	0.483	0.239	0.845	1.845	3.38	3.62	1.9	1.86
90	1.0	1.57	0.472	0.2228	0.817	1.817	3.31	3.529	1.88	1.88

Remarks D_0 values, k values, and $J_0(k)$ values are taken from Table 40.

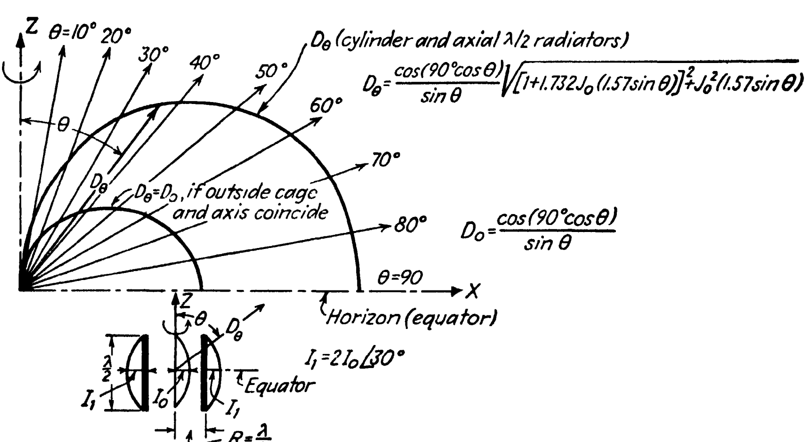


FIG. 216. Relative field amplitude directivity D_θ for a concentric cage radiating system.

145. Remarks on Circuits Based on Maxwell's Equations. So far, we have applied Maxwell's equations without any special reference to

impressed forces[1] as distinguished from forces arising from the circuit elements themselves. A distinction was made, however, indirectly; otherwise we should have violated concepts that are well known even in customary circuit analysis. We know that even at commercial frequencies, an applied driving voltage first has to overcome the ohmic resistance drop, besides any voltage drops due to inductive and capacitive effects. The electromagnetic relations are then

$$\left.\begin{aligned} &\mathcal{E}_i^{(\text{volts/m})} = \frac{J}{\sigma} + \nabla V + \frac{\partial \mathbf{A}}{\partial t} \quad \text{differential relation} \\ &\underbrace{\int_1^2 \mathcal{E}_i\, dl}_{E_i \text{ impressed volts}} = \int_1^2 \frac{J}{\sigma}\, dl + \int_1^2 \nabla V\, dl + \int_1^2 \frac{\partial \mathbf{A}}{\partial t}\, dl \quad \text{integral relation} \\ &\mathbf{A} = \frac{\mu}{4\pi} \int\!\!\int\!\!\int_\tau \frac{[J]}{r}\, d\tau \quad \text{volt}\cdot\text{sec/m}; \\ &\qquad V = \frac{1}{4\pi\kappa} \int\!\!\int\!\!\int_\tau \frac{[\rho]}{r}\, d\tau \quad \text{volts} \end{aligned}\right\} \quad (867)$$

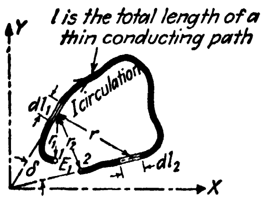
where E_i volts is impressed across the open input terminals 1 and 2 of a closed path as in Fig. 217a. The quantity $\mathbf{A}$ denotes the magnetic vector potential with the retarded current density $[J]$, and V is the scalar potential with the retarded space charge density $[\rho]$, where J is in amperes per square meter and ρ in coulombs per cubic meter. For a path from 1 to 2 consisting of a current filament, we deal throughout with line integrals. From the differential as well as the integral form, we note that an impressed electric field of $\mathcal{E}_i$ volts per meter refers generally to a term J/σ (ohmic) for a line element, a term grad (V) due to charges, and a term $\partial\mathbf{A}/\partial t$ due to a current of the line element dl.

As far as the mks system of units goes, the specific conductivity σ is expressed in mhos per meter, which is mhos across 1-sq-m parallel faces separated by 1 m distance. Since then the magnitude of dl is likewise expressed in meters, even though infinitesimal, σ is apt to vary within a cubic meter of material. Therefore, J/σ refers only to a point of a conductor or a point of the general medium that exhibits, besides μ and κ, also σ effects and the portion J/σ is due to the σ effect. Inasmuch as we also have $J = \sigma\mathcal{E}$, the field intensity $\mathcal{E}$ in volts per meter at a point must be the *resultant* electric field at a space point that is generally due to charges and currents as well as an impressed $\mathcal{E}_i$ field.

As far as short waves extending into the microwave region are of concern in this text, it would *not be correct* to take for granted that effects at one point of a network cause *simultaneous* effects at other points of

[1] Except in relations for impressed EM fields (see pp. 130–132, 235, 296, 650, 654–656.

said network, since the dimensions of a network along dl line elements as well between dl and dl' current path elements are generally an appreciable portion of the operating wavelength. This fact was mentioned on pages 166, 650, 656, in connection with a radiating half-wave dipole,

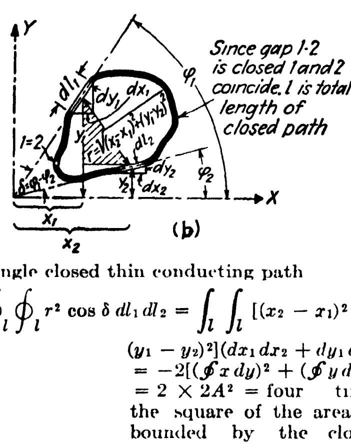


(a)

Terminals 1 and 2 are an infinitesimal gap for applying E_t volts.

$$R_{self} = R_r = \frac{\omega\mu}{4\pi}\oint_l\oint_l \frac{\sin(\beta r)\, dl_2 \cos\delta}{r}\, dl_1 \text{ ohms}$$

$$L = \frac{\mu}{4\pi}\oint_l\oint_l \frac{\cos(\beta r)\, dl_2 \cos\delta}{r}\, dl_1 \text{ henrys}$$

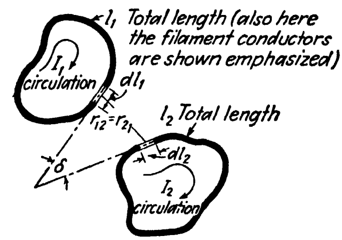


(b)

Single closed thin conducting path

$$\oint_l\oint_l r^2\cos\delta\, dl_1\, dl_2 = \int_l\int_l [(x_2 - x_1)^2 + (y_1 - y_2)^2](dx_1\, dx_2 + dy_1\, dy_2) = -2[(\oint x\, dy)^2 + (\oint y\, dx)^2] = 2 \times 2A^2$$

= four times the square of the area A bounded by the closed path l



(c)

$$R_m = \frac{\omega\mu}{4\pi}\oint_{l_1}\oint_{l_2}\frac{\sin(\beta r_{12})\cos\delta}{r_{12}}\, dl_1\, dl_2 \text{ ohms}$$

$$X_m = -\frac{\omega\mu}{4\pi}\int_{l_1}\oint_{l_2}\frac{\cos(\beta r_{12})\cos\delta}{r_{12}}\, dl_1\, dl_2 \text{ ohms}$$

$$\sin\varphi_1 = \frac{dy_1}{dl_1};\ \sin\varphi_2 = \frac{dy_2}{dl_2};\ \cos\varphi_1 = \frac{dx_1}{dl_1}$$

$$\cos\varphi_2 = \frac{dx_2}{dl_2}$$

$$\cos\delta = \cos(\varphi_1 - \varphi_2) = \cos\varphi_1\cos\varphi_2 \mp \sin\varphi_1\sin\varphi_2 = \frac{dx_1}{dl_1}\frac{dx_2}{dl_2} + \frac{dy_1}{dl_1}\frac{dy_2}{dl_2}$$

$$\oint_l\oint_l \cos\delta\, dl_1\, dl_2 = \oint_l\oint_l (dx_1\, dx_2 + dy_1\, dy_2) - \oint_l dx_1 \oint_l dx_2 + \oint dy_1 \oint dy_2 = 0$$

since $\oint_l dx_1 = 0 = \oint_l dx_2 = \oint_l dy_1 = \oint_l dy_2$

because total variation of dx_1, dx_2, dy_1, and dy_2, respectively, *must vanish* for a *closed* path l.

FIG. 217. Conditions of a single closed path and two coupled closed paths of negligible conductor cross section.

where the effect along the dipole at dl of the dipole, a distance of $\frac{1}{4}\lambda$ away from a dl' radiator element occurs 90 time degrees later. This is because we deal with *finite* velocities as far as actions due to current effect in a dl element are concerned on a dl' element and vice versa. There would be no difficulty in dealing with retarded scalar V and retarded magnetic vector potentials A if it were not for the charge ρ and current J distributions. It has, therefore, become customary to

assume a current distribution. From this, we find Λ and V† and the corresponding H and $\mathcal{E}$ fields at a distant field point. For a more accurate result, it would be necessary to make corrections for previous assumptions for the current distribution by the method of successive repetition.

As far as magnetic and electric field effects are concerned with assumed infinite propagation velocities (such velocities do not exist), it is true that respective fields merely oscillate between the source and the space about it. This means that just as much electromagnetic field energy leaves the source as returns to it. We have, therefore, what is known as "wattless" field energy exchanges. This is what occurs practically at low frequencies, because actions due to charge and current effects at one portion of a network cause at other portions of the network essentially simultaneous actions. The dimensions of the network are for low frequencies vanishingly small if taken in terms of the operating wavelength. Thus for a 60-cycle/sec commercial alternating current, we have a free-space wavelength of $3 \times 10^8/60 = 5$ million meters or $6.214 \times 10^{-4} \times 5 \times 10^6 = 3{,}107$ statute miles $= 3{,}107 \times 5{,}280 =$ 16.405 million feet. This is surely an immense dimension if compared with dimensions of an electrical network that is spread out on a laboratory table, for instance. Of course, 16.405×10^6 ft refers to a distance of 1 wavelength. Suppose we have a circular loop that carries a 60-cycle current and has a circumference of 10 ft. Such a circular distance would be only $10\lambda/(16.405 \times 10^6)$ or 6.25×10^{-7} wavelength long. It would be only $10^{+4} \times 6.25 \times 10^{-7}$ or $6.25 \times 10^{-3}\lambda$ long even for a 600-kc/sec current, which belongs to the standard broadcast range of carrier frequencies. This means that even for a 600-kc current flow in the circular conductor, the total current path is only 0.00625λ in length, which is only 0.625 per cent of a wavelength or $0.00625 \times 360° = 2.25$ electrical degrees. We have then practically a uniform current distribution along the circular conductor. This surely cannot be the case if the above-mentioned circular conductor is excited with UHF currents. Suppose we assume a uniform current distribution and apply the integral relation of (867), but in such a way that we have a closed circuit as in Fig. 217b. We find then

$$\int_1^{2=1} = \oint_1^1 \text{ which is a } closed\text{-line integral;} \qquad I = I_0\epsilon^{j\omega t}; \qquad [I] = I_0\epsilon^{j(\omega t - \beta r)} \qquad \text{amp}$$

$$d\Lambda = \frac{\mu[I]\,dl_2 \cos\delta}{4\pi r}$$ volt sec/m for the magnetic vector potential due to $I\,dl_2$ effect along element dl_1

$$d\Lambda \cong \frac{\mu I\,dl_2 \cos\delta}{4\pi r}$$ if the frequency is low and I constant along closed path l

From (867), we find

$$\oint_l \frac{J\,dl}{\sigma} = IlR_0 = RI \quad \text{resistive voltage drop;} \qquad \oint_l \text{grad } V\,dl = 0,$$

$$\text{since } \nabla V = \text{grad } V = \frac{\partial V}{\partial l}$$

$$\oint_l \frac{\partial V}{\partial l}\,dl = \oint_1^1 \frac{\partial V}{\partial l}\,dl = V_1 - V_1 = 0 \qquad \text{volts}$$

† Pages 164, 214–238, 253–258, 267–268, 290, 300, 306, 315, 356 359, 361, 496, 612–614, 617, 781, 782.

Assuming the closed conducting l path thin enough, all current density J is crowded into a *linear* current of I amperes along the conductor elements. We have then an R_0I resistive volt drop along a meter length of the closed path and an $lR_0I = RI$ volts drop for the entire closed loop. We have, therefore, in place of J/σ, the quantity R_0I volts. Moreover, in (867), we have

$$\underset{\cdot}{A} = \frac{\mu}{4\pi}\oint_l \frac{[\underset{\cdot}{I}]\,dl_2}{r} = \underbrace{\frac{\mu}{4\pi}\oint_l \frac{[I]\cos\delta\,dl_2}{r}}_{\substack{\text{magnitude of}\\ \text{retarded } I \text{ action}}} \cong \underbrace{\frac{\mu I}{4\pi}\oint_l \frac{\cos\delta}{r}\,dl_2}_{\text{magnitude}}$$

if the frequency is low enough so that approximately instantaneous I action occurs with respect to all current elements

$$\oint_l \frac{\partial \underset{\cdot}{A}}{\partial t}\,d\underset{\cdot}{l}_1 \cong \underbrace{\oint_l \frac{\partial\left(\frac{\mu I}{4\pi}\oint_l \frac{\cos\delta}{r}\,dl_2\right)}{\partial t}\,dl_1}_{\substack{\text{magnitude at low}\\ \text{frequencies}}} = \frac{dI}{dt}\underbrace{\frac{\mu}{4\pi}\oint_l\oint_l \frac{\cos\delta}{r}\,dl_1\,dl_2}_{\text{self-induction } L \text{ of closed loop}} \qquad \text{volts}$$

$$\oint \frac{\partial \underset{\cdot}{A}}{\partial t}\,d\underset{\cdot}{l}_1 = \frac{dI}{dt}\underbrace{\frac{\mu}{4\pi}\oint_l\oint_l \frac{\epsilon^{-j\beta r}\cos\delta}{r}\,dl_1\,dl_2}_{=L} \qquad \text{volts}$$

magnitude for fixed current along l path and frequencies for which the retarded effect is *not* negligible

For the closed current loop (Fig. 271b), according to (867),

$$\underbrace{\oint \mathcal{E}_i\,d\underset{\cdot}{l}_1}_{\substack{0 \text{ volts,}\\ \text{since 1 and}\\ \text{2 are same}\\ \text{terminals}}} = \underbrace{\oint_l \frac{\underset{\cdot}{J}}{\sigma}\,dl_1}_{\substack{RI \text{ (resist-}\\ \text{ance drop)}}} + \underbrace{\oint_l \text{grad } V\,d\underset{\cdot}{l}_1}_{=0} + \underbrace{\oint_l \frac{\partial \underset{\cdot}{A}}{\partial t}\,dl_1}_{\substack{L(dI/dt)\\ \text{(reactive}\\ \text{drop)}}}$$

Figure 217b has no provision for a localized impressed voltage of E_i volts, since the terminals 1 and 2 of Fig. 217a are connected together. The *closed* loop can be driven, however, by a coupled circuit that is energized or by an arriving electromagnetic wave. The impressed voltage is then *distributed* along the closed l path of Fig. 217b. The impressed distributed voltages are then such as to overcome the voltage drops just found. The result applies, however, if we imagine an infinitesimal gap between terminals 1 and 2 of Fig. 217a and apply across this gap, for instance, a sinusoidal voltage of E_i volts. We have then the well-known circuit relation

$$E_i = RI + L\frac{dI}{dt} \qquad \text{volts, differential form}$$

$$\underset{\cdot}{E}_i = R\underset{\cdot}{I} + j\omega L\underset{\cdot}{I} \qquad \text{volts, vector form for a time variation } \epsilon^{j\omega t}$$

A careless inspection of these relations gives the impression that we have only a joulean heat *loss* component and a wattless magnetic field component for which the field energy oscillates merely between the source that applies the voltage E_i and the space surrounding the conductor of

total length l. Formula (867a) shows, however, that the value of L is not so simple for the HF and surely not for the UHF case, even though we assume or provide a constant current distribution along the conducting path l from 1 to 2. We note that under the integral, we have now a factor $\epsilon^{-j\beta r} = \cos(\beta r) - j \sin(\beta r)$. This means that the self-induction L is of the form $a - jb$, and

$$E_i^{(\text{volts})} = RI + j\omega I(a - jb) = RI + \omega bI + j\omega aI = (R + \omega b)I + j\omega aI$$

We have then a dissipative voltage drop of RI volts to overcome by the driving E_i voltage and a useful voltage drop ωbI, since it has to do with the electromagnetic field power, which is now *in phase* with the exciting current and can no longer be wattless or reactive. It accounts for the *radiated* electromagnetic field energy. The component $-j\omega aI$ volts, due to the field only, causes a wattless field that produces reversible consecutive field energy exchanges between the loop radiator and the surrounding space. We can also account for the well-known circuit relation

$$E_i = RI + L\frac{dI}{dt} + \frac{1}{C}\int I\,dt \qquad \text{volts for any variation}$$

$$E_i = RI + j\omega LI - j\frac{I}{\omega C} \qquad \text{volts for a sinusoidal } I \text{ variation}$$

A localized capacitance of C farads means a *break* in a conducting path, so that instantaneous opposite charges can display themselves on the end faces of the conductors between which is the break or the dielectric gap. The conducting gap faces represent electromagnetically a *discontinuity*, which is exactly what is needed for explaining the $-jI/(\omega C)$ term in the above-mentioned classic network relation. Suppose we put one condenser plate at terminal 1 and another condenser plate at terminal 2 of Fig. 217a and imagine a driving E_i voltage in series with it. As far as the l path is concerned, everything is as before. We still have the IR voltage drop along l, in order to supply the joule in heat loss. We also can retain the $j\omega LI$ reactive drop due to the magnetic field effect around the conductor of length l, but we *can no longer* postulate that the line integral of $\nabla V\,dl$ in (867) vanishes. We have now a condenser *gap*, a *break*, *i.e.*, a *discontinuity* in the l path. This is equivalent to saying that we have now an *open*-line integral from 1 along the length l of the thin conductor to terminal 2, where 1 and 2 are the respective condenser plates, in order to facilitate the formation of $\pm q$ charges on these terminations. Consequently, on any conductor element dl_1, the opposite charges at 1 and 2 act over the distance r_1 and r_2 in addition to what the $I\,dl_2$ effect does to the dl_1 element. As mentioned, the $I\,dl_2$ effect causes what is expressed in (867a), even though this relation is based upon a closed-line integral $\oint_l$ and not upon an open-line integral $\int_l$, which we now have. The gap 1 to 2 of Fig. 217a is taken short, and we know from Maxwell that the gap current, even though a displacement current, is the continuation of the conductor current along the l path. Therefore, the result of (867a) is justified. However, for the *open*-line integral of $\Delta V\,dl_1$ of (867), we find now

$$\int_1^2 \nabla V\, dl_1 = \int_1^2 \frac{\partial V}{\partial l_1}\, dl_1 = V_2 - V_1 \quad \text{volts = instantaneous condenser voltage drop } \frac{1}{C}\int I\, dt$$

in series with the driving E_i voltage, where $V_2 - V_1 = kq$. The proportionality constant k, which is multiplied with the charge q in coulombs, is, as we know, the reciprocal of the C value, and

$$\int_1^2 \nabla V\, dl_1 = V_2 - V_1 = kq = \frac{q}{C} = \frac{\int I\, dt}{C}$$

At VHF's and UHF's, we *must* deal with *retarded* current effects $[I]$ and *retarded* charge effects $[\rho]$. We have no localized simple condition as just discussed but distributed $[I]$ and $[\rho]$ conditions. We have already found this in the discussion of Hallén distributions (pages 705–734). Even if we deal with thin conductors and a localized $\pm q$ charge effect, as just discussed, the effects of the concentrated charges at terminals 1 and 2 of Fig. 217*a* over respective distances r_1 and r_2 will *not* occur at the conductor element dl_1 *simultaneously*. Hence also here, we should expect, not a simple real value for the capacitance C, but a complex capacitance $C = d + je$, leading to

$$\begin{aligned} E_i &= RI + j\omega LI - j\frac{I}{\omega C} = RI + j\omega I(a - jb) - j\frac{I}{\omega(d + je)} \\ &= RI + j\omega(a - jb)I - j\frac{d - je}{\omega(d^2 + e^2)}I \\ &= RI + \omega bI + j\omega aI - \frac{e}{\omega(d^2 + e^2)}I - j\frac{d}{\omega(d^2 + e^2)}I \\ &= \Big[R + \underbrace{\omega b}_{A} - \underbrace{\frac{e}{\omega(d^2 + e^2)}}_{B}\Big]I + j\underbrace{\Big[\omega a - \frac{d}{\omega(d^2 + e^2)}\Big]}_{\text{reactance}}I \qquad (867a) \end{aligned}$$

where A represents the radiation effect due to the retarded I effect and B is the radiation effect due to the retarded effect of $\pm q$. We have then also a radiation resistance $-e/(\omega d^2 + \omega e^2)$ ohms, in addition to the radiation resistance $+\omega b$ ohms due to the retarded effects. We still have not included the effects due to stray capacitances. The radiation resistance terms just mentioned refer to the current value I, which is assumed fixed along the conducting path l, which is quite a limited field of application.

146. Application of the Closed-circuit Theory Based on Maxwell's Equations. We assume uniform longitudinal current distributions I_1 and I_2, which are in phase in respective closed current paths of Fig. 217*c*,

and the diameter of respective conductors is very small. To obtain such a distribution, we need suitable distributed impressed forces or very small loops in comparison with the operating wavelength. We find then the elementary magnetic vector potential due to an $I_2\,dl_2$ current element, *along* the element dl_1 of loop 1 as

$$d\Lambda_1 = \frac{\mu I_2 \epsilon^{-j\beta r_{12}}}{4\pi r_{12}}\, dl_2 \cos\delta \qquad \text{volt}\cdot\text{sec/m}$$

Therefore, the component along a loop element dl_1 due to all $I_2\,dl_2$ effects, must be

$$\Lambda_1 = \frac{\mu I_2}{4\pi}\int \frac{\epsilon^{-j\beta r_{12}}\cos\delta}{r_{12}}\, dl_2 \qquad \text{volt}\cdot\text{sec/m}$$

We have now a means of computing the induced voltage along the l_1 path, carrying current I_1, due to the field of all $I_2\,dl_2$ effects, and find

$$\oint_{l_1} \mathcal{E}_1\, dl_1 = -j\omega \oint_{l_1} \Lambda_1\, dl_1 = -j\,\frac{\omega\mu I_2}{4\pi}\oint_{l_1}\oint_{l_2} \frac{\epsilon^{-j\beta r_{12}}\cos\delta}{r_{12}}\, dl_1\, dl_2 \qquad \text{volts}$$

for

$$\oint_{l_1} \frac{\partial V}{\partial l_1}\, dl_1 = 0; \qquad \oint_{l_1} \mathcal{E}_1\, dl_1 \text{ is induced volts}$$

The impressed voltage $E_i = E_{i_1}$ volts, whether applied *distributed* along the conductor length l_1 or applied *localized* across a small gap of the loop 1, must overcome the induced voltage due to loop 2. This means that it must be in opposition. The mutual impedance Z_{12} is the impressed voltage E_{i_1} divided by the current I_2. We have then

$$\left.\begin{aligned}
Z_{12}^{(\text{complex ohms})} &= j\,\frac{\omega\mu}{4\pi}\oint\oint \frac{\epsilon^{-j\beta r_{12}}\cos\delta}{r_{12}}\, dl_1\, dl_2 \\
&= j\,\frac{\omega\mu}{4\pi}\left[\oint\oint \frac{\cos(\beta r_{12})\cos\delta}{r_{12}}\, dl_1\, dl_2 \right.\\
&\qquad \left. - j\oint\oint \frac{\sin(\beta r_{12})\cos\delta}{r_{12}}\, dl_1\, dl_2\right] \\
&= \underbrace{\frac{\omega\mu}{4\pi}\oint_{l_1}\oint_{l} \frac{\sin(\beta r_{12})\cos\delta}{r_{12}}\, dl_1\, dl_2}_{R_{12} = R_m \text{ mutual radiation resistance from 1 on 2 or 2 on 1)}} \\
&\qquad \underbrace{- j\omega\,\frac{\mu}{4\pi}\oint_{l_1}\oint_{l_2} \frac{\cos(\beta r_{12})\cos\delta}{r_{12}}\, dl_1\, dl_2}_{L_{12} = M \text{ (mutual inductance)} = L_{21}}
\end{aligned}\right\} \qquad (867b)$$

Inasmuch as

$$\sin(\beta r_{12}) = \beta r_{12} - \frac{(\beta r_{12})^3}{6} + \frac{(\beta r_{12})^5}{120} - \frac{(\beta r_{12})^7}{5{,}040} + - \cdots$$

$$\cos(\beta r_{12}) = 1 - \frac{(\beta r_{12})^2}{2} + \frac{(\beta r_{12})^4}{24} - \frac{(\beta r_{12})^6}{720} + - \cdots; \qquad \beta r_{12} = 6.28\frac{r_{12}}{\lambda} = \frac{\omega}{c} r_{12}$$

$$c = 3 \times 10^8 \text{m/sec}; \qquad \omega = 6.28 f^{(\text{cycles/sec})}; \qquad \mu \cong 1.257 \times 10^{-6} \text{ henry/m for air}$$

we have

$$R_m^{(\text{ohms})} = \frac{\mu}{4\pi}\left(\oint_{l_1}\oint_{l_2} \frac{\omega^2}{c}\cos\delta\, dl_1\, dl_2 - \oint_{l_1}\oint_{l_2} \frac{\omega^4 r_{12}^2}{6c^3}\cos\delta\, dl_1\, dl_2 + \oint_{l_1}\oint_{l_2} \frac{\omega^6 r_{12}^4}{120c^5}\cos\delta\, dl_1\, dl_2 - \oint_{l_1}\oint_{l_2} \frac{\omega^8 r_{12}^6}{5{,}040c^7}\cos\delta\, dl_1\, dl_2 + \cdots\right)$$

Inasmuch as there are *reciprocal* mutual actions in the coupled current loops of Fig. 217*c*, the value $2R_m$ is effective for both loop radiators. It is understood that uniform longitudinal and inphase current distributions are assumed along respective loops l_1 and l_2, and the radiation effect is based only on the magnetic retarded vector potential. In exactly the same way, we find for the network of Fig. 217*a*

$$d\Lambda = \frac{\mu}{4\pi r}[I]\cos\delta\, dl_2 \text{ volt}\cdot\text{sec/m}; \quad I = I_0\epsilon^{j\omega t}; \quad [I] = I_0\epsilon^{j(\omega t - \beta r)} \text{ amp}$$

where $d\Lambda$ is the magnitude for the magnetic vector potential of the fine conductor element dl_1 along the dl_1 direction and is due to the $I\, dl_2$ effect. The integral relation (867) then yields

$$\int \dot{\mathcal{E}}_i\, \dot{d}l_1 = \dot{E}_i \text{ volts}; \qquad \int_l \frac{J}{\sigma}\, \dot{d}l_1 = RI \text{ volts}; \qquad I = I_0\epsilon^{j\omega t};$$

$$[I] = I\epsilon^{-j\beta r} \text{ amp}; \qquad \int_l \nabla V\, \dot{d}l = 0$$

$$\left.\begin{aligned}
\int \frac{\partial\Lambda}{\partial t}\, \dot{d}l &= j\omega I \oint_l\oint_l \frac{\mu\epsilon^{-j\beta r}\cos\delta\, dl_2}{4\pi r}\, dl_1 \\
&= j\omega I\left[\frac{\mu}{4\pi}\oint_l\oint_l \frac{\cos(\beta r)\cos\delta}{r}\, dl_1\, dl_2 - j\frac{\mu}{4\pi}\oint_l\oint_l \frac{\sin(\beta r)\cos\delta}{r}\, dl_1\, dl_2\right] \\
&= \underbrace{I\frac{\mu\omega}{4\pi}\oint_l\oint_l \frac{\sin(\beta r)\cos\delta}{r}\, dl_1\, dl_2}_{\text{radiation resistance } R_r \text{ ohms}} \\
&\quad + j\omega I \underbrace{\frac{\mu}{4\pi}\oint_l\oint_l \frac{\cos(\beta r)\cos\delta}{r}\, dl_1\, dl_2}_{\text{self-induction } L \text{ in henrys}} \\
E_i &= (R + R_r)I + j\omega LI \text{ volts}; \\
Z_s &= (R + R_r) + jX_s; \qquad X_s = \omega L
\end{aligned}\right\} \qquad (867c)$$

We arrive then at an expression similar to the mutual impedance $Z_m = Z_{12}$ of (867b), since the l_2 path becomes coincident with the l_1 path. It is now up to the evaluation of respective integrals. In another publication,[1] the retarded current was expanded into a Taylor series. The sine and cosine series after (867b) are applied here. For the radiation resistance R_r of Fig. 217b and a, where an infinitesimal gap is used to apply the impressed voltage E_i, we have

$$R_r = \frac{\mu\omega}{4\pi}\oint_l\oint_l \frac{\sin(\beta r)\cos\delta}{r}\,dl_1\,dl_2 \quad \text{ohms}$$

$$R_r = \frac{\mu\omega}{4\pi}\oint_l\oint_l \left(\beta - \frac{\beta^3 r^2}{6} + \frac{\beta^5 r^4}{120} - \frac{\beta^7 r^6}{5{,}040} + - \cdots\right)\cos\delta\,dl\,dl_2 \quad \text{ohms}$$

But

$$\beta\mu\omega = \beta^2\mu\frac{\omega}{\beta} = \beta^2\mu\frac{\omega}{\omega/c} = \beta^2\mu c = \frac{\beta^2\mu}{\sqrt{\mu\kappa}} = \beta^2\sqrt{\frac{\mu}{\kappa}} = \beta^2 Z_0 = 376.7\beta^2;$$

$$\frac{\beta\mu\omega}{4\pi} = \frac{376.7}{12.56}\beta^2 = 30\beta^2$$

$$\frac{\beta^3}{6}\frac{\mu\omega}{4\pi} = \underbrace{\left(\frac{\beta\mu\omega}{4\pi}\right)}_{=30\beta^2}\frac{\beta^2}{6} = 5\beta^4; \qquad \frac{\beta^5}{120}\frac{\mu\omega}{4\pi} = \underbrace{\left(\frac{\beta\mu\omega}{4\pi}\right)}_{=30\beta^2}\frac{\beta^4}{120} = 0.25\beta^6;$$

$$\frac{\beta^7}{5{,}040}\frac{\mu\omega}{4\pi} = (30\beta^2)\frac{\beta^6}{5{,}040} = 0.005952\beta^8$$

Hence,

$$R_r^{(\text{ohms})} = \underbrace{30\beta^2\oint_l\oint_l \cos\delta\,dl_1\,dl_2}_{=0} - \underbrace{5\beta^4\oint_l\oint_l r^2\cos\delta\,dl_1\,dl_2}_{=-4A^2;\ A \text{ is area enclosed by boundary } l} + 0.25\beta^6\oint_l\oint_l r^4\cos\delta\,dl_1\,dl_2 - 0.00595\beta^8\oint_l\oint_l r^6\cos\delta\,dl_1\,dl_2 + - \cdots$$

where, according to the well-known solutions given in Fig. 217b, the first double integral vanishes and the second double integral is equal to the negative value of four times the square of the area A of the radiating loop in square meters. We have then

$$R_r = 20\beta^4A^2 + 0.25\beta^6\oint_l\oint_l r^4\cos\delta\,dl_1\,dl_2 - 0.00595\beta^8\oint_l\oint_l r^6\cos\delta\,dl_1\,dl_2 + - \cdots \quad \text{ohms} \qquad (867d)$$

where the $\beta = 6.28/\lambda$ value is computed in Table 43. The value $\beta = 1$ is obtained for $\lambda = 6.28$ m, corresponding to a frequency $f = 300/6.28 \cong 48$ Mc/sec. For such

[1] HUND, A., "Phenomena in High-frequency Systems," pp. 346–347, McGraw-Hill Book Company, Inc., New York.

TABLE 43

Frequency f, Mc/sec	10	100	200	300	500	1,000	3,000	10,000
λ, m	30	3	1 5	1	0 6	0 3	0 1	0 03
β, radians/m	0 2093	2 093	4 186	6 279	14 465	20 93	62 79	209 3
β^4	0 00191	19 1	308	$1\ 56 \times 10^3$	$4\ 42 \times 10^4$	$1\ 91 \times 10^5$	$1\ 56 \times 10^7$	$1\ 91 \times 10^9$

a frequency, all powers of β are unity; and below this frequency, the higher the power of β the *smaller* the value, while the opposite is true for frequencies above 48 Mc/sec. Therefore, if we stay below 48 Mc/sec, the first term of (867*d*) seems sufficient as long as the largest cross dimension of the loop remains under 1 m length. This condition, no doubt, must be met, since these formulas can apply only to very small loops for which a fixed longitudinal current distribution along the l path is approximated. If we have a circular loop, we find for a radius ρ in meters an area $A = \pi\rho^2$, $A^2 = \pi^2\rho^4$, and

$$\left.\begin{array}{ll} R_r^{(\text{ohms})} = 197.4\beta^4\rho^4 & \text{for a circular current loop of a radius of } \rho \text{ m} \\ R_r^{(\text{ohms})} = 20\beta^4 A^2 & \text{for any shape of plane area of } A \text{ sq m enclosed by a} \\ & \text{conductor of } l \text{ m} \end{array}\right\} \qquad (867e)$$

If, therefore, a circular loop has a length $l = \lambda/5$, we find $2\pi\rho = \lambda/5$,

$$\rho = \frac{\lambda}{5 \times 2\pi} = \frac{1}{5\beta}; \qquad \rho^4 = \frac{1}{625\beta^4}; \qquad R_r = \frac{197.4}{625} = 0.316 \text{ ohm}$$

147. Final Remarks on Unobstructed Radiation. There are several other practical applications in the chapter on Diffraction. They deal with wave-front radiators and are based upon the Kirchhoff-Huygens principle. It is then necessary to find a suitable function in order to satisfy the boundary conditions of an aperture through which radiation takes place. By "wave front" is then meant the field condition at the mouth of an electromagnetic horn, the opening plane of a wave guide as well as the *open* slots in metal sheets. This applies then to slot radiators instead of *metal* radiators. We have then to distinguish between the electromagnetic Fresnel radiation pattern and the electromagnetic Fraunhofer diffraction pattern. The subject of *diffraction* explains propagation phenomena and field conditions in regions of the space where the well-known electromagnetic ray theory, including atmospheric electromagnetic refraction, falls short of accounting for fields in certain portions of the geometrical shadow region. In recent years, we learned only *too often* that anomalous wave propagation is an occurrence which may last for hours. It may be seasonal and belong to certain geographical regions that yield coverages not expected by customary classic propagation concepts.

Section 107 shows what is meant by polarizations of electromagnetic waves. On page 401 (Fig. 230) of another publication[1] an illustration of an arriving wave with elliptical polarization is given. It is, therefore, important to know how in formulas we can account for the polarization in the case of a radiator as well as in the case of a receiving wave pickup system. This is important in short-wave measurements.

Since most formulas of this text are given in space (spherical) coordinates or some other suitable coordinate system, it would be possible to resolve the general field of elliptical polarization into two perpendicular linear polarizations if we did not have to account for the relative time phase of mentioned polarizations. Here then are the details.

From Sec. 107, we note that for a plane wave front in the XY plane, advancing in the Z direction, we have for the general case the two relations $x = \mathcal{E}_1 \cos(\omega t - \varphi_1)$ and $y = \mathcal{E}_2 \cos(\omega t - \varphi_2)$, since in radio engineering the linear polarized vector $\mathcal{E}$ is used to express the type of polarization. This assumes that the electric $\mathcal{E}$ and magnetic H vectors are perpendicular and in the XY plane, which is taken as the wave front at a certain instant. The values x and y are then the time values of respective linear $\mathcal{E}_1$ and $\mathcal{E}_2$ polarizations. The $\mathcal{E}_1$ electric vector varies then along the $\pm X$ directions and $\mathcal{E}_2$ along the $\pm Y$ directions after the particular wave front moves out of the initial XY plane along the $+Z$ direction with a velocity $v = 1/\sqrt{\mu\kappa}$ m/sec in a pure dielectric wave medium ($\sigma = 0$). For the amplitude $\mathcal{E}_1 = 0$, we have no *linear* polarization along the $\pm X$ directions and just a *single* linear electric polarization $y = \mathcal{E}_2 \cos(\omega t - \varphi_2)$. This would be the case if a *linear* radiator along the Y direction caused this polarization. This is then the simple case of a plane electromagnetic wave with the $\mathcal{E}_2$ variations along the $\pm Y$ directions and the H_2 variations along the $\pm X$ directions, since $\mathcal{E}_2$ and H_2 are normal to each other and the wave front $\mathcal{E}_2$, H_2 moves along Z. Exactly the same situation will occur when $\mathcal{E}_2$ is assumed to be zero in the general definitions above, except that the $\mathcal{E}_1$ vector is now *linearly* polarized along the $\pm X$ directions. In each case, the respective time phases do not play a part, since we have only a single linear $\mathcal{E}$ polarization in either case. Now suppose that we have both linear polarizations occurring simultaneously, as in the general definition, but so that $\varphi_1 = \varphi_2$. Then $x/y = \mathcal{E}_1/\mathcal{E}_2$ with a resultant amplitude $\sqrt{\mathcal{E}_1^2 + \mathcal{E}_2^2}$ and a *linear* polarization in a direction, given by $\tan^{-1}(\varphi_1/\varphi_2)$. It is, therefore, in a measurement possible to find this direction by means of a receiving pole aerial. Now, if we choose in the two general definitions for a *plane* wave, with two *linearly* polarized $\mathcal{E}$ components $\varphi_2 = \varphi_1 + 90°$ and $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, then $x^2 + y^2 = \mathcal{E}^2$ and we have a rotating vector in the moving wave front with *circular* symmetry. Inasmuch as the wave front moves with the velocity v along the Z direction as the resultant vector rotates with the angular velocity ω, we have, therefore, *circular* polarization as for a bay of a turnstile.[2] Hence, we deal with a *circular* resultant wave. For the general case, with two unequal *orthogonal* $\mathcal{E}_1$ and $\mathcal{E}_2$ *linear* polarizations, we find the extremity location of the resultant $\mathcal{E}$ vector by eliminating t with the result:

[1] HUND, A., "Phenomena in High-frequency Systems," McGraw-Hill Book Company, Inc., New York. Consult also Fig. 249 of "Short-wave Radiation Phenomena."

[2] For details consult A. Hund, "Frequency Modulation," pp. 341–344, McGraw-Hill Book Company, Inc., New York, 1942.

$$\frac{x^2}{\mathcal{E}_1^2} + \frac{y^2}{\mathcal{E}_2^2} - \frac{2xy}{\mathcal{E}_1\mathcal{E}_2} \cos (\varphi_1 - \varphi_2) = \sin^2 (\varphi_1 - \varphi_2) \tag{867f}$$

which is an ellipse, as was also found in Sec. 107. This ellipse is characterized by the two axes $2\mathcal{E}_1$, and $2\mathcal{E}_2$ and $\varphi_1 - \varphi_2$. Hence, the two general orthogonal $\mathcal{E}_1$ and $\mathcal{E}_2$ linear polarizations with a relative time phase $\varphi_1 - \varphi_2 = \varphi$ cause elliptical polarizations, while the resultant as well as single linear polarizations, described before, are only simplifications of the general case.

The case of *elliptical* polarization requires special parameters when dealing with patterns of aerial systems when receiving electromagnetic waves with *arbitrary* polarization. This has special reference to airborne transmitting and receiving antennas and also to the case of reception affected by the earth's crust.[1]

As far as wave propagation for such general *linear* orthogonal $\mathcal{E}_1$ and $\mathcal{E}_2$ polarizations is concerned, we must realize that the initial wave front in which $\mathcal{E}_1$ and $\mathcal{E}_2$ are located moves out normal to the plane of the $\mathcal{E}_1$, H_1 and $\mathcal{E}_2$, H_2 linear polarizations. We have then the general equations of motion for the two component waves:

$$\left.\begin{array}{c} x = \mathcal{E}_1 \cos (\omega t - \beta z - \varphi_1); \qquad y = \mathcal{E}_2 \cos (\omega t - \beta z - \varphi_2) \\ \varphi = \varphi_1 - \varphi_2 \end{array}\right\} \tag{867g}$$

Inasmuch as the wave front moves along the Z direction, during the wave motion we can have only circular resultant $\mathcal{E}$ vector rotation while the wave front moves along Z, and we have helical performance in space, which is that of a *right*-handed screw for $\varphi_2 = \varphi_1 - 90°$, and *left*-handed for $\varphi_2 = \varphi_1 + 90°$. This is also the case for elliptical polarization, except that the helix is flattened, where we pass through the smaller axis. The case of right- and left-handed polarizations (circular) plays a part in gyration.[2]

The *counterclockwise* and *clockwise* space screw[3] propagations *about* and *along* the Z axis have great practical significance. They give a means for received wave discrimination. For instance, if a circularly polarized wave is *incident* to a circularly polarized wave pickup aerial system, the pickup aerial can receive field energy only if it has the same (*exactly alike*) screw rotation and sense (left-handed or right-handed, respectively) as the arriving incident circularly polarized wave. The circularly polarized pickup aerial system, however, will *ignore* the reception of an incident circularly polarized wave of opposite space rotation about the Z axis. This is also the case for all angles around the Z axis in the XY plane to which the receiving aerial system is inclined (this

[1] Consult Sec. 134 on Ground, Radiation, and Aerial Resistance and Their Determination in "High-frequency Measurements," 1st ed., pp. 290–298, and also 2d ed., pp. 340–348.

[2] Consult "Phenomena in High-frequency Systems," pp. 394–400, McGraw-Hill Book Company, Inc., New York. See also "High-frequency Measurements," 2d ed., 1951, pp. 551–553.

[3] Space screw propagation (helical) can be accomplished by two properly phased and space-oriented linear as well as coil radiators.

means an $X'Y'$ plane is inclined against the XY plane of the incident XYZ coordinate system[1]). Moreover, since the *reflected* field energy from a highly conducting plane (copper surface) is *right*-handed circularly polarized for a *left*-handed *incident* direct field energy, a *left*-handed wave pickup aerial system *must discriminate against the reflected* polarized wave (*i.e.*, will *not* receive it). It is understood that a *solid* copper surface must be used as a reflector for discrimination. A copper reflector of parallel thin wires cannot be used, since such a wire reflector will reflect essentially only the $\mathcal{E}$ component parallel with the wires.

As far as the field energy is concerned, for the general definitions it is proportional to $x^2 + y^2$, where the wave impedance Z_0 is the factor by means o which we divide for an $\mathcal{E}$ polarization. For electromagnetic waves traveling in *arbitrary* directions instead of along the Z axis, we have generally three components of the $\mathcal{E}$ vector, and in such a case $x^2 + y^2 + z^2$ must be averaged for obtaining the value which is proportional to the moving field power. From the above discussion, we note that a single plane wave is actually a plane wave front with $\mathcal{E}$ and H vectors orthogonal and in the wave front which moves normal to this front. Therefore, $\mathcal{E}$ and H vectors experience linear polarizations which are normal to each other as the wave front travels normal outward in unobstructed radiation space If we deal with several plane waves of the same operating frequency with all waves moving in the same direction and with *arbitrary orientations* of the $\mathcal{E}$ as well as the H vectors, we have an *unpolarized* wave.

For taking polarization into account, certain factors as well as vectors have been proposed in the current literature[2] in which R. E. Burgess has suggested a generalized *equivalent* length $l\ (\theta,\ \varphi,\ \alpha)$ by means of the relation

$$\mathcal{E}(\alpha) = \frac{Z_0 Il}{2r\lambda}$$

where $\mathcal{E}(\alpha)$ is the field component of polarization α in volts per meter, r is the distance in the meters from the radiator to the field point, I is the terminal current in amperes, and $Z_0 \cong 377$ ohms and $\lambda^m = 300/f^{(\mathrm{Mc/sec})}$. Such a definition holds for the performance of a radiator and for the performance of an aerial receiving plane waves of *arbitrary linear* polarization. For the reception of plane waves of *arbitrary elliptical* polarization we must, however, use the concept of a *vector* $\underset{\sim}{l}$ or a *numeric* parameter l_n introduced by G. Sinclair (*op. cit.*). The case of a receiving field pickup for a plane polarized field of arbitrary polarization was also treated by C. W. Harrison, Jr. and R. King.[3] The *generalized* equivalent length $l\ (\theta,\ \varphi,\ \alpha)$, suggested by R. E. Burgess, as well as the general complex

[1] For details consult W. Sichak and S. Milazzo, *Proc. IRE*, **36**, 997-1001, August, 1948.

[2] BURGESS, R. E., Aerial Characteristics, *Wireless Eng.*, **21**, 145–160, April, 1944. SINCLAIR, G., The Transmission and Reception of Elliptical Polarized Waves, *Proc. IRE*, **38**, 148–151, February, 1950.

[3] *Proc. IRE*, **32**, 35–48, January, 1948.

vector $\underset{.}{l}$, suggested by G. Sinclair, simplify in the case of well-known antenna structures to formulas given already in this text and confirm the equivalent length l_e of such antennas. Thus, for a linear half-wave antenna along the vertical Z axis, the complex vector $\underset{.}{l}$ exhibits only a θ value in the spherical r, θ, φ coordinate system with the directivity D_θ given in Eq. (9) and l_e from page 166 to give a value

$$l_\theta = l_e D_\theta = \frac{2}{\pi} l D_\theta = 0.637 l D_\theta$$

since for a half-wave current distribution, the *actual* antenna length $l = 0.5\lambda$ and the customary equivalent length $l_e = 2l/\pi$. Hence, with such concepts, the value of the vector $\underset{.}{l}$ degenerates to the product of the customary *equivalent* linear antenna length l_e and the well-known directivity D_θ of a half-wave antenna. In the same way, a small current loop (small *magnetic* dipole) of area S in m^2 in the XY plane can exhibit only an l_φ value for the complex vector l with the result:

$$l_\varphi = -j \frac{6.28S}{\lambda} \sin \theta$$

The treatment of Sinclair is based on the Schelkunoff radiation vector N (Secs. 124 and 125) although customary procedures for finding the electric field intensity $\mathcal{E}$ could be used. If we deal then *only* with the *magnetic* retarded vector potential $\Lambda = \Lambda_m$ concepts, we have

$$\mathcal{E} = -j \frac{Z_0 N}{2r\lambda} \epsilon^{j(\omega t - \beta r)} = j \frac{Z_0 l I}{2r\lambda} \epsilon^{j(\omega t - \beta r)} \qquad \text{volts/m} \qquad (867h)$$

where $\underset{.}{l} = \dfrac{-N}{I}$ and I is in amperes and *actual* length l is in meters.

These expressions are based on the fact that N is proportional to the input current I for a radiator. For a receiving electrical antenna system, the ratio N/I yields a value that depends only on the configuration of the antenna and the application of the feeder. These expressions are also based on the fact that the radiation vector N, if a complex vector, must account for an electrical field $\mathcal{E}$ with *arbitrary elliptical* polarization if this should be the case. Generally, l has not only one component but l_θ as well as l_φ components. If ψ is the angle by which l_φ leads l_θ, we have $l_\varphi = |l_\theta|\epsilon^{j\psi}$. Hence, for a turnstile bay, with circular polarization, $\psi = \pm 90°$, depending on the rotation direction of the resultant $\mathcal{E}$ vector The *vector* l has a length in meters. It can be changed to a *numeric* length l_n, and we have for $l_n = l/\lambda$

$$\mathcal{E} = j \frac{Z_0 I l_n}{2r} \epsilon^{j(\omega t - \beta r)} \qquad \text{volts/m} \qquad (867i)$$